Review on long time asymptotics and collision of solitons for the quartic gKdV equation*

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Abstract

In these notes, we review recent nonlinear PDE techniques developed to address questions concerning solitons for the quartic generalized Korteweg-de Vries equation (gKdV) and other generalizations of the (KdV) equation. We draw a comparison between results obtained in this way and some elements of the classical integrability theory for the original (KdV) equation, which serve as reference in soliton and multi-soliton problems.

First, known results on stability and asymptotic stability of solitons for general (gKdV) equations are reviewed from several different sources. Second, we consider the problem of the interaction of two solitons for the quartic (gKdV) equation. We focus on recent results and techniques from [60] concerning the interaction of two almost equal solitons.

1 Introduction

The well-known (KdV) equation on the real line

$$\partial_t u + \partial_x(\partial_x^2 u + u^2) = 0 \quad t, x \in \mathbb{R}, \tag{1.1}$$

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was introduced by Korteweg and de Vries [38] as a model for long waves propagating in a channel. Later, in the sixties, the (KdV) equation and the modified (KdV) equation (i.e. a variant of the (KdV) equation with cubic nonlinearity, $p = 3$ in the (gKdV) equation below) were found to be relevant in a large number of different physical settings. These equations, as for example the nonlinear Schrödinger equations (NLS), are thus considered as universal models in Mathematical Physics.

Recall that both (KdV) and (mKdV) equations are completely integrable, which means that they have remarkable algebraic structures and consequently enjoy very special features for nonlinear partial differential equations (see e.g. Lax [42], Lamb [40], Miura [69] and Schuur [75]). We recall in Section 7 some of the most classical results on (KdV) concerning multi-solitons obtained using the inverse scattering transform.

In these notes, we consider the following classical generalizations of the Korteweg-de Vries equation:

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0 \quad t, x \in \mathbb{R}, \tag{1.2}$$

for $p \geq 2$ integer. In fact, we will mainly focus on the quartic case $p = 4$, for which the equation is not completely integrable. We review several results concerning stability and asymptotic stability of the family of solitons, existence, uniqueness and stability of multi-soliton type solutions and finally the vast problem of collision of solitons.

The inverse scattering transform led to results on the (KdV) equation which are remarkable but not accessible for non integrable equations and, in most situations, cannot be extended even to perturbations of the model. Moreover, the use of the inverse scattering transform usually prevents from obtaining results in the energy space – see Sections 2 and 7.

One of our main points is that, in contrast with results based on the inverse scattering transform and complete integrability, most of the theory discussed in these notes for (gKdV) with $p = 4$, based on nonlinear PDE tools, can be directly extended to the further generalizations

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0 \quad t, x \in \mathbb{R}, \tag{1.3}$$

under standard assumptions on the nonlinearity f, and even to other KdV type equations, such as the BBM equation or the Benjamin-Ono equation. There are also many similarities between (gKdV) equations and (NLS) equations. This motivates our interest in studying (gKdV) equations in a typical non integrable situation and in developing tools in the energy space rather than in smaller specific spaces where inverse scattering theory is applicable (see Eckaus and Schuur [17] and Cohen [10] about this point).

We now present the contents of these notes. After briefly recalling basic properties of the model (symmetries and invariants) and the Cauchy theory in Section 2, we introduce in Section 3 the main object of the study in this course, namely the solitons for (gKdV). In Section 3, we also discuss the spectral properties of the linearized operator around the solitons and deduce a simple proof of stability due to Weinstein [83]. These arguments are stationary in the sense that they only involve the invariants of the equation (mass and energy – see Section 2).

Starting from Section 4, we discuss more recent results on soliton dynamics, at the heart of the two authors' research on (gKdV) equations in the last ten years. Note that we exclude from the discussion the blow up problem for the critical (gKdV) equation (quintic case, i.e. $p = 5$ in (1.2)), studied in a series of papers [48], [65], [51], [52]. For a survey on these questions, we refer the reader to [53] and [78]. Nevertheless, we point out that these earlier works on blow up were decisive in studying soliton questions for the subcritical case ($p = 2$, 3 and 4 in (1.2) or (1.3) under suitable assumptions on f).

In Section 4, we present a result of asymptotic stability of solitons. Then, in Section 5, we recall monotonicity results of localization of conservation laws which revealed to be a fundamental tool for KdV-type equations. In Section 6, we sketch the proof of asymptotic stability by rigidity properties and discuss the generality of this approach.

We then discuss multi-soliton problems, i.e. questions on solutions containing several solitons. It is natural at this point to recall the existence and properties of the remarkable multi-solitons of the integrable case – see Section 7. Then, in Section 8, we recall existence and stability of multi-soliton solutions in non integrable situations. This leads to the fundamental question of the collision and interaction of several solitons in the non integrable case.

In the non integrable case, without special algebraic structure, the collision problem is very difficult, out of reach of current nonlinear PDE theory in general. In recent works [59], [60], the two authors have addressed the problem of collision of two solitons for the quartic (gKdV) equation in specific asymptotic situations where explicit computations are possible and provide a precise description of the phenomenon:

- The case of two solitons with very different sizes. See Section 9.
- The case of two solitons with almost same sizes. See Section 10.

In these two special cases, we claim that the two soliton structure is preserved for all time, similarly, at least at the main order, as in the integrable case. However, going further into the description of the collision, we characterize the lack of integrability of the quartic (gKdV) equation: by performing more precise computations, we prove that a small defect (and consequently some loss of mass) is produced by the collision, in contrast with the very special phenomenon displayed by the *pure* multi-solitons in the integrable case.

In these notes, we discuss more extensively the case where the two solitons have almost the same size, since the first situation has been already sketched in a previous review paper [57]. In Section 10, we recall the main result from [60] and we sketch the proof, by giving some insights on the construction of a relevant approximate solution to the problem.

Finally, we refer the reader to the following other review papers: de Bouard [8] and Martel [46] notes discuss long time stability and asymptotic stability problems both for (gKdV) and (NLS) models. Tzvetkov [78] and Martel and Merle [53] focus on the blow up problem for the critical (gKdV) equation. Tao's survey [73] presents problems of stability and asymptotic stability of solitons in a larger perspective.

Martel and Raphaël [67] provides a review at a more elementary level which aims at presenting the nonlinear PDE approach to soliton theory as a unifying concept for long time problems, stability and blow up in several dispersive equations.

Miura [69] and Schuur [75] are older works, presenting the integrability theory for the (KdV) equation and the relevance of this model in Physics.

Finally, we mention Craig et al. [15] which compares numerical computations for the water wave problem, real experiments in water tanks and the explicit multi-solitons of the (KdV) equation. Of course, this short references list is by no means exhaustive.

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2 First properties and Cauchy problem for (gKdV)

First, recall the symmetries of the gKdV equation:

- Scaling invariance: if $u(t,x)$ is solution of (1.2) then $u_{c_0}(t,x) = c_0^{\frac{1}{p-1}} u(c_0^{\frac{3}{2}} t, c_0^{\frac{1}{2}} x)$ is also solution, for any $c_0 > 0$;
- Translation invariance: if $u(t,x)$ is solution of (1.2) then $u_{t_0,x_0}(t,x) = u(t-t_0, x-x_0)$ is also solution, for any t_0, $x_0 \in \mathbb{R}$;
- Reversibility: if $u(t,x)$ is solution of (1.2) then $\tilde{u}(t,x) = u(-t,-x)$ is also solution.

Note that, for all $c_0 > 0$, $\|u_{c_0}\|_{L^2} = c_0^{\frac{1}{p-1}-\frac{1}{4}} \|u\|_{L^2}$ and $\frac{1}{p-1} - \frac{1}{4} > 0$ if and only if $1 < p < 5$. Thus, $p = 5$ in (1.2) is L^2–critical in the sense that the scaling of the equation does not affect the L^2 norm.

We consider solutions of (1.2) which converge to zero (as well as their derivatives) as $x \to \pm\infty$. Then, formally, the following quantities are conserved through time:

$$\int u(t) = \int u(0), \tag{2.1}$$

$$\int u^2(t) = \int u^2(0), \tag{2.2}$$

$$E(u(t)) = \frac{1}{2} \int (\partial_x u(t))^2 - \frac{1}{p+1} \int u^{p+1}(t) = E(u(0)). \tag{2.3}$$

Indeed, the first conservation is obtained by simply integrating the equation. The L^2 conservation follows from multiplying the equation by $u(t)$ and integrating. Finally, the energy conservation can be seen by multiplying the equation by $(\partial_x^2 u + u^p)$ and integrating by parts.

In the special cases $p = 2$ and 3, there are infinitely many such conserved quantities – see Miura [69]. These quantities involve higher order Sobolev norms.

From (2.2) and (2.3), $H^1 = \{u \in L^2,\ \partial_x u \in L^2\}$ is the energy space for (1.2), i.e. the most natural functional space in which looking for solutions. The local Cauchy problem for (1.2) in H^1 is now known to be well-posed but its complete resolution for all $p \geq 1$ has taken a long time and involves very sharp and delicate tools from harmonic analysis. We refer the interested reader to the following papers and the references therein: [30], [20], [32], [35]. We gather in the following statement results from Kenig, Ponce and Vega [35] restricted to the space H^1.

Theorem 1 (H^1 local Cauchy theory [35]). *Let $p \geq 2$.*

- *Well-posedness. For all $u_0 \in H^1$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in Z_T \cap C([0,T], H^1)$ of (1.2) on $[0,T)$.*
- *Lipschitz flow. For any $T' \in (0,T)$, there exists an H^1 neighborhood V of u_0 such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from V to $Z_{T'} \cap C([0,T'], H^1)$ is Lipschitz.*
- *Persistence of regularity. If $u_0 \in H^s(\mathbb{R})$ for $s > 1$, then $u \in C([0,T], H^s(\mathbb{R}))$.*

The space Z_T constructed in [35] actually depends on the value of p and involve several space-time $L^q L^p$ spaces. We refer to [35] for its definition. Note also that Kenig, Ponce and Vega actually proved a stronger result of well-posedness in H^s, for $0 \leq s_p \leq s < 1$, where s_p depends on p. The proofs in [35] are based on several delicate linear estimates on the Airy group (the Airy equation is $\partial_t u + \partial_x^3 u = 0$): Strichartz type estimates, sharp Kato smoothing effect and maximal function estimates (see (3.36), (3.6) and (3.9) in [35]).

Note that by continuous dependence of the solution upon the initial data, persistence of regularity and density arguments, one can approach any H^1 solution in the sense of Theorem 1 by a sequence of more regular solutions (say, solutions in $H^\infty = \cap_{s \geq 0} H^s$) to justify formal computations. In particular, this allows to prove L^2 norm and energy conservation for H^1 solutions.

