THE NON LINEAR HARMONIC OSCILLATOR WITH RANDOM INITIAL DATA
CANPDE CRASH-COURSE, EDINBURGH, 12-14 MARCH 2009

NICOLAS BURQ

1. Introduction, presentation of the problem

In this course I will present some recent results with L. Thomann and N. Tzvetkov on the behaviour of solutions to Schrödinger equations with random initial data. The main question I want to address is the following: Are solutions to Schrödinger equations better behaved when one consider initial data randomly chosen (in some sense) than what would be predicted by the deterministic theory? To my knowledge the first result known in this direction is due to Rademacher/Kolmogorov/Paley-Zygmund, and states that random series on the torus enjoy better L^p bounds that the deterministic bounds. These lectures are somehow a natural extension on the partial differential equations field of these harmonic analysis results. We shall use some basic results from probability theory. The non linear Schrödinger I will be interested in, is the following one dimensional non linear harmonic oscillator

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u = |u|^{r-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where $r > 1$ is the order of the non linearity. On a deterministic point of view, this equation is well posed in $L^2(\mathbb{R})$ as soon as $p \leq 5$ (see Proposition 4.5 below), and the assumption $p \leq 5$ is known to be optimal in some sense (see the works by Christ-Colliander-Tao [7] and Burq-Gérard-Tzvetkov [3, Appendix] in slightly different contexts). However, we shall prove, that for all non linearities $|u|^{p-1}u$, not only is the equation well posed for a large set of initial data whose Sobolev regularity is below L^2, but also that the flows enjoys very nice large time behaviour (in a probabilistic sense).

Theorem 1.1. *There exists a measure ρ defined on $\mathcal{D}'(\mathbb{R})$ such that*

- *The measure ρ is non trivial, and more precisely, its support is dense in any Sobolev space $\mathcal{H}^s(\mathbb{R})$, $\forall s < 0$.*
- *For any $s < 0$, the Sobolev space of functions with s derivatives in $L^2(\mathbb{R})$, $\mathcal{H}^s(\mathbb{R})$, (here the derivatives are measured with respect to the harmonic oscillator, see Section 3) is of full ρ measure*
- *The space $L^2(\mathbb{R})$ has ρ measure* 0
- *The flow of* (1.1) *is defined globally in time on a set of full ρ-measure and the measure ρ is invariant by this flow*

- *The time complexity of the flow is slowly increasing with time: there exists $C >$) such that ρ-almost surely there exists a constant $A > 0$ such that*
$$\|\Phi(t)u_0\|_{\mathcal{H}^s(\mathbb{R})} \leq C\log(A+t)^{1/2})$$
and
$$\rho(\{u_0; A(u_0) > \lambda)\} \leq Ce^{-c\lambda^2}$$

The purpose of these lectures is to present the main points leading to the proof of this result. In fact for the sake of simplicity most of the exposition will focus on the simpler case $r = 7$. The plan of the lectures is the following:

1.1. **Lecture 1.** I will first investigate the properties of random series. In particular, I will prove Rademacher/Kolmogorov/Paley/Zygmund's results: L^p norms of random series on the torus, large deviation estimates.Then I will extend these results to the case of random series on $\mathbb{R}$ using the harmonic oscillator on $\mathbb{R}$ and the Hermite functions.

1.2. **Lecture 2.** This section is devoted to the study of the well posedness theory of the non linear harmonic oscillator. I will first study the deterministic theory and Strichartz estimates leaing to a nice $L^2(\mathbb{R})$ theory. Then I will turn to a probabilistic theory below the L^2 level.

1.3. **Lecture 3.** I will introduce Gibbs measures for the non linear harmonic oscillator and show how, following Bourgain's ideas [1, 2], one can obtain nice large time estimates fot the solutions to the non linear harmonic oscillator on a probalistic set of initial data

1.4. **Tutorial: Christ-Kiselev Lemma.**

2. Paley Zygmund's result

Consider a square summable series $(u_n)_{n\in\mathbb{N}}$,
$$\sum_n |u_n|^2 < +\infty$$
and the trigonometric series on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$
$$u(\theta) = \sum_n u_n e^{in\theta}$$
Then it is standard that the sum of the series $u(\theta)$ is square integrable
$$\int_0^{2\pi} |u(\theta)|^2 d\theta < +\infty$$
but in general (generically for the l^2 topology), the function u is in no space $L^q(\mathbb{T}), q > 2$. Zygmund's result states that if one changes *randomly* and *independently* the signs of the coefficients u_n, the sum of the series is almost surely in $L^q(\mathbb{T})$ for all $q < +\infty$. In modern language, consider independent Bernouilli random variables b_n^ω on a probability space $(\Omega, \mathcal{P})$, i.e.
$$\mathbb{P}(b_n^\omega = \pm 1) = \frac{1}{2}$$

and consider the random series

$$u^\omega = \sum_n u_n b_n^\omega e^{in\theta}.$$

Theorem 2.1 (see [8]). *Almost surely* $u^\omega \in L^q(\mathbb{T}), \forall q < +\infty$

In fact, using modern technology, one can take more general random variables and precise the result.

Theorem 2.2 (see [5, 6]). *Assume that the random variable* b_n^ω *are*

(1) *independent,*
(2) *have mean equal to* 0*,*
(3) *have super-exponential decay*

$$\exists C, \delta > 0; \forall \alpha \in \mathbb{R}, \mathbb{E}(e^{\alpha |b_n^\omega|}) \leq C e^{\delta \alpha^2}. \tag{2.1}$$

Notice that this latter assumption is satisfied for Bernouilli, or more generally for families of random variables having a (fixed) compact support, or for standard Gaussian random variables.

Then,

(1) *almost surely,* $u_n^\omega \in L^p(\mathbb{T}), \forall q < +\infty$*,*
(2) *furthermore, the following large deviation estimate holds*

$$\forall q < +\infty, \exists C, \alpha_q > 0, \qquad \mathcal{P}(\|u^\omega\|_{L^q(\mathbb{T})} > \lambda) \leq e^{-\alpha_q \Lambda^2}.$$

The remaining of this section is devoted to the proof of Theorems 2.1 and 2.2. Even though, Theorem 2.1 is completly included in Theorem 2.2, we shall start giving an elementary proof of Theorem 2.1 in the special case $q = 4$.

2.1. Proof of Theorem 2.1 in the case $q = 4$.

We are going to prove that

$$\mathbb{E}(\|u^\omega\|^4_{L^4(\mathbb{T})}) < +\infty.$$

Notice that this readily implies that $\|u^\omega\|^4_{L^4(\mathbb{T})} < +\infty$ almost surely. Indeed, the finiteness of the expectancy i.e. the L^1 norm, implies that the function has to be finite almost surely (notice that the converse is false). In our analyse, the main points we shall use about the Bernouilly random variables is that according to the independence assumption,

$$n \neq m \Rightarrow \mathbb{E}(b_n^\omega b_m^\omega) = \mathbb{E}(b_n^\omega)\mathbb{E}(b_m^\omega)$$

and according to the fact that the Bernouilly ramdom variables have mean equal to 0, then

$$\mathbb{E}(b_n^\omega) = 0.$$

Now, compute, using Fubini's Theorem,

$$
\begin{aligned}
\mathbb{E}(\|u^\omega\|^4_{L^4(\mathbb{T})}) &= \int_\omega \int_\theta |u^\omega(\theta)|^4 d\theta dp_\omega \\
&= \int_\theta \int_\omega |u^\omega(\theta)|^4 dp_\omega d\theta \\
&= \int_\theta \int_\omega \sum_{n_1,\cdots,n_4} u_{n_1}u_{n_2}u_{n_3}u_{n_4} e^{i(n_1+n_2-n_3-n_4)\theta} b^\omega_{n_1}b^\omega_{n_2}b^\omega_{n_3}b^\omega_{n_4} dp_\omega d\theta \\
&= \int_\theta \sum_{n_1,\cdots,n_4} u_{n_1}u_{n_2}u_{n_3}u_{n_4} e^{i(n_1+n_2-n_3-n_4)\theta} \int_\omega b^\omega_{n_1}b^\omega_{n_2}b^\omega_{n_3}b^\omega_{n_4} dp_\omega d\theta
\end{aligned} \tag{2.2}
$$