From Theorem 1, it is standard to deduce that for any $u_0 \in H^1$, there exists a unique (in the above sense) maximal solution $u \in C([0,T^*), H^1)$ of (1.2), for some $T^* = T^*(u_0)$ satisfying the following alternative:

- Either $T^* = +\infty$, which means that the solution is global in time.
- Or $T^* < +\infty$ and then $\lim_{t \uparrow T^*} \|u(t)\|_{H^1} = +\infty$. In that case, we say that the solution blows up in finite time.

In particular, an a priori bound on the H^1 norm of a given solution is sufficient to prove that the solution is global. Using the so-called Gagliardo–Nirenberg inequality, for any $p > 1$,

$$\forall v \in H^1, \quad \int |v|^{p+1} \leqslant C_p \left(\int |\partial_x v|^2 \right)^{\frac{p-1}{4}} \left(\int |v|^2 \right)^{\frac{p+3}{4}}, \tag{2.4}$$

the conservation of mass and energy, we deduce the following well-known characterization of the L^2–critical case $p = 5$ with respect to global well-posedness:

- For $p = 2$, 3 and 4, any H^1 solution of (1.2) is globally defined and bounded in H^1. Indeed, for any t such that the solution $u(t)$ exists, we have

$$E(u(0)) = E(u(t)) \geq \frac{1}{2}\int (\partial_x u)^2(t) - \frac{C_p}{p+1}\left(\int (\partial_x u)^2(t)\right)^{\frac{p-1}{4}}\left(\int |u|^2\right)^{\frac{p+3}{4}}$$

  and for $0 < \frac{p-1}{4} < 1$, we obtain an a priori bound on the L^2 norm of $\partial_x u(t)$, which in particular prevents blow up.

- For $p = 5$, using a similar argument, we see that small solutions in L^2 are globally defined and bounded in H^1. However, blow up solutions do exist. See [66] and [52].

- For $p > 5$, the problem of global existence versus blow up is completely open, though there are strong evidences that blow up solutions should exist.

As mentioned in the introduction, we shall not discuss the critical or supercritical ($p \geq 5$) cases in these notes. Therefore, all considered solutions are global and bounded in H^1. Of course, this does not say much about their asymptotic behaviour in large time, which is the main question considered here in the context of soliton theory.

We conclude this section by an important remark concerning the continuity of the (gKdV) flow for weak H^1 convergence.

Theorem 2 (Weak continuity of the flow map)**.** *Let $p \geq 2$. Let $\{u_n\}_n$ be a sequence of H^1 solutions of* (1.2) *in $[0,T]$; assume that $u_n(0) \rightharpoonup u_0$ in H^1 weak. Assume also that $\|u_n(0)\|_{H^1} \leq A$, $\|u_0\|_{H^1} \leq A$, and $T \leq T(A)$ where $T(A)$ is defined as in Theorem 1. If $u(t)$ is the solution of* (1.2) *corresponding to $u(0) = u_0$, then*

$$\forall t \in [0,T], \quad u_n(t) \rightharpoonup u(t) \quad \text{in } H^1 \text{ weak.}$$

For $p = 2$, 3, 4 and 5 the result was proved in [48], [50]. Arguments developed in [31] for the Benjamin-Ono equation (see also [16], [22]) can be used to obtain simplified proofs; this was done in the Appendix of [11] for the case of the (gKdV) equation with $p > 5$.

Theorem 2 is a key tool to contruct limiting objects recurrent in the behavior of a given solution. In several situations, we will see that strong convergence cannot be obtained.

3 Solitons, linearized operator and stability

We look for travelling wave solutions of the form $u(t,x) = Q_c(x - ct)$, where $c \in \mathbb{R}$. We obtain the following equation for Q_c:

$$Q_c'' + Q_c^p = cQ_c.$$

There exist H^1 solutions of this equation only for $c>0$, as can be easily seen by Pohozaev's identity (multiply the equation by xQ_c' and integrate). In that case, the solution is unique up to translation and explicit:

$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x) \quad \text{where} \quad Q(x) = \left(\frac{p+1}{2}\cosh^{-2}\left(\frac{p-1}{2}x\right)\right)^{\frac{1}{p-1}}. \tag{3.1}$$

Thus, the only travelling wave solutions of (1.2) are $R_{c,x_0}(t,x) = Q_c(x-x_0-ct)$.

Now, we define the linearized operator around the soliton $Q_{c_0}(x-c_0t)$ by

$$\mathcal{L}_{c_0}\varepsilon = -\partial_x^2\varepsilon + c_0\varepsilon - pQ_{c_0}^{p-1}\varepsilon.$$

and we recall its spectral properties.

Lemma 3.1 (Properties of $\mathcal{L}_c$)**.** *Let $p>1$. Let $c>0$. The operator $\mathcal{L}_c$ defined in $L^2(\mathbb{R})$ by*

$$\mathcal{L}_c f = -f'' + cf - pQ_c^{p-1}f$$

is self-adjoint and satisfies the following properties:

(i) *First eigenfunction : $\mathcal{L}_c Q_c^{\frac{p+1}{2}} = -\frac{1}{4}(p-1)(p+3)cQ_c^{\frac{p+1}{2}}$;*

(ii) *Second eigenfunction : $\mathcal{L}_c Q_c' = 0$; the kernel of $\mathcal{L}_c$ is $\{\lambda Q_c', \lambda\in\mathbb{R}\}$;*

(iii) *For any function $h\in L^2(\mathbb{R})$ orthogonal to Q_c' for the L^2 scalar product, there exists a unique function $f\in H^2(\mathbb{R})$ orthogonal to Q_c' such that $\mathcal{L}_c f = h$; moreover, if h is even (respectively, odd), then f is even (respectively, odd).*

(iv) *Assume $1<p\le 5$. Then, for all $f\in H^1$,*

$$\text{if} \int f^2 = \int Q_c^2 \text{ and } E(f) = E(Q_c), \text{ then } f = Q_c(x-x_0) \text{ for some } x_0. \tag{3.2}$$

(v) *Assume $1<p<5$. Then, there exists $\lambda_c>0$ such that, if $f\in H^1(\mathbb{R})$ satisfies $\int Q_c f = \int Q_c' f = 0$, then*

$$(\mathcal{L}_c f, f) = \int f_x^2 + cf^2 - pQ_c^{p-1}f^2 \ge \lambda_c \|f\|_{H^1}^2. \tag{3.3}$$

Sketch of the proof of Lemma 3.1. Note first that, for all $p>1$,

$$pQ^{p-1}(x) = \frac{1}{2}p(p+1)\cosh^{-2}\left(\frac{p-1}{2}x\right),$$

and thus the operator $\mathcal{L}_{c_0}$ is a classical operator in Physics. Its spectral properties can be deduced from classical references in the subject, see e.g. Titchmarsh [77]. Note that (i) and (ii) can also be checked by direct computations and ODE arguments. Property (iii) can be deduced from (i) and (ii) by standard arguments (Fredholm alternative).

The arguments given above to justify (i) and (ii) depend heavily on the space dimension. Variational arguments are alternative arguments for the construction of Q_c, which can be extended to equation $\Delta Q_c - cQ_c + Q_c^p = 0$ on $\mathbb{R}^N$. In the subcritical and critical cases, we deduce from these variational arguments the characterization of Q_c given in (iv).

Property (v), true only in the subcritical case $1 < p < 5$, is a more subtle property since Q is not the eigenfunction related to the negative eigenvalue. It is related to the fact that the map $c \mapsto \int Q_c^2 = c^{\frac{5-p}{2(p-1)}} \int Q^2$ is increasing and $\mathcal{L}_c \left(\frac{dQ_c}{dc}\right) = -Q_c$. See [83]. □

Theorem 3 (Stability of the soliton for (gKdV))**.** *Let $p = 2$, 3 or 4. Let $c_0 > 0$. Let $u(t)$ be an H^1 solution of the (gKdV) equation* (1.2)*. There exists $K > 0$ such that if $\alpha_0 > 0$ is small enough and $\|u(0) - Q_{c_0}\|_{H^1} \leq \alpha_0$ then, for all $t \in \mathbb{R}$, there exists $x(t) \in \mathbb{R}$, such that*

$$\|u(t) - Q_{c_0}(\cdot - x(t))\|_{H^1} + |x'(t) - c_0| \leq K\alpha_0. \tag{3.4}$$

This notion of stability corresponds to orbital stability which means that the freedom due to the translation parameter $x(t)$ is necessary. Indeed, in general, it is not true in the context of Theorem 3 that $\|u(t) - Q_{c_0}(x - c_0 t)\|_{H^1}$ is small. For example, if $u(t) = Q_c(x - ct)$ with $c \neq c_0$ close to c_0, then $\|Q_c(x-ct) - Q_{c_0}(x-c_0t)\|_{H^1}^2 \to \|Q_c\|_{H^1}^2 + \|Q_{c_0}\|_{H^1}^2$ as $t \to +\infty$.

The proof of the above result does not really use the equation but only the two H^1 conserved quantities, the L^2 norm and the energy (2.2)–(2.3), and the variational information on Q_{c_0}. This stability result is due to Benjamin [2], Bona [5], Cazenave and Lions [9] and Weinstein [82]. Note also that the above formulation of the stability result for (gKdV) requires the local well-posedness of the Cauchy problem in H^1, i.e. Theorem 1, which was proved later by Kenig, Ponce and Vega [35] as recalled in the previous section.

Proof of Theorem 3. We repeat the proof from Weinstein [83]. By scaling arguments, we could assume $c_0 = 1$, but it is better to understand the dependence on c_0 to be able to deal with the case of several solitons with different scalings in the next sections. Also, with respect to [83], we use additional modulation arguments on the scaling for the same reason that it is used to handle the case of several solitons. Let $u(t)$ be a solution of (1.2) satisfying $\|u(0) - Q_{c_0}\|_{H^1} < \alpha_0$, for α_0 small.

1. Decomposition of the solution by modulation theory. We argue on a time interval $[0, t^*]$ so that, for all $t \in [0, t^*]$, $u(t)$ is close in H^1 to $Q_{c_0}(x - x_0(t))$ for some $x_0(t)$. It follows from the implicit function theorem that we can slightly modify the scaling and translation parameters c_0, $x_0(t)$ into $c(t)$, $x(t)$ such that

$$\varepsilon(t,x) = u(t,x) - R(t,x), \quad \text{where} \quad R(t,x) = Q_{c(t)}(x - x(t)), \tag{3.5}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), R(t)) = (\varepsilon(t), \partial_x R(t)) = 0, \tag{3.6}$$

where $(,)$ denotes the L^2 scalar product. We call this decomposition the modulation of the solution. This choice of orthogonality conditions is well adapted to the positivity properties on $\mathcal{L}$, see Lemma 3.1, and thus it is suitable to apply an energy method. Note that we have from $\|u(0) - Q_{c_0}\|_{H^1} \leq \alpha_0$,

$$\|\varepsilon(0)\|_{H^1} + |c(0) - c_0| \leqslant C\alpha_0. \tag{3.7}$$

2. Introduction of a functional adapted to the stability problem. Following Weinstein [83], we prove the stability property by using the conservation in time of a functional related to the two invariant quantities. We define

$$\mathcal{G}(u) = 2E(u) + c(0) \int |u|^2. \tag{3.8}$$

By expanding $u(t) = R(t) + \varepsilon(t)$ in the definition of $\mathcal{G}(u(t))$, we obtain the following.