Notice now that in the expression above, according to the independence and mean-0 assumption, all integrals with respect to the ω variables vanish unless the indexes n_1, n_2, n_3 and n_4 are pairwise equal. To fix ideas, let us consider the contribution of the case $n_1 = n_3 = n$ and $n_2 = n_4 = m$ to the sum above. Notice first that since $\int_\omega (b^\omega_n)^2 dp_\omega = \int_\omega (b^\omega_n)^4 dp_\omega = 1$, one has

$$
\int_\omega (b^\omega_n)^2 (b^\omega_m)^2 = \begin{cases} \int_\omega (b^\omega_n)^2 dp_\omega \int_\omega (b^\omega_m)^2 dp_\omega = 1 & \text{if } n \neq m \\ \int_\omega (b^\omega_n)^4 dp_\omega = 1 & \text{if } n = m \end{cases} \tag{2.3}
$$

As a consequence, we obtain for the contribution of the case $n_1 = n_2 = n$ and $n_3 = n_4 = m$ to the sum above,

$$
\int_\theta \sum_{n,m} |u_n|^2 |u_m|^2 d\theta \leq C(\sum_n |u_n|^2)^2. \tag{2.4}
$$

The contribution of the two other cases ($n_1 = n_3, n_2 = n_4$ and $n_1 = n_4, n_2 = n_3$) are dealt with similarly. This concludes the proof of Theorem 2.1 in the case $q = 4$. □

2.2. **Proof of Theorem 2.2.** The proof relies on

Proposition 2.3. *[large deviation estimate] Assume that the random variables satisfy the assumptions of Theorem 2.2. Then there exists $\delta > 0$ such that for any $\Lambda > 0$, and any sequence $v_n \in \ell^2$,*

$$
\mathcal{P}(|\sum_n v_n b^\omega_n| > \Lambda) \leq e^{-\delta \frac{\Lambda^2}{\sum_n |v_n|^2}}
$$

2.2.1. *Proof of Theorem 2.2 assuming Proposition 2.3.* Recall that according to Fubini's Theorem, we have

Lemma 2.4. *For any random variable $f : \omega \in \Omega \mapsto f(\omega) \in \mathbb{R}$, we have*

$$
\int_\omega |f(\omega|^q dp_\omega = \int_{\lambda=0}^{+\infty} q\lambda^{q-1} \mathcal{P}(|f| > \lambda) d\lambda
$$

Indeed,

$$\begin{aligned}
\int_0^{+\infty} q\lambda^{q-1}\mathcal{P}(|f|>\lambda)d\lambda &= \int_0^{+\infty} q\lambda^{q-1}\int_\omega 1_{|f(\omega)|>\lambda}dp_\omega,\\
&= \int_\omega dp_\omega \int_{\lambda=0}^{+\infty} 1_{|f(\omega)|>\lambda} q\lambda^{q-1}d\lambda,\\
&= \int_\omega dp_\omega \int_{\lambda=0}^{|f(\omega)|} q\lambda^{q-1}d\lambda,\\
&= \int_\omega dp_\omega |f(\omega|^q.
\end{aligned} \tag{2.5}$$

□

Let us go back to the proof of Theorem 2.2. Fix $r \geq q$. Remark that the norm of an integral is always smaller that the integral of the norm. As a consequence,

$$\begin{aligned}
\| \|u^\omega(\theta)\|_{L^q_\theta}\|_{L^r_\omega} &= \Big(\| \int_\theta |u^\omega(\theta)|^q d\theta\|_{L^{r/q}_\omega}\Big)^{1/q},\\
&\leq \Big(\int_\theta \| |u^\omega(\theta)|^q\|_{L^{r/q}_\omega} d\theta\Big)^{1/q},\\
&= \| \|u^\omega(\theta)\|_{L^r_\omega}\|_{L^q_\theta}.
\end{aligned} \tag{2.6}$$

Applying (2.6) to the function $u^\omega(\theta) = \sum_n u_n e^{in\theta}$, we get

$$\| \|u^\omega(\theta)\|_{L^q_\theta}\|_{L^r_\omega} \leq \| \|u^\omega(\theta)\|_{L^r_\omega}\|_{L^q_\theta}.$$

Applying Lemma 2.4, we obtain (θ is a fixed parameter)

$$\|u^\omega(\theta)\|_{L^r_\omega} = \int_0^{+\infty} r\lambda^{r-1}\mathcal{P}(|u^\omega(\theta)|>\lambda)d\lambda,$$

and according to Proposition 2.3 (and the change of variables $\mu = \frac{\sqrt{2}\delta^{1/2}}{(\sum_n |u_n e^{in\theta}|^2)^{1/2}}$)

$$\begin{aligned}
\|u^\omega(\theta)\|^r_{L^r_\omega} &\leq C\int_0^{+\infty} r\lambda^{r-1}e^{-\delta\frac{\lambda^2}{\sum_n |u_n e^{in\theta}|^2}}d\lambda,\\
&\leq (C\sum_n |u_n|^2)^{r/2} r\int_0^{+\infty} \mu^{r-1}e^{-\frac{\mu^2}{2}}d\mu,\\
&\leq (C\sum_n |u_n|^2)^{r/2} \times r \times r-2 \times \cdots \times 1,\\
&\leq (C' r\sum_n |u_n|^2)^{r/2}.
\end{aligned} \tag{2.7}$$

Notice now that the norm with respect to the θ parameter is harmless (as the bound does not depend on θ). As a conclusion, we just proved

$$\| \|u^\omega(\theta)\|_{L^q_\theta}\|_{L^r_\omega} \leq (C' r\sum_n |u_n|^2)^{1/2}.$$

To conclude, let us recall Tchebytchev inequality:

$$\forall \lambda, \qquad \lambda \mathcal{P}(f^\omega > \Lambda) \leq \mathbb{E}(f).$$

Apply this inequality to the random variable $f^\omega = \|u^\omega(\theta)\|^r_{L^q_\theta}$ and $\lambda = \Lambda^r$. We get

$$\begin{aligned}
\mathcal{P}(\|u^\omega(\theta)\|_{L^q_\theta} > \Lambda) &= \mathcal{P}(\|u^\omega(\theta)\|^r_{L^q_\theta} > \Lambda^r = \lambda) \\
&\leq \frac{1}{\Lambda^r}\mathbb{E}(\|u^\omega(\theta)\|^r_{L^q_\theta}) = \frac{1}{\Lambda^r}\|\|u^\omega(\theta)\|_{L^q_\theta}\|^r_{L^r_\omega} \\
&\leq \Big(\frac{(C' r \sum_n |u_n|^2)}{\Lambda^2}\Big)^{r/2}
\end{aligned} \tag{2.8}$$

Now we optimize this inequality by choosing r so that

$$\frac{(C' r \sum_n |u_n|^2)}{\Lambda^2} = \frac{1}{2}$$

which gives

$$\mathcal{P}(\|u^\omega(\theta)\|_{L^q_\theta} > \Lambda) \leq \Big(\frac{1}{2}\Big)^{r/2} = e^{-\delta \frac{\Lambda^2}{\sum_n |u_n|^2}}$$

which ends the proof of Theorem 2.2 □

2.2.2. *Proof of Proposition 2.3.* In the special case where the random variables g_n are gaussian random variables of variance 1, the result is straightforward. Indeed, $\sum_n v_n g_n$ is a Gaussian random variable of variance $\sum_n |v_n|^2$ and the result follows. In the general case, it is enough to prove

$$\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) \leq e^{-\delta \frac{\lambda^2}{\sum_n |v_n|^2}}.$$

Indeed, the estimate for the other part, $\mathcal{P}(\sum_n v_n b_n^\omega < -\lambda)$ is obtained by changing v_n to $-v_n$. Let us fix $t > 0$ and compute (using the fact that the random variables are independent)

$$\begin{aligned}
\mathbb{E}(e^{t\sum_n v_n b_n^\omega}) &= \mathbb{E}(\prod_n e^{t v_n b_n^\omega}) = \prod_n \mathbb{E}(e^{t v_n b_n^\omega}) \\
&\leq \prod_n e^{\delta t^2 |v_n|^2} \leq e^{t^2 \sum_n |v_n|^2}
\end{aligned} \tag{2.9}$$

where in the last but one inequality, we used the super-exponential decay assumption(2.1). Now, using Tchebytchev inequality,

$$\begin{aligned}
\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) &= \mathcal{P}(e^{t\sum_n v_n b_n^\omega} > e^{t\lambda}), \\
&\leq e^{-t\lambda}\mathbb{E}(e^{t\sum_n v_n b_n^\omega}), \\
&\leq e^{\delta t^2 \sum_n |v_n|^2 - t\lambda}.
\end{aligned} \tag{2.10}$$

Optimize by choosing $\delta t^2 \sum_n |v_n|^2 = t\lambda/2$, i.e. $t = \lambda/(2\delta \sum_n |v_n|^2)$, which gives

$$\mathcal{P}(\sum_n v_n b_n^\omega > \lambda) \leq e^{-\alpha \frac{\lambda^2}{\sum_n |v_n|^2}}$$

which ends the proof of Proposition 2.3 and consequently the proof of Theorem 2.2. □

2.3. **Sobolev embeddings.** Let us recal that in space dimension 1, Sobolev embeddings take the form

$$2 \leq p < +\infty \Rightarrow \|u\|_{L^p(\mathbb{T})} \leq C\|u\|_{H^{1/2-1/p}(\mathbb{T})},$$

where the Sobolev space on the torus are defined by

$$u = \sum_n e^{in\theta} \in H^s(\mathbb{T}) \Leftrightarrow \sum_n (1+n^2)^s |u_n|^2 = \|u\|_{H^s}^2 < +\infty.$$

As a consequence, to be sure that our results are not trivial, one has to check that our randomization procedure did not lead to a smoothing in the Sobolev scale (in which case Theorems 2.1 and 2.2 would be simple consequences of Sobolev embeddings). This is sumarized in

Theorem 2.5. *Assume that the random variables satisfy the assumptions of Theorem 2.2. Then almost surely, the random series u^ω belongs to $L^2(\mathbb{T})$. More precisely*

$$\mathcal{P}(\|u^\omega\|_{L^2(\mathbb{T})} > \lambda) \leq Ce^{-\delta\Lambda^2}$$

(notice that in the case of Bernouilli r.v, $\|u^\omega\|_{L^2(\mathbb{T})} \leq (\sum_n |u_n|^2)^{1/2}$) Assume now that

1. *the random variables b_n^ω do not accumulate at 0, i.e.*

$$\exists c > 0, \delta > 0; \mathcal{P}(|b_n^\omega| > c) \geq \delta,$$

2. *the sequence u_n does not have additional decay, i.e. $\sum_n (1+n^2)^s |u_n|^2 = +\infty$.*

Then

$$\mathcal{P}(u^\omega \in H^s(\mathbb{T})) = 0.$$

The proof of the first part of Theorem 2.5 is very similar to the proofs of Theorems 2.1, 2.2 above and I will consequently skip them. Let us instead focus on the last (most important) part. We are going to prove

Lemma 2.6. *We have*

$$\mathbb{E}(e^{-\|u^\omega\|_{H^s}^2}) = 0 \tag{2.11}$$

Since the function $e^{-\|u^\omega\|_{H^s(\mathbb{T})}^2}$ is non negative, the fact that its expectancy is equal to 0 implies readily that the function is almost surely equal to 0 i.e

$$\|u^\omega\|_{H^s(\mathbb{T})}^2 = +\infty, \text{almost surely.}$$

Compute, using independence of the random variables,

$$\begin{aligned}\mathbb{E}(e^{-\|u^\omega\|^2_{H^s}}) &= \mathbb{E}(e^{-\sum_n (1+n^2)^s|u_n|^2|b_n^\omega|^2}),\\ &= \prod_n \mathbb{E}(e^{-(1+n^2)^s|u_n|^2|b_n^\omega|^2}),\\ &\leq \prod_n (1\times \mathcal{P}(|b_n^\omega|\leq c) + e^{-(1+n^2)^s|u_n|^2c^2}\mathcal{P}(|b_n^\omega)\geq c0,\\ &\leq \prod_n (1-\delta_n+\delta_n e^{-(1+n^2)^s|u_n|^2c^2}),\\ &\leq \prod_n (1-\delta_n+\delta_n(1-(1+n^2)^s|u_n|^2c^2/2)),\\ &\leq \prod_n (1-\delta_n(1+n^2)^s|u_n|^2c^2/2),\\ &\leq \prod_n (1-\delta(1+n^2)^s|u_n|^2c^2/2),\end{aligned} \tag{2.12}$$

where using the assumption in Theorem 2.5, we have $\delta_n = \mathcal{P}(|b_n^\omega|\geq c)\geq \delta$. Notice that in the last but one estimate above we used the fact that $e^{-x}\leq 1-x/2$, which is true if $x=(1+n^2)^s|u_n|^2c^2\geq 0$ is small enough. Otherwise, we have to replace the estimate by $e^{-x}\leq \alpha$ for a fixed $0<\alpha<1$. Now the end of the proof of Lemma 2.6 is straightforward as the assumption

$$\sum_n (1+n^2)^s|u_n|^2 = +\infty$$

ensures that the infinite product diverges to 0:

$$\prod_n (1-\delta(1+n^2)^s|u_n|^2c^2/2) = 0$$

(take the logarithm and use standard results to show that the logarithm of the infinite product which is an infinite sum is equal to $-\infty$). This ends the proof of Theorem 2.5. □

3. Random data on $\mathbb{R}$

Let us now turn to the study of random series on $\mathbb{R}$. The starting point is the following result on the eigenfunctions of the harmonic oscillator, that we shall not prove (see [4])

Proposition 3.1. *There exists an eigenbasis of $L^2(\mathbb{R})$ formed by the eigenfunctions of the harmonic oscillator, h_n, associated to eigenvalues $2n+1$:*

$$Hh_n = -h_n''(x) + x^2 h_n(x) = (2n+1)h_n(x) = \lambda_n^2 h_n(x)$$

Furthermore, these eigenfunctions are explicit:

$$h_n(x) = c_n e^{\frac{x^2}{2}}\partial_x^n(e^{-x^2})$$

and if the constant c_n is chosen to normalize the L^2 norm to 1, we have

$$\|h_n\|_{L^4(\mathbb{R})}\leq C\lambda_n^{-\frac14+\epsilon}, \qquad \|h_n\|_{L^\infty}\leq C\lambda_n^{-\frac16} \tag{3.1}$$

Define the Sobolev space (for $s \in \mathbb{R}, 1 \le p \le +\infty$)

$$\mathcal{W}^{s,p}(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}); H^{s/2}u \in L^p(\mathbb{R})\}$$

endowed with its natural norm

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p(\mathbb{R})}.$$

Notice that for $u = \sum_n u_n h_n(x)$,

$$H^\alpha u = \sum_n \lambda_n^{2\alpha} u_n h_n(x).$$

Lemma 3.2. *Assume that $s \in \mathbb{R}$ and $1 < p < +\infty$. Then the $\mathcal{W}^{s,p}$ norm is equivalent to*

$$\|u\|_{s,p} = \||D_x|^s u\|_{L^p(\mathbb{R})} + \|\langle x\rangle u\|_{L^p(\mathbb{R})}$$

Proof. Indeed, for $s = 0$, this result is trivial. Let us prove it for $s \in 2\mathbb{N}$ and $p = 2$:

$$\|H^p u\|_{L^2}^2 = \Big((-\partial_x^2 + x^2)^q u, (-\partial_x^2 + x^2)^p u\Big)$$

and clearly

$$\|u\|_{\mathcal{W}^{2q,2}} \le C\|u\|_{2q,2}$$

to prove the reverse inequality, we proceed by induction. Assume the result is true for $s \le 2(q-1)$. Then, applying the induction assumption, we get

(3.2)
$$\|H^{2q}u\|_{L^2}^2 \ge C\|Hu\|_{2(q-1),2}^2 \ge C\|(-\partial_x^2)^{q-1}(-\partial_x^2 + x^2)u\|_{L^2}^2 + \|(x^2+1)^{q-1}(-\partial_x^2 + x^2)u\|_{L^2}^2$$

But

$$\begin{aligned}(3.3)\quad &\|(-\partial_x^2)^{q-1}(-\partial_x^2 + x^2)u\|_{L^2}^2 \\ &= \|(-\partial_x^2)^q u\|_{L^2}^2 + \|(-\partial_x^2)^{q-1}x^2 u\|_{L^2}^2 + 2\Big((-\partial_x^2)qu, (-\partial_x^2)^{(q-1)}x^2 u\Big)_{L^2}\end{aligned}$$

Now, to estimate the last term, we notice that it is equal to

$$2\Big(x^2(-\partial_x^2)^{(2(q-1)}u, u\Big)_{L^2} = 2\Big(x(-\partial_x^2)^{(2(q-1)}xu, u\Big)_{L^2} + 2\Big(x[x, (-\partial_x^2)^{2(q-1)}]u, u\Big)_{L^2}$$

The first term is non negative, whereas the second one is equal to

$$2\Big(x[x, (-\partial_x^2)^{2(q-1)}]u, u\Big)_{L^2} = 2\Big(2(q-1)x\partial_x^{2q-3}u, u\Big)_{L^2}$$

and it is consequently bounded (in absolute value) by

$$C\|u\|_{2q-2,2}^2.$$

Similarly, the second term in the r.h.s. of (3.2) controls $\|(x^2+1)^{q-1}(x^2)u\|_{L^2}^2$ (and hence $\|(x^2+1)^q u\|_{L^2}^2$) modulo an error which is bounded by $C\|u\|_{2q-2,2}^2$. As a consequence, we obtain

$$\|H^{2q}u\|_{L^2}^2 \ge c\|u\|_{2q,2}^2 - C\|u\|_{2q-2,2}^2,$$

which implies Lemma 3.2 for $s \in 2\mathbb{N}, p = 2$.