Lemma 3.2. *The following holds*

$$\mathcal{G}(u(t)) = \mathcal{G}(Q_{c(0)}) + (\mathcal{L}_{c(0)}\varepsilon(t), \varepsilon(t)) + \|\varepsilon(t)\|_{H^1}^2 \beta(\|\varepsilon(t)\|_{H^1}) + O(|c(t) - c(0)|^2),$$

with $\beta(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Note that we have used $\mathcal{G}(Q_{c(t)}) = \mathcal{G}(Q_{c(0)}) + O(|c(t) - c(0)|^2)$, which comes from $\frac{d}{dc}\mathcal{G}(Q_c) = 2\frac{d}{dc}E(Q_c) + c\frac{d}{dc}\int Q_c^2 = 0$ applied at $c(0)$.

3. Control of $|c(t) - c(0)|$. We prove that $|c(t) - c(0)|$ is quadratic in $\varepsilon(t)$. Note that by the conservation of $\int u^2(t)$, and the orthogonality condition $\int R\varepsilon = 0$, we have

$$\int Q_{c(t)}^2 - \int Q_{c(0)}^2 = \left(c^{\frac{5-p}{2(p-1)}}(t) - c^{\frac{5-p}{2(p-1)}}(0)\right) \int Q^2 = -\int |\varepsilon(t)|^2 + \int |\varepsilon(0)|^2. \tag{3.9}$$

Thus, by linearization, we deduce

$$|c(t) - c(0)| \leqslant C\|\varepsilon(t)\|_{L^2}^2 + C\|\varepsilon(0)\|_{L^2}^2. \tag{3.10}$$

4. Control of $\|\varepsilon(t)\|_{H^1}$. Since $\mathcal{G}(u(t))$ is the sum of two conserved quantities, we have $\mathcal{G}(u(t)) = \mathcal{G}(u(0))$. Thus, by Lemma 3.2, it follows that

$$\begin{aligned}(\mathcal{L}_{c(0)}\varepsilon(t), \varepsilon(t)) \leqslant (\mathcal{L}_{c(0)}\varepsilon(0), \varepsilon(0)) + C|c(t) - c(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2 \beta(\|\varepsilon(0)\|_{H^1}) \\ + C\|\varepsilon(t)\|_{H^1}^2 \beta(\|\varepsilon(t)\|_{H^1}).\end{aligned}$$

By Lemma 3.1, and since $(\mathcal{L}_{c(0)}\varepsilon(0), \varepsilon(0)) \leqslant C\|\varepsilon(0)\|_{H^1}^2$, we obtain:

$$\lambda_1 \|\varepsilon(t)\|_{H^1}^2 \leqslant (\mathcal{L}_{c(0)}\varepsilon(t), \varepsilon(t)) \leqslant C|c(t) - c(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2 + C\|\varepsilon(t)\|_{H^1}^2 \beta(\|\varepsilon(t)\|_{H^1}),$$

which gives

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant C|c(t) - c(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2, \tag{3.11}$$

for $\|\varepsilon(t)\|_{H^1}$ small enough.

5. Conclusion. Combining (3.11) and (3.10), and then (3.7), we obtain

$$\|\varepsilon(t)\|_{H^1}^2 + |c(t) - c(0)| \leqslant C\|\varepsilon(0)\|_{H^1}^2 \leqslant C\alpha_0^2,$$

for some constant $C > 0$, for $\|\varepsilon(t)\|_{H^1}$ and $|c(t) - c(0)|$ small enough. Thus, for α_0 small enough,

$$\begin{aligned}\|u(t) - Q_{c_0}(x - x(t))\|_{H^1} &\leqslant \|u(t) - R(t)\|_{H^1} + \|R(t) - Q_{c_0}(x - x(t))\|_{H^1}\\ &\leqslant \|\varepsilon(t)\|_{H^1} + C|c(t) - c_0| \leqslant \|\varepsilon(t)\|_{H^1} + C|c(t) - c(0)| + C|c(0) - c_0| \leqslant C\alpha_0.\end{aligned}$$

This completes the proof of stability of one solitary wave. □

4 Asymptotic stability of solitons

After the stability property stated in Theorem 3, we investigate asymptotic stability properties of the family of solitons. Indeed, in the context of Theorem 3, estimate (3.4) does not tell the whole story about the behavior of $u(t)$ for large time. It is legitimate to wonder whether $u(t)$ should actually converge to a soliton in some sense.

In looking for convergence, one has to keep in mind the following observations:

- Assume that $u(t)$ is an H^1 solution of (1.2) such that $\lim_{t\to+\infty} \|u(t) - Q_c(\cdot - x(t))\|_{H^1} = 0$ for some $x(t)$. This implies $\int u^2 = \int Q_c^2$ and $E(u) = E(Q_c)$. It then follows from (3.2) that $u(t)$ is exactly a soliton. Therefore, one cannot expect asymptotic stability of solitons in such a strong H^1 sense. In fact, we have to restrict the convergence to some local in space norm.

- Let $c_0 > 0$. For $c \neq c_0$ close to c_0, $\|Q_c - Q_{c_0}\|_{H^1}$ is small, but $\|Q_c - Q_{c_0}\|_{H^1} \geq K(c - c_0)$ and thus $Q_c(x - ct)$ does not converge to $Q_{c_0}(x - c_0t)$ even up to a translation parameter. This means that a correct notion of asymptotic stability has to allow freedom both in the translation parameter and the scaling parameter.

We now state an asymptotic stability result for the (gKdV) equation (1.2) in the subcritical case.

Theorem 4 (Asymptotic stability of solitons for (gKdV) [50], [54], [45])**.** *Let $p = 2$, 3 or 4. Let $c_0 > 0$. There exists $K_0 > 0$, and for any $\beta > 0$, there exists $\alpha_0 = \alpha_0(\beta) > 0$ such that the following is true. Let $u(t)$ be an H^1 solution of (1.2) satisfying $\|u(0) - Q_{c_0}\|_{H^1} \leq \alpha_0$. Then, there exist $c^+ > 0$ with $|c^+ - c_0| \leq K_0\alpha_0$ and a C^1 function $x : [0, +\infty) \to \mathbb{R}$ such that*

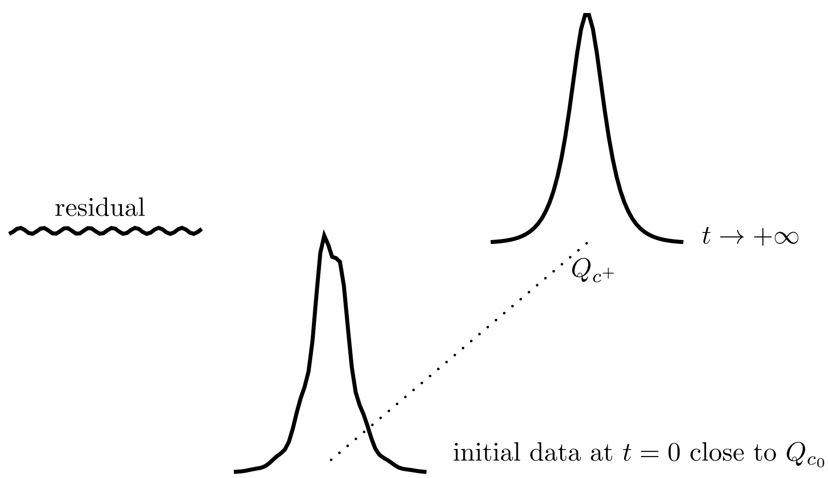
$$v(t, x) = u(t, x) - Q_{c^+}(x - x(t)) \quad \textit{satisfies} \quad \lim_{t\to+\infty} \|v(t)\|_{H^1(x>\beta t)} = 0. \tag{4.1}$$

Moreover, $\lim_{t\to+\infty} \frac{dx}{dt}(t) = c^+$.

This result means that by taking α_0 small enough, we know the behavior of $u(t)$ on the space time region $x > \beta t$, for any $\beta > 0$ (α_0 depending on β). The space region where the convergence holds in Theorem 4 is sharp, since in the integrable case $p = 2$, there exists an explicit solution which behaves asymptotically as $t \to +\infty$ as $Q(x-t) + Q_c(x-ct)$, where $0 < c < 1$ (see Section 7 on multi-solitons for the (KdV) case). In particular, choosing $c > 0$ small, the H^1 norm of $Q_c(x-ct)$ is small, and this soliton travels on the line $x = ct$. This explains the necessity of a positive β in the convergence result. Moreover, one expects in general some loss of L^2 norm as dispersion for $x < 0$.

We refer to Pego and Weinstein [72] for previous asymptotic stability result in Sobolev spaces with exponential weights.

Here is a shematic representation of the asymptotic stability result, and of how a solution initially close to a soliton eventully simplifies to a soliton plus a small residue.



A very natural question concerns the behavior of $v(t)$ in the region $x < \beta t$. By the stability result, we know that $v(t)$ is small in H^1. However, the behavior of $v(t)$ is completely open in general. The only notable result concerns the quartic case $p = 4$ and is due to Tao [73]. The result states that if the initial data is close to a soliton in H^1 and in the critical homogeneous $\dot{H}^s$ space ($s < 0$), then $v(t)$ scatters at infinity.

This result has been recently refined by Koch and Marzuola [37]. For related questions, we also refer to Côte's results [13], [14] concerning the construction of some special solutions, with prescribed asymptotic behavior, in the cases $p = 4$ and $p = 5$.

5 Kato identity and monotonicity properties

A fundamental tool in the proof of asymptotic stability and in later developments involving several solitons is a *monotonicity* result of local L^2 norm. For these notes, we restrict ourselves to the case of a solution close to Q in the sense of Theorem 3. Let $K > 0$. For $x \in \mathbb{R}$, let

$$\phi(x) = \frac{2}{\pi} \arctan(\exp(x/K)),$$

so that $\lim_{+\infty}\phi = 1$, $\lim_{-\infty}\phi = 0$ and for all $x \in \mathbb{R}$, $\phi(-x) = 1 - \phi(x)$. Note that by direct calculations

$$\phi'(x) = \frac{1}{K\pi \cosh(x/K)}, \quad \phi'''(x) \le \frac{1}{K^2}\phi'(x). \tag{5.1}$$

Let $0 < \sigma < 1/2$, $x_0 > 0$. We define, for $t_0 \in \mathbb{R}$, for all $t \le t_0$:

$$I_{x_0,t_0}(t) = \int u^2(t,x)\, \phi\left(x - x(t_0) + \sigma(t_0 - t) - x_0\right) dx.$$

We claim the following

Proposition 5.1 (Monotonicity result [50])**.** *For all $0 < \sigma < \frac{1}{2}$, for all $K > \sqrt{\frac{2}{\sigma}}$, if $\alpha_0 > 0$ is small enough in Theorem 3, then for all $t, t_0 \in \mathbb{R}$, $t \le t_0$, and all $x_0 > 0$,*

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \le C \exp\left(-\frac{x_0}{K}\right), \tag{5.2}$$

for some constant $C > 0$.

The function $I_{x_0,t_0}(t)$ is thus almost decreasing. It plays the role of a Liapunov functional to study the convergence.

Proof of Proposition 5.1. We repeat the proof from [50] and [54]. By simple calculations, for $f : \mathbb{R} \to \mathbb{R}$ of class C^3, we obtain Kato's identity, see [30]:

$$\frac{d}{dt}\int u^2 f = \int \left(-3u_x^2 + \frac{2p}{p+1}u^{p+1}\right) f' + \int u^2 f'''. \tag{5.3}$$

Recall that initially, in [30], this identity is applied to derive local smoothing effect (L^2 solutions are automatically H^1 locally in space and time).

Let $0 < \sigma < \frac{1}{2}$, $x_0 > 0$, $t_0 \in \mathbb{R}$, and $K > \sqrt{\frac{2}{\sigma}}$. Let

$$\psi(t,x) = \phi(x - x(t_0) + \sigma(t_0 - t) - x_0).$$

We obtain from (5.3), for all $t \le t_0$,

$$\frac{d}{dt}\int u^2 \psi = -\int \left(3u_x^2 + \sigma u^2 - \frac{2p}{p+1}u^{p+1}\right)\psi_x + \int u^2 \psi_{xxx}.$$

Using (5.1) and $\frac{1}{K^2} \le \frac{\sigma}{2}$, we obtain

$$\frac{d}{dt}\int u^2 \psi \le -\int \left(3u_x^2 + \frac{\sigma}{2}u^2 - \frac{2p}{p+1}|u|^{p+1}\right)\psi_x.$$

Let $R_0 > 0$ to be chosen later. For t, x such that $|x - x(t)| \geq R_0$, by $|Q(x)| \leq Ce^{-|x|}$, the conclusion of Theorem 3 and the well-known inequality $\|v\|_{L^\infty}^2 \leq C\|v_x\|_{L^2}\|v\|_{L^2}$, we have

$$|u(t,x)| \leq R(t,x) + \|u(t) - R(t)\|_{L^\infty} \leq Ce^{-\frac{R_0}{2}} + C\alpha_0.$$

Therefore, for α_0 small enough and R_0 large enough, we have, for such t, x:

$$\frac{2p}{p+1}|u(t,x)|^{p-1} \leq \frac{\sigma}{4}.$$

Now, R_0 is fixed.

If $|x - x(t)| \leq R_0$ then $|x - x(t_0) + \sigma(t_0 - t) - x_0| \geq -|x - x(t)| + |x(t) - x(t_0) + \sigma(t_0 - t) - x_0| \geq -R_0 + \frac{t_0 - t}{2} + x_0$, and so

$$|\psi_x(t,x)| \leq Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}}.$$

Therefore, by $\int |u|^{p+1} \leq C$, we obtain

$$\frac{d}{dt}\int u^2\psi \leq -\int \left(3u_x^2 + \frac{\sigma}{4}u^2\right)\psi_x - Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}} \leq -Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}}. \tag{5.4}$$

By integration between t and t_0, we obtain (5.2). □

There exist variants of Proposition 5.1, in different contexts. In particular, one can write similar results with energy type quantities. See e.g. [47].

6 Strategy of the proof of asymptotic stability

We recall the main steps of the proof of asymptotic stability in [50] based on the concept of L^2 compact solutions, i.e. solutions of the equation which have uniformly in time localized L^2 norm. This notion is fundamental in asymptotic problems because of the following:

First, a general fact for dispersive equations is that an L^2 compact solution can be extracted from the behavior of any solution $u(t)$ (i.e. obtained as the weak limit of a subsequence of $u(t)$). For the (gKdV) equation, the procedure is especially successful thanks to additional use of monotonicity arguments, such as in Section 5, which allows to prove convergence for all the sequence of time.

Second, L^2 compact solutions are very special solutions, which can be classified in some circumstances. This classification gives in return information on the original solution $u(t)$.

We point out that this approach has been used in the pioneering work [23] for (NLS), then in [48], [50] for (gKdV), and in [68], [33], [34], [36] for (NLS) and the wave equation, among other works.

6.1 General properties of L^2 compact solutions of (gKdV)

We say that a solution $\widetilde{u}(t)$ of the (gKdV) equation is L^2-compact if

$$\forall \epsilon_0,\ \exists A_0 > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>A_0} \widetilde{u}^2(t, x + \widetilde{x}(t))dx \le \epsilon_0, \tag{6.1}$$

for some $\widetilde{x}(t)$ with $\widetilde{x}'(t) > 0$.

By a remarkable property of the flow of the (gKdV) equation, condition (6.1) implies that $\widetilde{u}(t)$ is completely smooth (i.e. $\widetilde{u}(t)$ is C^∞ in time and space) and decays exponentially in x as well as all its derivatives.

Proposition 6.1 (Regularity and decay of L^2 compact solutions [50], [48], [41])**.** *Let $p \ge 2$ integer. Let $\widetilde{u}(t)$ be a global H^1 solution of (1.2), globally bounded in H^1. Assume that there exist $0 < \beta_1 \le \beta_2$ and a C^1 function $\widetilde{x}(t) : \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$, $\beta_1 \le \widetilde{x}'(t) \le \beta_2$, and such that (6.1) holds. Then $\widetilde{u} \in C^\infty(\mathbb{R} \times \mathbb{R})$. Moreover, there exist constants $\gamma > 0$ and $C_k > 0$ such that*

$$\forall k \in \mathbb{N}, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad \left|\partial_x^k \widetilde{u}(t, x + \widetilde{x}(t))\right| \le C_k \exp(-\gamma|x|). \tag{6.2}$$

This result is reminiscent of Kato smoothing effect, see [30]. Proposition 6.1 follows from techniques introduced in Martel and Merle [48] and [50] while proving the asymptotic stability result (see [41] for a complete proof).

From this typical result, one sees that L^2 compact solutions are very specific solutions of dispersive equations, with similar properties than entire solutions of elliptic problems.

6.2 Classification of L^2 compact solutions close to solitons

We now state a Liouville property, which is the main tool in the proof of Theorem 4.

Theorem 5 (Liouville property close to Q [50])**.** *Let $p = 2$, 3 or 4. Let $\widetilde{u}(t)$ be a global H^1 solution of* (1.2)*. There exists $\alpha_0 > 0$ such that, if $\|\widetilde{u}(0) - Q\|_{H^1} \le \alpha_0$ and if $\widetilde{u}(t)$ satisfies (6.1), then there exist $c_1 > 0$, $x_1 \in \mathbb{R}$ such that*

$$\widetilde{u}(t, x) = Q_{c_1}(x - x_1 - c_1 t).$$

From Proposition 6.1, assumption (6.1) is equivalent to an exponential decay assumption. It is a strong assumption on the solution. Theorem 5 states that if an H^1 solution of the (gKdV) equation is close in H^1 to a soliton, and has uniform exponential decay in x, then it is exactly a soliton.

We recall the main steps of the proof of Theorem 5. Under the assumptions of Theorem 5, one sees that $\widetilde{u}(t)$ can be decomposed with $\widetilde{\varepsilon}(t)$, $\widetilde{c}(t)$, $\widetilde{x}(t)$ and $\widetilde{R}(t)$ as in (3.5), (3.6). Moreover, $\widetilde{\varepsilon}$, $\widetilde{c}(t)$, $\widetilde{x}(t)$ satisfy the following equation:

$$\partial_t\widetilde{\varepsilon} + \partial_x^3\widetilde{\varepsilon} + \partial_x\big((\widetilde{\varepsilon} + \widetilde{R})^p - \widetilde{R}^p\big) = -\frac{\widetilde{c}_t(t)}{2\widetilde{c}(t)}\left(\frac{2\widetilde{R}}{p-1} + (x - \widetilde{x}(t))\partial_x\widetilde{R}\right) + (\widetilde{x}_t(t) - \widetilde{c}(t))\partial_x\widetilde{R}.$$

Note also that by (6.1), Proposition 6.1 on $\widetilde{u}(t)$ and the exponential decay of Q, $\widetilde{\varepsilon}(t,x)$ also satisfies a uniform exponential decay property.

Note that the conclusion of Theorem 5 is equivalent to obtain $\widetilde{\varepsilon} \equiv 0$ provided that α_0 is small enough. We argue by contradiction on the smallness of α_0, assuming that there exists a sequence of solutions $(\widetilde{u}_n)$ which are not solitons, but satisfy $\|\widetilde{u}_n(0) - Q\|_{H^1} \to 0$ as $n \to +\infty$ and the assumptions of Theorem 5. Using Kato identity (5.3) with $f = x$, one proves the following estimate on $\widetilde{\varepsilon}_n(t)$ holds (see [50] for more details):

$$\sup_{t\in\mathbb{R}} \|\widetilde{\varepsilon}_n(t)\|_{H^1} \leq C \sup_{t\in\mathbb{R}} \|\widetilde{\varepsilon}_n(t)\|_{L^2}.$$

This allows us to prove that a renormalized version of the sequence $(\widetilde{\varepsilon}_n(t))$ converges to a solution $w(t) \not\equiv 0$ of the following linear problem

$$\partial_t w - \partial_x(\mathcal{L}w) = \alpha(t)\Big(\tfrac{2Q}{p-1} + xQ'\Big) + \beta(t)Q', \tag{6.3}$$

where $\alpha(t)$ and $\beta(t)$ are given functions of t and $\mathcal{L}w = -\partial_x^2 w + w - pQ^{p-1}w$. Formally, the equation of $w(t)$ is found by keeping only linear terms in the equation of $\widetilde{\varepsilon}(t)$.

Moreover, w also satisfies an exponential decay property for both $y > 0$ and $y < 0$:

$$\forall y \in \mathbb{R}, \forall s \in \mathbb{R}, \quad |w(s,y)| \leq Ce^{-\theta|y|}, \tag{6.4}$$

and the same orthogonality conditions as $\widetilde{\varepsilon}_n(t)$.

To conclude the proof, it suffices to prove that necessarily $w \equiv 0$, which implies the desired contradiction. Therefore the last step of the proof of the asymptotic stability result is the classification of bounded and localized solutions of the linearized equation around Q.