By interpolation, we deduce the same estimate for $s \in \mathbb{R}^+, p = 2$, and by duality for $s \in \mathbb{R}, p = 2$. Consider now arbitrary $2 < q < +\infty$. Writing , with $\theta = \frac{2}{q}$ and $\sigma = \frac{sq}{2}$

$$\frac{\theta}{2} = \frac{1}{q}, \qquad s = \theta\sigma,$$

we see that Lemma 3.2 follows by interpolating between the cases $(0, \infty)$ and $(2, \sigma)$. The other cases follow by duality. □

We deduce from Lemma 3.2 and the analog result on the usual $W^{s,p}$ spaces, the following

Lemma 3.3. *For any u, v, we have, if $1 < p_i, q_i < +\infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} = \frac{1}{q_1} + \frac{1}{p_2}$*

$$\|uv\|_{\mathcal{W}^{s,p}} \leq C(\|u\|_{\mathcal{W}^{s,p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|v\|_{\mathcal{W}^{s,q_2}})$$

Consider now the random series

$$u^\omega = \sum_n u_n b_n^\omega e^{in\theta},$$

where b_n are random variables. and $u_n \in \ell^2$. Then the same proof as above shows (using (3.1)

Theorem 3.4. *Assume that the random variable b_n^ω satisfy the assumptions of Theorem 2.2 above. Then*

1. *Almost surely, $u_n^\omega \in \mathcal{W}^{\frac{1}{4},q}(\mathbb{R})$,*
2. *(large deviation estimate)*

$$\exists C, \alpha_q > 0, \qquad \mathcal{P}(\|u^\omega\|_{\mathcal{W}^{\frac{1}{4},4}(\mathbb{R})} > \lambda) \leq e^{-\alpha_q \Lambda^2}.$$

Notice now that we can add an additional time dependence and obtain (with the same proof), if we denote by

$$S(t)u_0 = e^{itH} u_0$$

the solution to the linear Schrödinger equation,

$$i\partial_t u + Hu = 0, \qquad u\mid_{t=0} = u_0,$$

the following result.

Theorem 3.5. *Assume that the random variable b_n^ω satisfy the assumptions of Theorem 2.2 above. Then*

1. *Almost surely for any $4 \leq q < +\infty$, $S(t)u_n^\omega \in L^q((0, 2\pi)_t; \mathcal{W}^{\frac{1}{4},4}(\mathbb{R}))$,*
2. *(large deviation estimate)*

$$\forall q < +\infty, \exists C, \alpha_q > 0, \qquad \mathcal{P}(\|S(t)u^\omega\|_{L^q((0,2\pi)_t; \mathcal{W}^{\frac{1}{4},4}(\mathbb{R}))} > \lambda) \leq e^{-\alpha_q \Lambda^2}$$

4. Cauchy theory for the harmonic oscillator

4.1. **Deterministic theory.** The key result on the deterministic point of view, is the Strichartz estimate

Theorem 4.1. *Consider u solution to*

$$i\partial_t u + Hu = f_1 + f_2, \qquad u\mid_{t=0} = u_0$$

Then

$$\|u\|_{L^\infty((0,2\pi);L^2_x))} + \|u\|_{L^4((0,2\pi);L^\infty_x)} \leq C\Big(\|u_0\|_{L^2} + \|f_1\|_{L^1((0,2\pi);L^2_x)} + \|f_2\|_{L^{4/3}((0,2\pi);L^1_x)}\Big)$$

Notice that the estimates n Theorem 4.1 are stated on a time interval of length 2π (which is natural as the operator e^itH is 2π periodic), but it is clearly enough to prove the estimates on a time interval of length 1. The proof of this result relies on Hardy-Littlewood-Sobolev inequalities (see [9, VIII.4.2]:

Lemma 4.2. *Consider $2 \leq p \leq +\infty$ and T a convolution operator*

$$Tf(x) = \int_{-\infty}^{+\infty} K(x-y)f(y)dy$$

with $K \in L^1_{loc}$. Then the norm of T as an operator from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ is bounded by $\|K\|_{L^{\frac{p}{2}}}$. Furthermore, if $K(x) = \frac{1}{|x|^{2/p}}$ and $p \neq 2$, then the operator T is still bounded from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ (despite the logarithmic divergence of $\|K\|_{L^{\frac{p}{2}}(\mathbb{R})}$).

Indeed, let us first deal with the contribution of u_0, $S(t)u_0$. The fact that $T : u_0 \mapsto S(t)u_0$ sends L^2 to $L^\infty((0,1)_t; L^2_x)$ (with norm 1), is simply due to the conservation of the L^2 norm. The fact that it sends L^2 to $L^4((0,1); L^\infty(\mathbb{R})$ is equivalent to the fact that its dual T^* sends $L^{4/3}((0,1); L^1(\mathbb{R})$ to L^2 (notice that L^1 is not the dual of L^∞, but it is dense in the dual of L^∞ and this fact is enough for the equivalence), and since $\|T^*f\|^2_{L^2} = (T^*f, T^*f) = (TT^*f, f)$, it is easy to see that this latter fact is equivalent to the fact that TT^* is continuous from $L^{4/3}((0,1); L^1(\mathbb{R})$ to $L^4((0,1); L^\infty(\mathbb{R}))$. But, according to Duhamel formula,

$$TT^*f = \int_0^{2\pi} S(t-s)f(s)ds$$

Now the distribution kernel of the operator $S(t-s)$ is given by

$$K(t-s,x,y) = \frac{C}{|\sin\left(\frac{t-s}{2}\right)|^{1/2}} e^{\frac{i}{\sin((t-s)/2)}\left(\frac{x^2+y^2}{2}\cos((t-s)/2) - xy\right)}$$

and consequently its norm as an operator from L^1_y to L^∞_x which is equal to the $L^\infty_{x,y}$ norm of this kernel, is bounded by (notice that $t, s \in (0,1)$)

$$\frac{C}{|\sin\left(\frac{t-s}{2}\right)|^{1/2}} \leq \frac{C}{|t-s|^{1/2}}$$

and the estimate for the contribution of u_0 follows from Lemma 4.2. The contribution of f_1 is easily deduced from this by Minkowski inequality whereas the contribution from f_2 follows from a duality and TT^* argument. Indeed, according to our study of the contribution of u_0, we know that the operator

$$f \mapsto \int_0^1 e^{-(t-s)H} f(s) ds$$

is continuous from $L^{4/3}_t; L^1_x$ to $L^\infty_t; L^2_x \cap L^4_t; L^\infty_x$ Notice now that the contribution of f_2 is given by

$$\int_0^t e^{i(t-s)H} f(s) ds = \int_0^1 1_{s<t} e^{i(t-s)H} f(s) ds$$

As a consequence, Theorem 4.1 will follows from Christ and Kiselev Lemma (see the tutorial for the statement and the proof of this result)

Lemma 4.3. *Consider two Banach spaces X, Y, an interval $I \subset \mathbb{R}$, $1 \leq p < q \leq +\infty$, and $K : L^p(I; X) \to L^q(I; Y)$ a bounded linear operator with distribution kernel $k(t,s)$ is given by a function locally integrable with values bounded operators from X to Y, $k \in L^1_{loc}((I \times I); \mathcal{L}(X,Y))$. Consider the operator R whose distribution kernel is given by $r(t,s) = k(t,s) 1_{s<t}$. Then the operator R is bounded from $L^p(I; X)$ to $L^q(I; Y)$, with norm*

$$\|R\|_{\mathcal{L}(L^p(I;X);L^q(I;Y))} \leq \frac{1}{1 - 2^{\frac{1}{q} - \frac{1}{p}}} \|K\|_{\mathcal{L}(L^p(I;X);L^q(I;Y))} \tag{4.1}$$