Theorem 6 (Linear problem related to (gKdV) [50], [47])**.** *Let $p > 1$. Let $w(t,x) \in C(\mathbb{R}, H^1) \cap L^\infty(\mathbb{R}, H^1)$ be a solution of*

$$\partial_t w = \partial_x(\mathcal{L}w) + \alpha(t)\left(\tfrac{2}{p-1}Q + xQ'\right) + \beta(t)Q' \quad \text{on } \mathbb{R}\times\mathbb{R}, \tag{6.5}$$

where $\alpha(t)$ and $\beta(t)$ are two continuous and bounded functions. Assume (6.4)*.*

Then, for all $t \in \mathbb{R}$,

$$w(t) \equiv a(t)\left(\tfrac{2}{p-1}Q + xQ'\right) + b(t)Q', \tag{6.6}$$

for some C^1 bounded functions $a(t)$ and $b(t)$ satisfying $a'(t) = \alpha(t)$, $b'(t) = -2\,a(t) + \beta(t)$.

Using this result and the orthogonality conditions on $w(t)$, we see that $a(t) = b(t) = 0$ and thus $w \equiv 0$.

Theorem 6 has first been proved by Martel and Merle in [48] for $p = 5$ and [50] for $p = 2$, 3 and 4 and then simplified and extended in [47] and [55]. The proof presented in

[47] is based on the fact that $w(t)$ such as in Theorem 6 is smooth, and $W(t) = \mathcal{L}w(t)$ satisfies

$$\partial_t W = \mathcal{L}(\partial_x W) - 2\alpha(t)Q.$$

It turns out that the equation of $W(t)$ is easier to handle than the equation of $w(t)$. In particular, we have the following remarkable algebraic fact on $W(t)$:

$$\frac{1}{3}\left(\frac{p+1}{p-1}\right)\frac{d}{dt}\int W^2\frac{Q'}{Q} = \int Z_x^2 Q^{p+1} \geq 0,$$

where $Z = W/Q$. This identity is the key tool to prove Theorem 6. See details in [55].

6.3 Conclusion of the proof of asymptotic stability

We briefly explain how to obtain Theorem 4 from Theorem 5. Let $u(t)$ be a solution as in Theorem 4. By the uniform bound of $u(t)$ in H^1, one considers a sequence $t_n \to +\infty$, a function $\widetilde{u}(0) \in H^1$ and $\widetilde{c}_0 > 0$ such that

$$c^{-\frac{1}{p-1}}(t_n)u\Big(t_n, \tfrac{1}{\sqrt{c(t_n)}}. + x(t_n)\Big) \rightharpoonup \widetilde{u}(0) \quad \text{and} \quad c(t_n) \to \widetilde{c}_0 \quad \text{as } n \to +\infty.$$

Let $\widetilde{u}(t)$ be the solution of (1.2) defined for all $t \in \mathbb{R}$ corresponding to $\widetilde{u}(0)$. Up to scaling and translation, the solutions $u(t_n + t)$ and $\widetilde{u}(t)$ are related by weak convergence using Theorem 2.

The solution $\widetilde{u}(t)$ is an asymptotic solution, and the fact that it comes asymptotically from the behavior of a solution $u(t)$ implies that $\widetilde{u}(t)$ is L^2 compact in the sense (6.1). The proof of this property uses Proposition 5.1 (see [50] for a detailed proof).

Being a weak limit of $u(t)$ close to Q, $\widetilde{u}(t)$ is close to Q and thus by Theorem 5, $\widetilde{u}(t)$ is exactly $Q_{\widetilde{c}_0}(x - \widetilde{c}_0 t)$. This proves asymptotic stability for a subsequence in H^1 weak. By monotonicity arguments (variants of Proposition 5.1), we prove that $\widetilde{c}_0$ does not depend on the sequence chosen, and thus convergence holds for the whole sequence of time.

Finally, H^1 convergence in the sense of Theorem 4 is obtained by using further monotonicity results.

7 Multi-solitons in the integrable case

The first remarkable observations related to soliton collision are due to pioneering numerical works of Fermi, Pasta and Ulam [19] and Zabusky and Kruskal [85]. The theory of complete integrability was just after discovered and led to many further developments. See Lax [42] and the review paper Miura [69].

We briefly recall the properties of the remarkable multi-soliton solutions of the (KdV) equation (1.1). We shall not discuss their construction in these notes, see [69].

For any given parameters $0 < c_1 < \cdots < c_N$, $\delta_1, \ldots, \delta_N \in \mathbb{R}$, there exists an explicit solution $U(t,x)$ of (1.1) which satisfies:

$$\lim_{t\to+\infty}\left\|U(t,x)-\sum_{j=1}^{N}Q_{c_j}(x-c_jt-\delta_j)\right\|_{H^1}=0.$$

Moreover, $U(t)$ also satisfies

$$\lim_{t\to-\infty}\left\|U(t,x)-\sum_{j=1}^{N}Q_{c_j}(x-c_jt-\delta_j')\right\|_{H^1}=0,$$

for some δ_j' such that $\delta_j-\delta_j'$ depends on all the (c_k). Such a solution $U(t)$ is called a multi-soliton. It describes the elastic interaction of N given solitons of the equation. It is especially remarkable that the sizes of the solitons as $t\to+\infty$ and $t\to-\infty$ are exactly the same. Moreover, these solutions do not disperse, in the sense that all the mass (and energy) of the solution is contained in the solitons:

$$\int U^2=\sum_{j=1}^{N}\int Q_{c_j}^2,\quad E(U)=\sum_{j=1}^{N}E(Q_{c_j}).$$

Recall that Maddocks and Sachs [44] proved that the N-soliton solutions of (1.1) are stable in H^N, which means that if a solution $u(t)\in H^N(\mathbb{R})$ of the (KdV) equation is close to a multi-soliton in $H^N(\mathbb{R})$ for some time, then for all time it is close in $H^N(\mathbb{R})$ to an N-soliton solution with same speeds (but possibly different positions – the same restriction as for the orbital stability of one soliton). We refer to the original paper for a more precise statement. The exponent N of the Sobolev norm has to be larger or equal to the number of solitons since the proof of the stability result uses the first $N+1$ conserved quantities of the (KdV) equation. In particular, the result cannot be obtained in the energy space H^1 by this technique. Moreover, there is not hope to extend the strategy of the proof to non integrable equations.

Finally, we recall another striking result in the integrable case concerning multi-solitons.

Theorem 7 ([10], [17], [75]). *Let $u(t)$ be a solution of* (1.1) *such that*

$$u(0)\in C^4(\mathbb{R}),\quad k=0,\ldots,4,\ \forall x\in\mathbb{R},\quad \left|\frac{\partial^k u(0,x)}{\partial x^k}\right|\le\frac{C}{|x|^{10}}.$$

Then, there exist $N\ge 0$, $c_j>0$ and $x_j\in\mathbb{R}$ such that for $x>0$, as $t\to+\infty$,

$$u(t,x)-\sum_{j=1}^{N}Q_{c_j}(x-x_j-c_jt)\to 0.$$

This result means that the asymptotic behavior of any solution (sufficiently regular and decaying) is governed by a finite number of solitons. The number N of solitons emerging is related to a spectral property of the operator $-f'' - 2u(0)f$. We refer to the original paper and book for a precise convergence result.

Of course, this result should convince the reader of the fundamental role played by multi-solitons for the (KdV) equation.

8 Multi-solitons asymptotics for (gKdV)

For the non integrable case, no such explicit multi-solitons are known. The objective in this section is to mimick their behavior as $t \to -\infty$.

Theorem 8 (Asymptotic multi-solitons for (gKdV) [45])**.** *Let $p = 2$, 3 or 4. Let $N \in \mathbb{N}$, $0 < c_1 < c_2 < \cdots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$. There exists one and only one function $U \in C(\mathbb{R}, H^1)$, which is an H^1 solution of (1.2) in the sense of Theorem 1, and such that*

$$\lim_{t\to-\infty} \left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t) \right\|_{H^1} = 0. \tag{8.1}$$

This result means that one can indeed construct a solution that behaves as $t \to -\infty$ as the sum of several solitons in H^1. Moveover this solution is in fact completely smooth and converges exponentially in time in all Sobolev norms. Thus, the situation is quite similar to the one of the integrable cases $p = 2$ and $p = 3$, with the explicit multi-soliton solutions, *as far as asymptotic behavior is concerned.* However, in general, in the non integrable case $p = 4$, we do not know the behavior of the solution $U(t)$ for $t \to +\infty$ when the solitons begin to interact. See Sections 9 and 10 for partial results.

Theorem 8 also contains a uniqueness result which was new even in the integrable case since results using the inverse scattering transform are for decaying and regular solutions. The proof of Theorem 8 is a refinement of the arguments used in the proof of the next theorem. We shall not discuss it further in these notes.

Now, we extend to multi-soliton situation the stability and asymptotic stability results of Theorems 3 and 4.

Theorem 9 (Asymptotic stability of the sum of N solitons [63])**.** *Let $p = 2$, 3 or 4. Let $0 < c_1^0 < \cdots < c_N^0$. There exist $\gamma_0, A_0, L_0, \alpha_0 > 0$ such that the following is true. Let $u(t)$ be an H^1 solution of (1.2), and assume that there exist $L > L_0$, $0 < \alpha < \alpha_0$, and $x_1^0 < \cdots < x_N^0$, such that*

$$\left\| u(0) - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j^0) \right\|_{H^1} \le \alpha, \quad \text{and } x_j^0 > x_{j-1}^0 + L, \text{ for all } j = 2, \ldots, N. \tag{8.2}$$

Then, there exist $x_1(t), \ldots, x_N(t)$ such that *(i) Stability of the sum of N decoupled solitons.*

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j(t)) \right\|_{H^1} \leq A_0 \left(\alpha + e^{-\gamma_0 L} \right). \tag{8.3}$$

(ii) Asymptotic stability of the sum of N solitons.
There exist $c_1^+, \ldots, c_N^+$, with $|c_j^+ - c_j^0| \leq A_0 \left(\alpha + e^{-\gamma_0 L} \right)$, such that

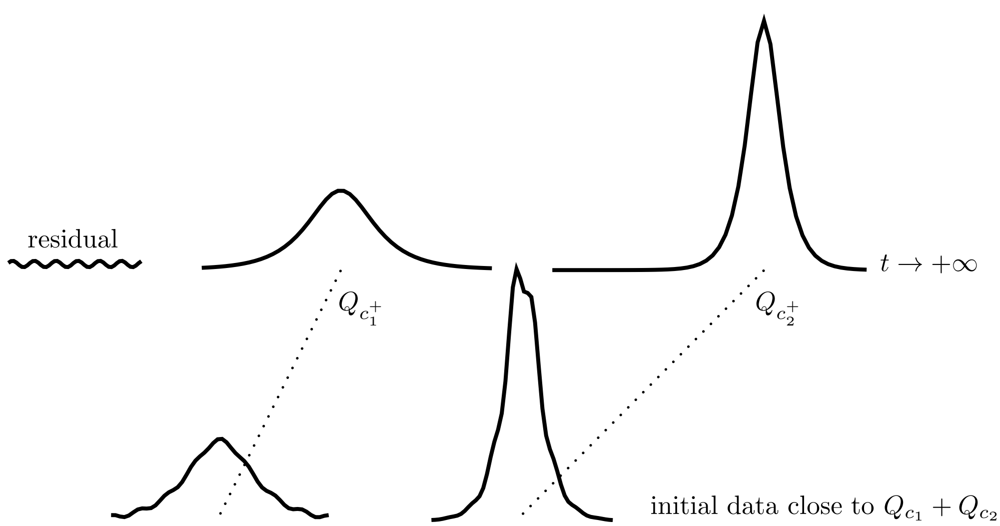
$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j^+}(\cdot - x_j(t)) \right\|_{H^1(x > c_1^0 t/10)} \to 0, \quad x_j'(t) \to c_j^+ \quad \textit{as } t \to +\infty. \tag{8.4}$$

As in Theorem 4, one cannot expect the convergence to hold in $H^1(x > 0)$. Indeed, assumption (8.2) on the initial data allows the existence in $u(t)$ of an additional soliton of size less than α (thus travelling at arbitrarily small speed). Moreover, convergence in $H^1(\mathbb{R})$ would imply that $u(t)$ is one of the special solutions $U(t)$ as constructed in Theorem 8, thus it cannot be true in general.

Note that a stability result similar to (8.3) in Theorem 9 is the main tool to prove the existence result in Theorem 8.

Remark also that $\sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t))$ is not a solution of the (gKdV) equation, because of interactions between solitons of order $e^{-\gamma_0(L+t)}$. It is thus natural to find an error term of this nature in (8.3). However, if instead of comparing $u(t)$ with the sum of N solitons, we measure its distance to the familly of multi-solitons constructed in Theorem 8, then the error term is only $A_0 \alpha$; see [60].

The following picture illustrates the behavior in large time of a solution initially close to the sum of two decoupled solitons.



As a direct corollary of Theorem 1, for $p = 2$ and $p = 3$, the explicit multi-soliton solutions are stable and asymptotically stable, see Corollary 1 in [50]. This improves the result in [44] mentionned in Section 7.

We now sketch the proof of Theorem 9. The main elements of the proof were already introduced in these notes: Weinstein's proof of the stability theorem in Section 3, the monotonicity result in Section 5 and finally, to prove (ii), the arguments introduced in Section 6, which are easily extended to the case of multi-solitons.

In the context of Theorem 9, we define the functional

$$\mathcal{G}_N(u) = 2E(u) + \int c(t,x)|u|^2, \tag{8.5}$$

where

$$c(t,x) = c_1^0 + \sum_{j=2}^{N} \left(c_j^0 - c_{j-1}^0\right) \phi(x - m_j(t)), \quad m_j(t) = \frac{1}{2}\left(x_j(t) + x_{j-1}(t)\right).$$

In other words, the functional $\mathcal{G}_N$, locally around each soliton $Q_{c_j^0}$, is the Weintein's functional (3.8). Moreover, the variation in time of this functional will be handled through the computations of Section 5 since $m_j(t)$ is far away from any soliton. For the solution $u(t,x)$, we introduce a decomposition similar to (3.5) in the multi-soliton case

$$\varepsilon(t,x) = u(t,x) - R(t,x), \quad \text{where} \quad R(t,x) = \sum_{j=1}^{N} Q_{c_j(t)}(x - x_j(t)), \tag{8.6}$$

and $\varepsilon(t)$ satisfies $2N$ orthogonality conditions similar to (3.6). As in the proof of Theorem 3 in Section 3, we first prove the following quadratic control of the scaling parameters

$$\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leq C\left(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma L}\right).$$

The proof of this estimate requires the use of monotonicity results on $N-1$ quantities similar to $\mathcal{G}_N(u)$.

The conclusion of Theorem 9 then comes from the combination of the following facts:

$$\mathcal{G}_N(u(t)) \geq \lambda \|\varepsilon(t)\|_{H^1}^2, \tag{8.7}$$

$$\mathcal{G}_N(u(t)) - \mathcal{G}_N(u(0)) \leq Ce^{-\gamma L} + O(\|\varepsilon\|_{H^1}^3). \tag{8.8}$$

Indeed, using in addition $\mathcal{G}_N(u(0)) \leq \|\varepsilon(0)\|_{H^1}^2$, we obtain $\|\varepsilon(t)\|_{H^1}^2 \leq C'\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\gamma L}$, and we finish the proof as the one of Theorem 3.

To conclude, just note that estimate (8.7) is an extension of the positivity property (3.3) to the multi-soliton case, whereas estimate (8.8) is obtained by refining the monotonicity arguments of Section 5.

9 Collision of two solitons with very different sizes for the quartic (gKdV)

9.1 Introduction to the general collision problem

As we have seen in Section 7, in the integrable case, the explicit 2-soliton solutions give a precise description of the interaction between several solitons.

Apart from integrability theory, soliton interaction problems have been studied since the 60's from both experimental and numerical points of view.

Since the celebrated works of Fermi, Pasta and Ulam [19], Zabusky and Kruskal [85] and Zabusky [84] later justified by the theory of integrability, many other systems have been studied numerically.

There is also an extensive literature devoted to experiments on water tanks. A key question is whether or not the collision between two solitary waves is elastic (equivalently, whether the collision is pure or generates dispersion). From experiments related to wave propagation in shalow water (see Weidman and Maxworthy [80], Hammack et al. [25], Craig et al. [15]), it seems that collisions are inelastic but very close to be elastic, for solitary waves of different amplitudes.

For equations of KdV type, Bona et al. [6], and Kalisch and Bona [29], studied numerically the problem of collision of two solitary waves for the Benjamin and the BBM equations. Shih [76] studied the case of the gKdV equation (1.2) with some half-integer values of p. Craig et al. [15] report on numerics for the water wave problem. In all these works, numerics match experiments and show that the collision of two solitary waves fails to be elastic by a very small dispersion.

9.2 Main result on collision of very different solitons from [59]

In this section, we briefly recall the main results in [59], which are the first rigorous results concerning collision of solitons for a non integrable system, and in particular, give the first proof of non existence of pure multi-soliton solutions for the quartic (gKdV) equation. From now on, we restrict ourselves to the quartic case

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}. \tag{9.1}$$

We consider the case where one soliton is much smaller than the other one. By scaling invariance of the equation, we restrict ourselves to the case where the large soliton is Q without loss of generality.

Theorem 10 ([59]). *Assume $0 < c \ll 1$. Let $U(t)$ be the solution of (9.1) such that*

$$\lim_{t\to-\infty} \left\| U(t) - (Q(.-t) + Q_c(.-ct)) \right\|_{H^1} \to 0.$$

- *Global stability of the 2-soliton structure and asymptotics as $t \to +\infty$.*

There exist $c_1^+ \underset{c\to 0}{\sim} 1$, $c_2^+ \underset{c\to 0}{\sim} c$, $\rho_1(t)$, $\rho_2(t)$ *and* $K>0$ *such that*

$$w^+(t,x) = U(t,x) - \left(Q_{c_1^+}(x-\rho_1(t)) + Q_{c_2^+}(x-\rho_2(t))\right),$$

satisfies

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x\geq \frac{c}{10}t)} = 0, \quad \sup_{t\in\mathbb{R}} \|w^+(t)\|_{H^1} \leq Kc^{\frac{1}{3}}.$$

- *Inelasticity of the collision. For some* $K, K' > 0$,

$$c_1^+ - 1 \geq K\, c^{\frac{17}{6}}, \quad 1 - \frac{c_2^+}{c} \geq K\, c^{\frac{8}{3}},$$

$$0 < K\, c^{\frac{17}{12}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2} \leq K' c^{\frac{11}{12}}, \quad \textit{for } t \textit{ large.}$$

Comments on Theorem 10.

1. The first part of the theorem means that the two solitons are preserved through the collision, even the smaller one, which is surprizing. The size of the perturbative term $w^+(t)$ satisfies, for c small,

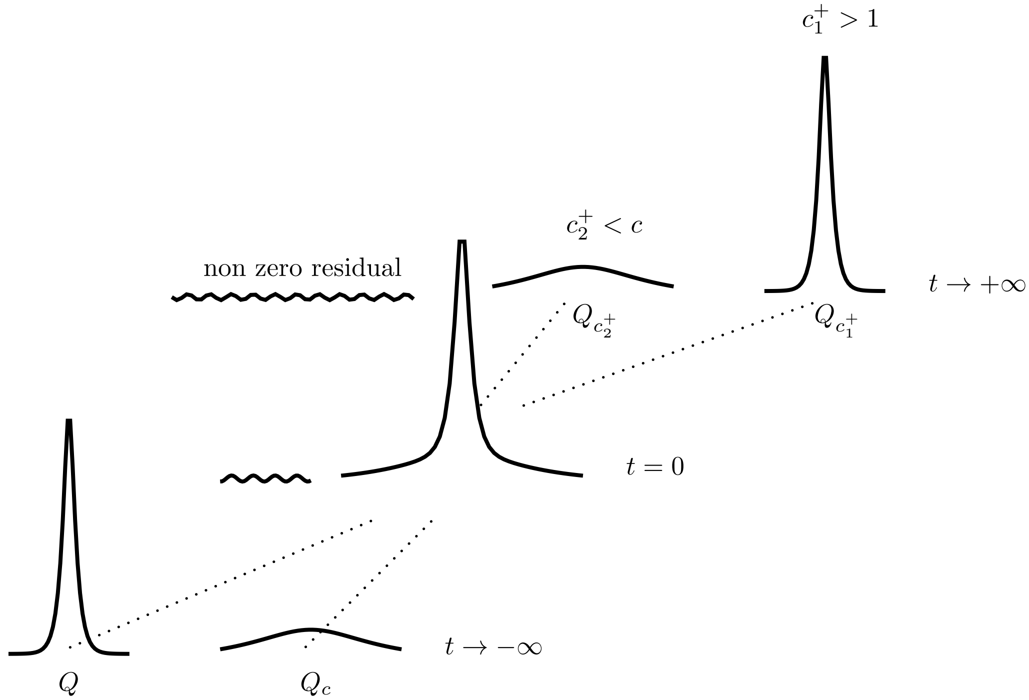
$$\sup_t \|w^+(t)\|_{H^1} \leq Kc^{\frac{1}{3}} \quad \text{whereas} \quad \|Q_c\|_{H^1} \sim Kc^{\frac{1}{12}}.$$

2. Precise upper bounds on $c_1^+ - 1$ and $c - c_2^+$ in terms of powers of c are available, see Theorem 1 in [59].

3. The second part of Theorem 10 proves the nonexistence of a pure 2-soliton solution in this regime, since $\liminf_{t\to+\infty} \|w^+(t)\|_{H^1} > 0$. *The collision is thus not elastic.* This is the main point in [59].