Remark 4.4. *Remark that the estimate* (4.1) *the constants depend only on the values of the indexes p, q. As a consequence, it is possible to apply the result, even though the assumption $k \in L^1_{loc}((I \times I); \mathcal{L}(X,Y))$ is* not satisfied. *Indeed, it is enough to regularize the operetor K (and hence the kernel), prove estimates uniform with respect to the regularization parameter, an then pass to the limit.*

We now deduce from Theorem 4.1

Proposition 4.5. *Assume that $r < 5$. Then, for any $u_0 \in L^2(\mathbb{R})$, there exists $T > 0$ and a unique solution of* (1.1) *in $C^0_t; L^2(\mathbb{R}) \cap L^4_{loc,t}; L^\infty(\mathbb{R})$. Furthermore, the time existence T is bounded from below by $C\|u_0\|_{L^2}^{-\gamma}$, $\gamma > 0$.*

Indeed, we are looking for a solution of the following fixed point:

$$u = e^{itH} u_0 + \int_0^t e^{i(t-s)H} (|u|^{r-1} u)(s) ds = K(u)$$

in the space $L^r((0,T); L^{2r}(\mathbb{R}))$. According to Theorem 4.1 (notice that $\frac{r-1}{2r} + \frac{1}{2r} = \frac{1}{2}$), we have

$$\|K(u)\|_{L^{\frac{4r}{r-1}}((0,T);L^{2r}(\mathbb{R}))} \leq C(\|u_0\|_{L^2} + \||u|^r\|_{L^1((0,T);L^2)}) \leq C(\|u_0\|_{L^2} + \|u\|^r_{L^r((0,T);L^{2r})})$$

The assumption $r < 5$ implies $\frac{4r}{r-1} > r$ and consequently, according to Holder inequality (with respect to the time variable), there exists $\delta > 0$ such that

$$\|K(u)\|_{L^r((0,T);L^{2r}(\mathbb{R}))} \leq CT^\delta (\|u_0\|_{L^2} + \|u\|^r_{L^r((0,T);L^{2r})}).$$

As a consequence, if T is small enough, the map K sends the unit ball of $L^r((0,T);L^{2r})$ into itself. The same calculation shows

$$\|K(u)-K(v)\|_{L^r((0,T);L^{2r}(\mathbb{R}))} \leq CT^\delta \|u-v\|_{L^r((0,T);L^{2r})}(\|u\|^r_{L^r((0,T);L^{2r})}+\|v\|^r_{L^r((0,T);L^{2r})}).$$

and consequently the map K is a contraction on the unit ball of $L^r((0,T);L^{2r})$, which proves Proposition 4.5. Notice that the use of Strichartz estimates restricts the range of admissible non linearities to $p\leq 5$ (if one want to work at the L^2 level). To deal with higher power non linearities, one has to perform the fixed point in higher Sobolev spaces. From now on we are going to focus on the model case of harmonic non linear Schrödinger equation with a non linearity of order 7.

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u = |u|^6 u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{4.2}$$

Proposition 4.6. *There exists $s<\frac{1}{6}$ such that for any $u_0\in\mathcal{H}^s(\mathbb{R})$, there exists $T>0$ and a unique solution of (4.2) in $C^0_t;\mathcal{H}^s(\mathbb{R})\cap L^8_{(0,T);\mathcal{W}^{s,4}(\mathbb{R})}$. Furthermore, the time existence T is bounded from below by $C\|u_0\|^{-\gamma}_{\mathcal{H}^s}$, $\gamma>0$.*

According to Lemma 3.3, we have

$$\begin{aligned} \||u|^6u\|_{\mathcal{H}^s} \leq C\|u\|_{\mathcal{W}^{s,4}}\|u^5\|_{L^4} + C\|u\|_{L^{24}}\||u|^6\|_{\mathcal{W}^{s,\frac{24}{5}}} \leq \cdots \\ \cdots \leq C\|u\|_{\mathcal{W}^{s,4}}\|u\|^6_{L^{24}} \leq C\|u\|^7_{\mathcal{W}^{s,4}} \end{aligned} \tag{4.3}$$

and

$$\||u|^6u-|v|^6v\|_{\mathcal{H}^s} \leq C\|u-v\|_{\mathcal{W}^{s,4}}(\|u\|^6_{\mathcal{W}^{s,4}}+\|v\|^6_{\mathcal{W}^{s,4}} \tag{4.4}$$

as soon as

$$\mathcal{W}^{s,4}\hookrightarrow L^{24}$$

(notice that according to Sobolev embeddings, this condition is satisfied if $\frac{5}{24}\leq s<1/4$. Let us now revisit the proof of Proposition 4.5. We are looking for a solution of the following fixed point:

$$u = e^{itH}u_0 + \int_0^t e^{i(t-s)H}(|u|^6u)(s)ds = K(u)$$

on the space $L^7((0,T);\mathcal{W}^{s,4}(\mathbb{R}))$. According (4.3), we have

$$\|K(u)\|_{L^8((0,T);\mathcal{W}^{s,4})} \leq C(\|u_0\|_{\mathcal{H}^s}+\||u|^6u\|_{L^1((0,T);\mathcal{H}^s)}) \leq C(\|u_0\|_{\mathcal{H}^s}+\|u\|^7_{L^7((0,T);\mathcal{W}^{s,4})})$$

and consequently, according to Holder inequality, there exists $\delta>0$ such that

$$\|K(u)\|_{L^7((0,T);\mathcal{W}^{s,4})} \leq CT^\delta(\|u_0\|_{\mathcal{H}^s}+\|u\|^7_{L^7((0,T);\mathcal{W}^{s,4})}).$$

As a consequence, if T is small enough, the map K maps the unit ball of $L^r((0,T);L^{2r})$ into itself. The same calculation shows

$$\|K(u)-K(v)\|_{L^7((0,T);\mathcal{W}^{s,4})} \leq CT^\delta\|u-v\|_{L^7((0,T);\mathcal{W}^{s,4}}(\|u\|^6_{L^7((0,T);\mathcal{W}^{s,4}}+\|v\|^6_{L^7((0,T);\mathcal{W}^{s,4})}).$$

and consequently the map K is a contraction on the unit ball of $L^7((0,T);\mathcal{W}^{s,4}$, which proves Proposition 4.6. Let us now notice the following refinement of Proposition 4.6 which will be usefull in the sequel: we do not actually need $u_0 \in \mathcal{H}^s$ but rather

$$e^{itH}u_0 \in L^8((0,2\pi);\mathcal{W}^{s,4}),$$

which we deduced from the assumption $u_0 \in \mathcal{H}^s$ and Strichartz estimates, but is clearly weaker.

Definition 4.7. *For $s \in \mathbb{R}$, let us denote by Y^s the space of temperate distributions on $\mathbb{R}$, such that*

$$\|u_0\|_{Y^s} = \|e^{itH}u_0\|_{L^8((0,2\pi);\mathcal{W}^{s,4})} < +\infty$$

Proposition 4.8. *There exists $s < \frac{1}{6}$ such that for any $u_0 \in Y^s$, if*

$$\|u_0\|_{Y^s} < R,$$

then there exists $T(R) > 0$ and a unique solution of (4.2) in $e^{itH}u_0 + C^0_t;\mathcal{H}^s(\mathbb{R}) \cap {L^4}_{loc,t};\mathcal{W}^{s,\infty}(\mathbb{R})$. Furthermore, the time existence T is bounded from below by $CR^{-\gamma}$, $\gamma > 0$, and we have

$$\|u(T)\|_{Y^s} \leq R + R^{-1}$$

Indeed, let us revisit the proof of Proposition 4.6: We now write $u = e^{itH}u_0 + v$ and are seeking for a solution v of

$$(i\partial_t + H)v = |e^{itH}u_0 + v|^6(e^{itH}u_0 + v), \qquad v\mid_{t=0} = 0$$

which we write as a fixed point

$$v = \int_0^t e^{i(t-s)H}(|e^{itH}u_0 + v|^6(e^{itH}u_0 + v)) = K(v)$$

and the assumption on $e^{itH}u_0$, and the same estimates as in the proof of Proposition 4.6) show that if $T \leq C\|e^{itH}u_0\|^{-\gamma}_{L^8((0,2\pi);\mathcal{W}^{s,4})}$, $\gamma > 0$, the map K has a unique fixed point in the unit ball of the space $L^7((0,T);\mathcal{W}^{s,4})$. The rest of the proof is easy. Notice that the last estimate on $\|u(T)\|_{Y^s}$ is obtained simply by possibly shrinking the existence time (i.e. taking γ larger).

4.2. **Probabilistic theory, definition of the measures.** Now write $c_n = a_n + ib_n$. For $N \geq 1$, consider the probability measure on $\mathbb{R}^{2(N+1)}$ defined by

$$\mathrm{d}\mu_N = d_N \prod_{n=0}^{N} e^{-\frac{\lambda_n^2}{2}(a_n^2+b_n^2)} da_n db_n,$$

where d_N is such that

$$\frac{1}{d_N} = \prod_{n=0}^{N} \int_{\mathbb{R}^2} e^{-\frac{\lambda_n^2}{2}(a_n^2+b_n^2)} da_n db_n = (2\pi)^{N+1} \prod_{n=0}^{N} \frac{1}{\lambda_n^2} = (2\pi)^{N+1} \prod_{n=0}^{N} \frac{1}{2n+1}\,.$$

The measure μ_N defines a measure on E_N via the map

$$(a_n, b_n)_{n=0}^N \longmapsto \sum_{n=0}^{N} (a_n + ib_n)e_n,$$

which will still be denoted by μ_N. Let $\sigma > 0$. Then (φ_N) is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}))$ which defines

$$\phi(\omega, x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x), \tag{4.5}$$

as the limit of (φ_N). Indeed, the map

$$\Phi : \omega \longmapsto \Phi(\omega) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x),$$

defines a (Gaussian) measure on $\mathcal{H}^{-\sigma}(\mathbb{R})$ which will be denoted by μ:

$$\mu(A) = \mathcal{P}(\Phi^{-1}(A))$$

Notice that μ_N may be seen as the distribution of the E_N valued random variable

$$\omega \longmapsto \sum_{n=0}^{N} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) e_n(x) \equiv \varphi_N(\omega, x), \tag{4.6}$$

where $(g_n)_{n=0}^N$ is a system of independent, centered, L^2 normalised complex gaussians. Notice also that the measure μ_N can be seen as a measure on $\mathcal{D}'(\mathbb{R})$ via the map

$$u \in \mathcal{D}'(\mathbb{R}) \mapsto \Pi_N(u) \in E_N.$$

Finally, for $\chi \in C_0^\infty(\mathbb{R})$, equal to 1 near 0, we define

$$\text{if } u = \sum_n u_n h_n(x), \qquad \begin{aligned} S_N u &= \chi\big(\frac{H}{N^2}\big) u = \sum_n \chi\Big(\frac{2n+1}{N^2}\Big) u_n h_n(x) \\ \Pi_n u &= 1_{\frac{H}{N^2} < 1} u = \sum_{n; 2n+1 < N^2} u_n h_n(x) \end{aligned}$$

Let us recall the following basic L^q uniform continuity property of the operators S_N.

Lemma 4.9. *For any $1 < q < +\infty$, there exists $C > 0$ such that for any $N \in \mathbb{R}^+$,*

$$\|S_N\|_{L^q(\mathbb{R}) \to L^q(\mathbb{R})} \leq C.$$

Proposition 4.10. *We define the Gibbs measures by*

$$d\rho(u) = \exp\Big(-\frac{1}{4}\|u\|^4_{L^4(\mathbb{R})}\Big) d\mu(u), \qquad d\rho_N(u) = \exp\Big(-\frac{1}{4}\|S_N(u)\|^4_{L^4(\mathbb{R})}\Big) d\mu_N(u),$$

and the measure is nontrivial. Moreover the sequence $d\rho_N$ converges weakly to $d\rho$ as N tends to infinity.

5. From local to global well posedness: Bourgain's argument

According to our previous results, we know that for any initial data in the support of the measures μ and ρ, there exists a nice local solution to the system (4.2). Our aim in this section is to prove that this solution is global in time. For this Bourgain's argument is the following: introduce first a family of approximate problems for which one knows that the solutions are global in time. Then prove estimates uniform with respect to the approximating parameter, and pass finally to the limit. Of course, in this program, the major step is the second one.

5.1. **The Galerkin approximation.** We consider the following finite dimensional approximation of (4.2)

$$(i\partial_t - H)u = S_N\big(|S_N u|^6 S_N u\big), \quad u(0,x) = \Pi_N(u(0,x)) \in E_N. \tag{5.1}$$

which is a finite dimensional approximation of the non linear Schrödinger equation. For $u \in E_N$, write

$$u = \sum_{n=0}^{N} c_n h_n,$$

then we have the following result.

Lemma 5.1. *The equation* (5.1) *is a Hamiltonian ODE, with Hamiltonian*

$$J(c_0, \overline{c_0}, \cdots, c_N, \overline{c_N}) = \frac{1}{2}\sum_{n=0}^{N} \lambda_n^2 |c_n|^2 + \frac{1}{8}\int_{-\infty}^{\infty} \Big| S_N\Big(\sum_{n=0}^{N} c_n h_n(x)\Big)\Big|^8 dx. \tag{5.2}$$

Moreover the mass

$$\|\Pi_N u\|^2_{L^2(\mathbb{R})} = \sum_{n=0}^{N} |c_n|^2, \tag{5.3}$$

is conserved under the flow of (5.1). As a consequence, (5.1) *has a well-defined global flow* Φ_N*.*

Proof. • First we prove (5.2). Multiply equation (5.1) with $\partial_t \Pi_N \overline{u}$ and integrate over $\mathbb{R}$. Then deduce that J defined by (5.2) is conserved by the equation. Then it is straightforward to check that the equation (5.1) can be rewritten in the Hamiltonian form

$$\frac{\mathrm{d}}{\mathrm{d}t} c_n = -i\frac{\partial J}{\partial \overline{c_n}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\overline{c_n} = i\frac{\partial J}{\partial c_n}, \quad 1 \le n \le N.$$

• We turn to the proof of (5.3). Multiply the equation (5.1) with $\Pi_N \overline{u}$ and integrate over $\mathbb{R}$.
First, by an integration by parts, we have

$$-\int_{\mathbb{R}} \Pi_N \overline{u}\, H u = \int_{\mathbb{R}} |H^{1/2}\Pi_N u|^2 \in \mathbb{R}. \tag{5.4}$$

Then, using the self adjointness of the operators S_N, we deduce that

$$\begin{aligned} \int_{\mathbb{R}} S_N\Big(|S_N u|^6 S_N u\Big)\Pi_N \overline{u} &= \int_{\mathbb{R}} S_N\Big(|S_N u|^6 S_N u\Big) S_N \Pi_N \overline{u} \\ &= \int_{\mathbb{R}} \Big(|S_N u|^6 S_N u\Big) S_N \overline{u} \in \mathbb{R}. \end{aligned} \tag{5.5}$$

Hence, from (5.4) and (5.5) we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi_N u\|^2_{L^2(\mathbb{R})} = 0.$$

□

Recall that $\Phi_N(t) : E_N \longrightarrow E_N$ denotes the flow of the ordinary differential equation (5.1). We now state an invariance result.

Proposition 5.2. *The measure ρ_N as defined in above is invariant under the flow Φ_N of* (5.1).

Proof. By Liouville's theorem, the measure $\mathrm{d}a\mathrm{d}b = \prod_{n=0}^N a_n b_n$ is invariant under Φ_N. Then, as the Hamiltonian $J = J(a_1, b_1, \dots, a_N, b_N)$ is conserved, the measure

$$e^J \prod_{n=0}^N \mathrm{d}a_n \mathrm{d}b_n = \frac{1}{d_N} \exp\Big[-\frac{1}{8} \int_{\mathbb{R}} \Big| S_N\Big(\sum_{n=0}^N (a_n + ib_n) h_n(x) \Big) \Big|^8 \Big] \mathrm{d}\mu_N,$$

is also invariant by the flow of (5.1). □

5.2. **Use of the invariant measure argument: a heuristic approach.** Before getting into details, let us now describe the heuristics of the strategy involved in proving the global existence result. The three main ingredients are

- A nice local Cauchy theory stating that for initial data of size smaller than R (measured in a suitable norm), the solution exists (and enjoys nice estimates for $t \in [0, T], T \sim CR^{-\gamma}$.
- A measure, ρ, on the space of initial data which is (at least formally) invariant by the flow of the equation, $\Phi(t)$, for any time $t \in \mathbb{R}$.
- Large deviations estimates on this measure which state that the set of initial data of size larger than λ has measure smaller than $e^{-c\lambda^2}$ for this invariant measure.