4. Since $\|w^+(t)\|_{L^2} \leq K\|Q_c\|_{L^2}^7$, the collision is almost elastic, in the sense that the defect is of much higher order of c than $\|Q_c\|_{L^2}$.

The behavior of the special solution $U(t)$ is represented schematically by the following picture.



The first part of Theorem 10 has been extended to (1.3) with general nonlinearity f for which solitons are stable in Weinstein's sense. The second part, i.e. the inelastic nature of the collision, has been proved for general non integrable nonlinearities for small solitons by Muñoz [71] (in that work, both solitons are small, and one is much smaller than the other one).

The proof of Theorem 10 in [59] is long and involved and beyond the scope of this course. We just sketch the main steps of the proofs in [59], and refer to the review paper [57] and to the original paper for details.

1. First, we construct an approximate solution to the problem in the collision region, i.e. in the time interval $[-c^{-(\frac{1}{2})^+}, c^{-(\frac{1}{2})^+}]$. The approximate solution has the form of a series in terms of $c = c_2/c_1$ and involves a delicate algebra.

2. Second, using asymptotic arguments similar to the ones presented in the previous sections, we justify that the solution $U(t)$ is close to the approximate solution (so that the description of the collision given by the approximate solution is relevant) and we control the solution in large time, i.e. for $|t| > c^{-(\frac{1}{2})^+}$.

3. Finally, we prove the inelastic character of the collision by a further analysis of the approximate solution. The defect is due to a nonzero extra term in the approximate solution after recomposition of the series. Thus, the defect is a direct consequence of the algebra underlying the construction of the approximate solution.

10 Collision of two solitons with almost same sizes for quartic (gKdV)

10.1 Previous results for almost equal solitons

As in the previous section, we focus on the case of the quartic (gKdV) equation (9.1). However, the first intuition on the problem of two solitons with almost same sizes actually comes from analysing the explicit multi-solitons of the integrable case.

LeVeque [43] gave a precise description of the behavior of the multi-soliton $U_{c_1^-,c_2^-}$ of (1.1) satisfying

$$\lim_{t\to\pm\infty}\left\|U_{c_1^-,c_2^-}(t)-Q_{c_1^-}(.-c_1^-t-y_1^\pm)-Q_{c_2^-}(.-c_2^-t-y_2^\pm)\right\|_{H^1}=0, \qquad (10.1)$$

in the special asymptotics where $\mu_0=\frac{c_2^- - c_1^-}{c_1^- + c_2^-}>0$ is small, i.e. for nearly equal solitons. In [43], the following estimate is proved for some explicit functions $c_j(t)$, $y_j(t)$:

$$\sup_{t,x\in\mathbb{R}}\left|U_{c_1^-,c_2^-}(t,x)-Q_{c_1(t)}(x-y_1(t))-Q_{c_2(t)}(x-y_2(t))\right|\leq C\mu_0^2. \qquad (10.2)$$

Moreover, it is proved that

$$\min_{t\in\mathbb{R}}(y_1(t)-y_2(t))=2|\ln\mu_0|+O(1). \qquad (10.3)$$

This means that the minimum separation between the two solitons goes to ∞ as $\mu_0\to 0$. It follows that the two solitons do not cross if they have sufficiently close speeds. In particular, the solution has two maximum points for all time. The interaction is repulsive, the solitons exchange their sizes and speeds at large distance and consequently avoid the collision.

We now recall Mizumachi's results [70] in the non integrable case $p=4$, for which they are especially striking: let $u(t)$ be a solution of (9.1) for which $u(0)$ is close to the sum of two solitons $Q(x)+Q_c(x+Y_0)$, where $Y_0>0$ is large and $0<c-1\leq e^{-\frac{1}{2}Y_0}$, so that the quicker soliton is initially on the left of the other soliton. Mizumachi proved for (9.1) (see Theorem 1.1 in [70]) that the interaction of the two solitons is repulsive: the two solitons remain separated for all positive time and eventually $u(t)$ behaves as

$$u(t)=Q_{c_1^+}(.-c_1^+t-y_1^+)+Q_{c_2^+}(.-c_2^+t-y_2^+)+w(t,x), \qquad (10.4)$$

for large time, for some $c_1^+>c_2^+$ close to 1 and w small in some sense. The analysis part in [70] relies on scattering techniques due to Hayashi and Naumkin [26, 27] and on the use of spaces of exponentially decaying functions (introduced in this context in [72]).

The situation for almost equal solitons of the quartic (gKdV) is thus at the main order similar to the one described in the integrable case by LeVeque [43]. However, after Mizumachi's work, two important questions were remaining open in this regime for the non integrable case: the global stability in the energy space H^1 of the 2-soliton structure and the existence or non existence of pure 2-soliton solutions.

10.2 Main result on interaction of almost equal solitons from [60]

In [60], we describe the interaction of two solitons with almost equal speeds for (9.1) and in particular we answer in this specific regime the two main questions raised above:

1. The 2-soliton structure is stable globally in time in the energy space H^1.
2. There does not exist a pure 2-soliton solution.

Before presenting the main result from [60], for simplicity, we first change variables. For $c_2^- - c_1^- > 0$ small, and any x_1^-, x_2^-, let $u(t)$ be the unique solution of (9.1) such that

$$\lim_{t\to-\infty} \|u(t) - Q_{c_1^-}(. - c_1^- t - x_1^-) - Q_{c_2^-}(. - c_2^- t - x_2^-)\|_{H^1} = 0. \tag{10.5}$$

Let

$$c_0 = \frac{c_1^- + c_2^-}{2}, \quad \mu_0 = \frac{c_2^- - c_1^-}{c_1^- + c_2^-}, \quad y_1^- = x_1^- \sqrt{c_0}, \quad y_2^- = x_2^- \sqrt{c_0}. \tag{10.6}$$

Then

$$U(t,x) = c_0^{-1/3} u\left(c_0^{-3/2} t, c_0^{-1/2}(x+t)\right) \tag{10.7}$$

solves

$$\partial_t U + \partial_x(\partial_x^2 U - U + U^4) = 0, \quad t, x \in \mathbb{R}, \tag{10.8}$$

and it is the unique solution of (10.8) satisfying

$$\lim_{t\to-\infty} \left\|U(t) - Q_{1-\mu_0}(. + \mu_0 t - y_1^-) - Q_{1+\mu_0}(. - \mu_0 t - y_2^-)\right\|_{H^1} = 0. \tag{10.9}$$

This means that, from the general case (10.5), we can reduce ourselves to a symmetric situation for the asymptotic speeds at $-\infty$. We are interested by the asymptotics where μ_0 is small.

Theorem 11 (Inelastic interaction of two nearly equal solitons)**.** *Assume* $0 < \mu_0 \ll 1$. *Let* $U(t)$ *be the unique solution of* (10.8) *such that*

$$\lim_{t\to-\infty} \left\|U(t) - Q_{1-\mu_0}(. + \mu_0 t + \tfrac{1}{2}Y_0 + \ln 2) - Q_{1+\mu_0}(. - \mu_0 t - \tfrac{1}{2}Y_0 - \ln 2)\right\|_{H^1} = 0, \tag{10.10}$$

where $Y_0 = |\ln(\mu_0^2/\alpha)|$ *and* $\alpha = 12(10)^{2/3}(\int Q^2)^{-1}$. *Then*

- *Global stability of the* 2*-soliton structure and asymptotics as* $t \to +\infty$.

  There exist $\mu_1(t), \mu_2(t), y_1(t), y_2(t)$ *of class* C^1 *such that*

  $$w(t,x) = U(t) - Q_{1+\mu_1(t)}(. - y_1(t)) - Q_{1+\mu_2(t)}(. - y_2(t))$$

  satisfies, for all $t \in \mathbb{R}$,

  $$\|w(t)\|_{H^1} \le C\mu_0^{2^-}, \quad \lim_{t\to+\infty} \|w(t)\|_{H^1(x>-(99/100)t)} = 0, \tag{10.11}$$

  $$\left|\min_{t\in\mathbb{R}}(y_1(t) - y_2(t)) - Y_0\right| \le C\mu_0^{\frac{3}{2}^-}, \tag{10.12}$$

  $$\lim_{+\infty} \mu_1 = \mu_1^+ \sim \mu_0, \quad \lim_{+\infty} \mu_2 = \mu_2^+ \sim -\mu_0. \tag{10.13}$$

- *Inelasticity of the interaction.*

$$\liminf_{t\to+\infty} \|w(t)\|_{H^1} \geq c\mu_0^3, \tag{10.14}$$

$$\mu_1^+ \geq \mu_0 + c\mu_0^5, \quad \mu_2^+ \leq -\mu_0 - c\mu_0^5. \tag{10.15}$$

It follows immediately from the lower bound (10.14) that *no pure 2-soliton exists*, which was a new result in this regime.

Comments on the results:

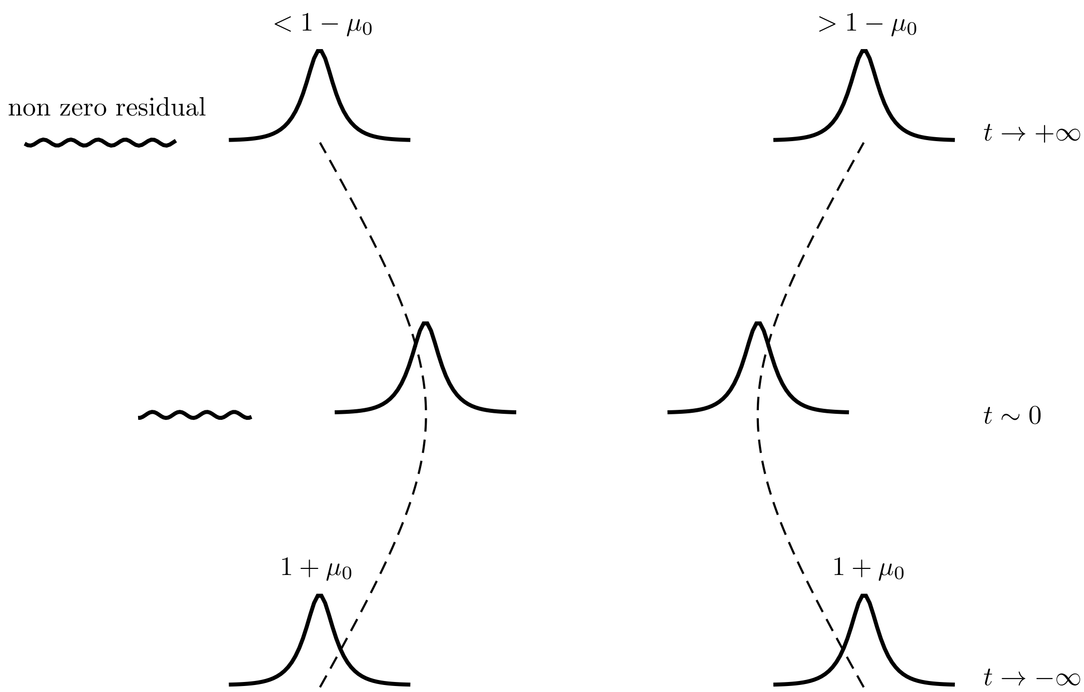
1. For the specific solution $U(t)$ considered in Theorem 11, the dynamics of the parameters $\mu_j(t)$, $y_j(t)$ are closely related to the function

$$Y(t) = Y_0 + 2\ln(\cosh(\mu_0 t)) \text{ which solves } \ddot{Y} = 2\alpha e^{-Y},\ \lim_{\pm\infty} \dot{Y} = \pm 2\mu_0,\ \dot{Y}(0) = 0. \tag{10.16}$$

More information on the behavior of $U(t)$ and the parameters $\mu_j(t)$, $y_j(t)$ in terms of $Y(t)$ is available in [60].