Let us fix a target time N, and prove that the solution exists for a set of full ρ measure on the time interval $[0, N]$. Let $m \in \mathbb{N}$. According to Proposition 4.8, the equation (4.2) has a nice solution on the time interval $[0, \frac{N}{m}]$, as soon as the size of the initial data is smaller than $C(N/m)^{1/\gamma}$ (in a appropriate norm). As a consequence, if we denote by E_1, the set of initial data for which we cannot ensure the existence of a nice solution on the time interval $[0, \frac{N}{m}]$, we have

$$\rho(E_1) \le Ce^{-c(\frac{N}{m})^{1/\gamma}}.$$

Now denote by E_2 the set of initial data for which we cannot ensure the existence of a nice solution on the time interval $[0, 2\frac{N}{m}]$. It is clear that if $u_0 \in E_2$ (i.e. we cannot solve up to time $2\frac{N}{m}$), then either $u_0 \in E_1$ or $u_0 \notin E_1$ but $\Phi(\frac{N}{m})u_0 \in E_1$ (i.e, we can solve up to time $\frac{N}{m}$ but after that lapse of time, the solution lies in E_1 and consequently we cannot solve on $[\frac{N}{m}, 2\frac{N}{m}]$). Sumarizing

$$E_2 \subset E_1 \cup \{u_0 \in E_1^c; \Phi(\frac{N}{m})(u_0) \in E_1\}$$

(where E_1^c denotes the complement of E_1). Since the measure ρ is invariant by the flow,

$$\rho(\{u_0 \in E_1^c; \Phi(\frac{N}{m})(u_0) \in E_1\}) \le \rho(\{u_0; \Phi(\frac{N}{m})(u_0) \in E_1\}) = \rho(E_1)$$

and consequently

$$\rho(E_2) \le 2\rho(E_1)$$

We can reiterate the procedure and obtain, if E_k is the set of initial data for which we cannot solve the equation up to time $k\frac{N}{m}$,

$$\rho(E_k) \leq k\rho(E_1) \leq kCe^{-c(\frac{N}{m})^{1/\gamma}}$$

Taking $k = m$, the set of initial data for which we cannot solve up to time N has ρ measure smaller than

$$Cme^{-c(\frac{N}{m})^{1/\gamma}}$$

Letting m tend to infinity, we see that this measure tends to 0 and consequently, we can solve (4.2) on the time interval $[0, N]$, on a set of full ρ measure, for all $N \in \mathbb{N}$.

Notice that there is an obvious flaw in the argument above. Indeed, to prove the existence of the flow, we assume that there exists an invariant measure ρ, but to show that the measure ρ is invariant, we need to prove first that the flow is defined...

As a consequence, to implement a rigorous argument, we are going to replace (4.2) by the family of approximating systems (5.1) for which the global existence of the flow is automatically ensured (and which enjoy invariant measures, ρ_N). Then we shall replace in the heuristics above the "nice" solutions of (4.2) by families of solutions for the approximate systems (5.1), enjoying uniform estimates with respect to the approximating parameter, apply the measure arguments to these families, and finally pass to the limit.

5.3. **Global existence.**

Proposition 5.3. *Fix s_0 as in Proposition 4.8. For all $i, N \in \mathbb{N}^*$, there exists a ρ_N measurable set $\Sigma_N^i \subset E_N$ so that*
For all $i, N \in \mathbb{N}^*$

$$\rho_N(E_N \backslash \Sigma_N^i) \leq 2^{-i}.$$

For all $\sigma > 0$, $f \in \Sigma_N^i$ and $t \in \mathbb{R}$

$$\|e^{-i\tau H}(\Phi_N(t)f)\|_{L^8_{\tau\in(0,2\pi)}L^4(\mathbb{R})} + \|\Phi_N(t)f\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq C\Big(i + \log(1 + |t|)\Big)^{\frac{1}{2}}.$$

Proof. The proof is inspired from the work by Bourgain [1, 2]. Here we set, for i, j integers ≥ 1,

$$B_N^{i,j}(D) \equiv \Big\{u \in E_N \,:\, \|e^{-itH}u\|_{L^8_{2\pi};\mathcal{W}^{s_0,4}(\mathbb{R})} + \|u\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq D(i + j)^{\frac{1}{2}}\Big\}.$$

where the number $D \gg 1$ (independent of i, j, N) will be fixed later. Thanks to Proposition 4.8 (notice that Proposition 4.8 was stated only for the system (4.2). However, using Lemma 4.9, it is easy to see that the same proof shows that the result holds, with uniform constants, for the family of systems (5.1)), there exist $c > 0$, $C > 0$, $\gamma > 0$ only depending on α such that if we set $\tau \equiv cD^{-\gamma}(i + j)^{-\gamma/2}$ then for every $t \in [-\tau, \tau]$,
(5.6)

$$\Phi_N(t)\big(B_N^{i,j}(D)\big) \subset \Big\{u \in E_N \,:\, \|e^{-itH}u\|_{L^8_{2\pi}L^4(\mathbb{R})} + \|u\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq D(i+j)^{\frac{1}{2}} + D^{-1}(i+j)^{-\frac{1}{2}}\Big\}.$$

Let us now define

$$\Sigma_N^{i,j}(D) \equiv \bigcap_{k=-[2^j/\tau]}^{[2^j/\tau]} \Phi_N(-k\tau)(B_N^{i,j}(D)),$$

where $[2^j/\tau]$ stays for the integer part of $2^j/\tau$. By Proposition 5.2, the measure ρ_N is invariant by the flow Φ_N. Hence

$$\begin{aligned}\rho_N\big(E_N\backslash\Sigma_N^{i,j}(D)\big) &\leq (2[2^j/\tau]+1)\rho_N\big(E_N\backslash B_N^{i,j}(D)\big)\\ &\leq C2^jD^\gamma(i+j)^{\gamma/2}\rho_N\big(E_N\backslash B_N^{i,j}(D)\big)\,.\end{aligned}$$

Now, by the large deviation bounds of Theorem 3.5,

$$\rho_N\big(E_N\backslash\Sigma_N^{i,j}(D)\big)\leq C2^jD^\gamma(i+j)^{\gamma/2}e^{-cD^2(i+j)}\leq 2^{-(i+j)}, \tag{5.7}$$

provided $D\gg 1$, independently of i,j,N. Thanks to (5.6), we obtain that for $u_0\in\sigma_N^{i,j}(D)$, the solution of (5.1) with data f satisfies

$$\begin{aligned}\big\|e^{-i\tau H}\big(\Phi_N(t)(f)\big)\big\|_{L^8((0,2\pi);\mathcal{W}^{s_0,4}(\mathbb{R}))}&+\|\Phi_N(t)(f)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})}\\ &\leq D(i+j)^{\frac12}+D^{-1}(i+j)^{-\frac12}\leq D(i+j+1)^{\frac12},\quad |t|\leq 2^j\end{aligned} \tag{5.8}$$

Indeed, for $|t|\leq 2^j$, we may find an integer $k\in[-[2^j/\tau],[2^j/\tau]]$ and $\tau_1\in[-\tau,\tau]$ so that $t=k\tau+\tau_1$ and thus $u(t)=\Phi_N(\tau_1)\big(\Phi_N(k\tau)(f)\big)$. Since $f\in\Sigma_N^{i,j}(D)$ implies that $\Phi_N(k\tau)(f)\in B_N^{i,j}(D)$, we may apply (5.6) and arrive at (5.8). Next, we set

$$\Sigma_N^i=\bigcap_{j=1}^{\infty}\Sigma_N^{i,j}(D)\,.$$

Thanks to (5.7), $\rho_N(E_N\backslash\Sigma_N^i)\leq 2^{-i}$. In addition, using (5.8), we get that there exists C such that for every i, every N, every $u_0\in\Sigma_N^i$, every $t\in\mathbb{R}$,

$$\big\|e^{-i\tau H}\big(\Phi_N(t)(f)\big)\big\|_{L^8(\tau\in(0,2\pi);L^4(\mathbb{R}))}+\|\Phi_N(t)(f)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})}\leq C(i+\log(1+|t|))^{\frac12}\,. \tag{5.9}$$

Indeed for $t\in\mathbb{R}$ there exists $j\in\mathbb{N}$ such that $2^{j-1}\leq 1+|t|\leq 2^j$ and we apply (5.8) with this j. This completes the proof of Proposition 5.3. □