2. Theorem 11 answers the two questions raised before concerning the interaction of two solitons of almost equal speeds. The lower bounds in estimates (10.14) and (10.15) measure the defect of $U(t)$ at $+\infty$; in other words, they quantify in the energy space H^1 the inelastic character of the collision of 2 solitons of (10.8) in the regime where μ_0 is small.

The behavior of the solution $U(t)$ considered in Theorem 11 is represented schematically by the following picture:



10.3 Sketch of the proof of Theorem 11

The strategy of the proof follows, as in [59], from the combination of three different types of arguments, the main one being the construction of a suitable approximate solution to the problem in the interaction region.

(1) We construct an approximate solution to the problem in terms of a series in $e^{-y(t)}$ where $y(t) = y_1(t) - y_2(t)$ is the distance between the two solitons. We explain the construction from Section 2 of [60], simplifying the exposition and the notation.

We look for an approximate solution of $V_t + (V_{xx} - V + V^4)_x = 0$ under the form

$$V_{\mu_1(t),\mu_2(t),y_1(t),y_2(t)}(x) = R_1(t,x) + R_2(t,x) + w(t,x),$$

where

$$R_j(t,x) = Q_{1+\mu_j(t)}(x - y_j(t)), \quad j = 1, 2,$$

and $(\mu_1(t), \mu_2(t), y_1(t), y_2(t))$ are time dependent parameters to be ajusted when using the approximate solution.

By direct computations, using

$$Q''_{1+\mu_j} + Q^4_{1+\mu_j} = (1+\mu_j)Q_{1+\mu_j},$$

$$\left(\frac{d}{d\mu}Q_{1+\mu}\right)_{|\mu=\mu_j} = \frac{1}{1+\mu_j}\left(\frac{1}{3}Q_{1+\mu_j} + \frac{1}{2}xQ'_{1+\mu_j}\right) = \Lambda Q_{1+\mu_j},$$

and the notation $\Lambda R_j = \Lambda Q_{1+\mu_j}(x - y_j(t))$, we find

$$V_t + (V_{xx} - V + V^4)_x = E + F + G(w) + H(w),$$

where

$$\begin{aligned}
E &= \sum_{j=1,2} \dot{\mu}_j \Lambda R_j + \sum_{j=1,2} (\dot{y}_j - \mu_j)(R_j)_x,\\
F &= \left((R_1+R_2)^4 - R_1^4 - R_2^4\right)_x,\\
G(w) &= \left(w_{xx} - w + 4\left(R_1^3 + R_2^3\right) w\right)_x,\\
H(w) &= w_t + \Big((R_1 + R_2 + w)^4 - \left((R_1+R_2)^4 + 4\left(R_1^3 + R_2^3\right) w\right)\Big)_x.
\end{aligned}$$

These terms are organized as follows. First, E contains the time derivative of $R_1(t)$ and $R_2(t)$ which depend on t only through the parameters $\mu_j(t)$, $y_j(t)$. Second, F contains the nonlinear interaction terms between R_1 and R_2. We will see below how F can be decomposed. Next, G contains linear terms of order 1 in w, in particular it involves the linearized operator around $R_1 + R_2$. Finally, H contains what are expected to be higher order terms. In particular, w should be of slow variation, so that w_t is of higher order and other terms are either quadratic in w or linear in w but containing quadratic interactions of R_1 and R_2.

Now, we expand the term F, denoting $x_j = x - y_j$. By direct computations using the asymptotics of Q in (3.1),

$$F = F_A + F_1 \quad \text{where } F_A = 4(10)^{1/3} e^{-y} \left(e^{-x_1} Q^3(x_1) + e^{x_2} Q^3(x_2)\right),$$

and F_1 is of higher order in e^{-y} and μ_1, μ_2 (μ_1, μ_2 are approximately of size μ_0). We explain how to remove the term F_A from F using a suitable w. We take $w = w_A + w_1$, where

$$w_A = e^{-y} \left(A_1(x_1) + A_2(x_2)\right).$$

For a suitable choice of functions A_j and constants α, a (e.g. A_1 solves $(-\mathcal{L}A_1) + \alpha\Lambda Q + aQ' = g_{A_1}$ for g_{A_1} related to F_A) , we find that

$$|F_A + G(w_A) + \alpha e^{-y}\Lambda R_1 - \alpha e^{-y}\Lambda R_2 + ae^{-y}(R_1)_x + ae^{-y}(R_2)_x| = o(e^{-y}),$$

where $o(e^{-y})$ is negligible with respect to e^{-y}. In fact, it is in some sense of order $e^{-\frac{3}{2}y}$. The underlying idea for the choice of w_A is the method of separation of variables: the independent variables in the problem being $x_1 = x - y_1$, $x_2 = x - y_2$ and e^{-y}, as can be seen in the definition of F_A.

The constants a and α are not zero and they are necessary to solve the problem. They are needed to deal with the resonance (in some sense) and the kernel of the operator $(\mathcal{L})_x$. Turning back to the equation of V and the expression of E, we see that this forces the following equations on μ_j, y_j:

$$\begin{aligned} &\dot\mu_1 \sim \alpha e^{-y}, &\qquad &\dot\mu_2 \sim -\alpha e^{y},\\ &\dot y_1 \sim \mu_1 - a e^{-y}, &\qquad &\dot y_2 \sim \mu_2 - a e^{-y}. \end{aligned}$$

Thus, we are able to choose w_A and the time parameters so that $V_A = R_1 + R_2 + w_A$ solves the equation at the order e^{-y}. In fact, we can compute explicitly all error terms and continue looking for an approximate solution at a higher order. There is virtually no limit at the order of the approximate. However, for the proof of Theorem 11, we only need to compute an approximate solution V up to order $e^{-2y(t)}$ in some sense, meaning that $V_t + (V_{xx} - V + V^4)_x$ is controlled by e^{-2y} around the interaction. Then, the parameters $\mu_j(t)$, $y_j(t)$ are bound to satisfy a refined dynamical system of the form

$$\begin{aligned} &\dot\mu_1 \sim \alpha\, e^{-y} + \beta\, \mu_1 y e^{-y},\\ &\dot\mu_2 \sim -\alpha\, e^{-y} - \beta\, \mu_2 y e^{-y},\\ &\dot y_1 \sim \mu_1 - a\, e^{-y} - b_1\, \mu_1 y e^{-y},\\ &\dot y_2 \sim \mu_2 - a\, e^{-y} - b_2\, \mu_2 y e^{-y}. \end{aligned} \tag{10.17}$$

The limit properties of A_1 and A_2, $\lim_{+\infty} A_1 = \lim_{-\infty} A_2 = 0$ and $\lim_{-\infty} A_1 = \lim_{+\infty} A_2 \neq 0$, are consequences of the explicit resolution of the equations of A_1 and A_2. It follows that w_A belongs to L^2 but contains a flat part of size of order $e^{-y(t)}$ between the two solitons (as in the (KdV) case), which is relevant in the description of the exact solution.

An extra difficulty appears at order $e^{-\frac{3}{2}y}$ since it turns out that the approximate solution at this order, denoted by w_B, has a nonzero limit at $-\infty$, and thus w_B is not in L^2. After a suitable cut-off, we can get round this difficulty, which is clearly related to non integrability, see point (3) below.

Note that these computations improve the ansatz of [70], limited to $V = R_1 + R_2$.

(2) After the approximate solution is constructed, we introduce the following decomposition of the solution $U(t)$ defined in Theorem 11:

$$U(t,x) = V_{\mu_1(t),\mu_2(t),y_1(t),y_2(t)}(x) + \varepsilon(t,x),$$

where $V_{\mu_1(t),\mu_2(t),y_1(t),y_2(t)}$ is the modulated approximate solution and $\varepsilon(t)$ is a rest term. To prove stability of the 2-soliton structure, we have to control both the parameters $\mu_j(t)$ and $y_j(t)$ and the rest term $\varepsilon(t)$.

The control of the rest term $\varepsilon(t)$ uses variants of techniques discussed in the previous sections for large time stability and asymptotic stability of solitons and multi-solitons for the (gKdV) equations in the energy space. Note that these techniques apply only in situations where the solitons are decoupled, but this is indeed the case in the present situation, since $y(t) = y_1(t) - y_2(t)$ is large as μ_0 is small.

As a conclusion of this analysis on the rest term $\varepsilon(t)$ and of the control of the dynamical system satisfied by the parameters, we obtain the stability results of Theorem 11. In particular, we prove $\|\varepsilon(t)\|_{H^1} \leq e^{-\frac{5}{4}^{-} Y_0}$ for all t.

(3) Finally, we sketch the most delicate part of the proof, which is the inelasticity. We refer to [60] for details.

Recall that we mentionned the need of introducing an L^2 approximation of V, using a suitable cut-off function, since the natural approximate solution at order $e^{-\frac{3}{2}y(t)}$ has a nonzero limit as $x \to -\infty$. Note that, in the integrable case $p = 2$, one obtains V in L^2 using the same strategy – this difficulty is thus clearly related to non integrability (see [61]). It is thus only at order $e^{-\frac{3}{2}y(t)}$ in the construction of the approximate solution that our analysis points out a deep difference between (KdV) and non integrable (gKdV).

The proof of inelasticity follows from a contradiction argument and a refined analysis of the dynamical system of parameters (10.17). Assume that $U(t)$ is a pure 2-soliton solution. Then a contradiction follows from combining the following two facts.

On the one hand, by uniqueness properties (Theorem 8), $U(t)$ satisfies $U(t,x) = U(-t,-x)$ up to translation in space and time. As a consequence, the parameters $\mu_j(t)$, $y_j(t)$ which are related to a certain decomposition of $U(t)$ also have certain symmetry properties under the transform $t \to -t$.

On the other hand, the dynamical system satisfied by $\mu_j(t)$, $y_j(t)$ is not symmetric by the transformation $x \to -x$, $t \to -t$ at order $e^{-\frac{3}{2}y(t)}$. Indeed, from the algebra in the quartic case, we have $b_1 \neq b_2$ in (10.17).

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