For integers $i\geq 1$ and $N\geq 1$, we define the cylindrical sets

$$\tilde\Sigma_N^i\equiv\big\{u\in\mathcal{H}^{-\sigma}(\mathbb{R})\,:\,\Pi_N(u)\in\Sigma_N^i\big\}.$$

Next, for integers $i\geq 1$ and $M\geq 0$, we set

$$\begin{aligned}\Sigma^i=\Big\{u\in\mathcal{H}^{-\sigma}(\mathbb{R})\,:\exists\,N_k\to+\infty,&\\ N_k\in\mathbb{N},\ \exists\,u_{N_k}\in\tilde\Sigma_{N_k}^i,&\,u_{N_k}\to u\text{ in }\mathcal{H}^{-\sigma}(\mathbb{R})\Big\}\end{aligned}$$

Observe that $\Sigma^i(M)$ is a closed subset of $\mathcal{H}^{-\sigma}(\mathbb{R})$, and that we have the following inclusions

$$\limsup_{N\to+\infty}\tilde\Sigma_N^i=\bigcap_{N=1}^{\infty}\bigcup_{N_1=N}^{\infty}\tilde\Sigma_{N_1}^i\subset\Sigma^i.$$

Therefore

$$\rho\big(\Sigma^i\big)\geq\rho\big(\limsup_{N\to+\infty}\tilde\Sigma_N^i\big). \tag{5.10}$$

Using Fatou's lemma, we get

$$\rho(\limsup_{N\to\infty}\tilde{\Sigma}_N^i) \geq \limsup_{N\to\infty}\rho(\tilde{\Sigma}_N^i)\,. \tag{5.11}$$

Let $G_N(u) = e^{-\frac{1}{4}\|S_N(u)\|_{L^4(\mathbb{R})}}$ and $G(u) = e^{-\frac{1}{4}\|u\|_{L^4(\mathbb{R})}}$. We have that

$$\rho(\tilde{\Sigma}_N^i) = \int_{\tilde{\Sigma}_N^i} G(u)\mathrm{d}\mu(u),$$

and

$$\rho_N(\Sigma_N^i) = \int_{\Sigma_N^i} G_N(u)\mathrm{d}\mu_N(u) = \int_{\tilde{\Sigma}_N^i} G_N(u)\mathrm{d}\mu(u).$$

Therefore, we get

$$\lim_{N\to\infty}\big((\rho(\tilde{\Sigma}_N^i) - \rho_N(\Sigma_N^i)\big) = 0,$$

and we obtain

$$\limsup_{N\to\infty}\rho(\tilde{\Sigma}_N^i) = \limsup_{N\to\infty}\rho_N(\Sigma_N^i) \geq \limsup_{N\to\infty}\big(\rho_N(E_N) - 2^{-i}\big) = \rho\big(\mathcal{H}^{-\sigma}(\mathbb{R})\big) - 2^{-i}. \tag{5.12}$$

Collecting (5.10), (5.11) and (5.12), we obtain that

$$\rho\big(\Sigma^i\big) \geq \rho\big(\mathcal{H}^{-\sigma}(\mathbb{R})\big) - 2^{-i}. \tag{5.13}$$

Now, we set

$$\Sigma^i \equiv \bigcup_{M=0}^{\infty}\Sigma^i(M)\,,$$

and

$$\Sigma \equiv \bigcup_{i=1}^{\infty}\Sigma^i, \qquad \widetilde{\Sigma} \equiv \bigcup_{i=1}^{\infty}\Sigma^i(0), \tag{5.14}$$

Then, by (5.13), the sets Σ and $\widetilde{\Sigma}$ are of full ρ (or μ) measure. It turns out that one has global existence for an initial condition $f \in \Sigma$ (resp. $f \in \widetilde{\Sigma}$).

Proposition 5.4. *Let $\sigma > 0$, $i \in \mathbb{N}^*$ and $M \in \mathbb{N}$. Then the local solution of (4.2) with initial condition $f \in \Sigma^i(M)$ is globally defined. Moreover, there exists $C > 0$ so that for every $f \in \Sigma^i(M)$*

$$\|e^{-i\tau H}(u(t))\|_{L^8_{\tau\in(0,2\pi)}L^4(\mathbb{R})} + \|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq C\Big(i + \log(1 + M + |t|)\Big)^{\frac{1}{2}}.$$

Moreover, if $(f_k)_{k\geq 0} \in \Sigma^i_{N_k}$, $N_k \to +\infty$ are so that

$$f_k \longrightarrow f \quad \textit{in} \quad \mathcal{H}^{-\sigma}(\mathbb{R}), \quad \textit{when} \quad k \to +\infty,$$

then

$$\|u(t) - \Phi_{N_k}(t)(f_k)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \longrightarrow 0, \quad \textit{when} \quad k \to +\infty.$$

Indeed, to prove this result, one simply use that having uniform estimates on the solutions to the approximating system, and knowing that the initial data converge (in a very weak sense, i.e. for the $\mathcal{H}^{-\sigma}$ topology), is essentially enough to obtain that the solution of approximating system converge in a weak sense and (passing to the limit in the estimate) that the solution of the full system (4.2) enjoys the same estimates.

Appendix A. Tutorial: Proof of Christ-Kiselev Lemma

Lemma A.1. *Consider two Banach spaces X, Y, an interval $I \subset \mathbb{R}$, $1 \leq p < q \leq +\infty$, and $K : L^p(I;X) \to L^q(I;Y)$ a bounded linear operator with distribution kernel $k(t,s)$ is given by a function locally integrable with values bounded operators from X to Y, $k \in L^1_{loc}((I \times I); \mathcal{L}(X,Y))$. Consider the operator R whose distribution kernel is given by $r(t,s) = k(t,s)1_{s<t}$. Then the operator R is bounded from $L^p(I;X)$ to $L^q(I;Y)$, with norm*

$$\|R\|_{\mathcal{L}(L^p(I;X);L^q(I;Y))} \leq \frac{1}{1 - 2^{\frac{1}{q}-\frac{1}{p}}} \|K\|_{\mathcal{L}(L^p(I;X);L^q(I;Y))}. \tag{A.1}$$

A.1. **Preliminaries.**

(1) By considering $K_n = 1_{|t|<n} K 1_{|s|<n}$, show that we can assume that the interval I is bounded.
(2) Show that if $E \subset L^q(I;Y)$ is a dense subset, it is enough to prove

$$\forall f \in E, \qquad \|Rf\|_{L^q(I;Y)} \leq \frac{\|K\|_{\mathcal{L}(L^p(I;X);L^q(I;Y))}}{1 - 2^{\frac{1}{q}-\frac{1}{p}}} \|f\|_{L^p(I;X)}. \tag{A.2}$$

(3) Show that the set

$$E = \{f \in L^q(I;Y); \|f\|_Y(s) > 0 \text{ for almost all } s \in I\}$$

is dense in $L^q(I;Y)$.
(4) Finally, show that we can reduce the study to the case $\|f\|_{L^p(I;X)} = 1$

A.2. **Dyadic partition.** In this section we consider $f \in E$, $\|f\|_{L^p(I;X)} = 1$.

(1) Show that the map

$$F : s \mapsto \int_0^s \|f(s)\|_Y^p ds$$

defines a bijection $I \to (0,1)$
(2) Consider for $n \in \mathbb{N} \setminus \{0\}$ and $j = 0, \cdots, 2^n - 2$, the sets

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}[, \qquad J_{n,j} = [(j+1)2^{-n}, (j+2)2^{-n}[.$$

Show that

$$\{(x,y) \in (0,1)^2; y < x\} = \cup_{n=1}^{+\infty} \cup_{j=0}^{2^n-2} J_{n,j} \times I_{n,j}$$

(draw a picture).

(3) Define the functions

$$f_{n,j}(s) = 1_{F(s)\in J_{n,j}}, \qquad g_{n,j}(t) = 1_{F(t)\in I_{n,j}}.$$

Show that

$$1_{s<t} = \sum_{n=1}^{+\infty}\sum_{j=0}^{2^n-2} g_{n,j}(t) f_{n,j}(s).$$

(4) Deduce that

$$R(f) = \sum_{n=1}^{+\infty}\sum_{j=0}^{2^n-2} g_{n,j}(t) f_{n,j}(s).$$

A.3. **Conclusion.**

(1) Compute

$$\|f_{n,j}\|^p_{L^p(I;X)}.$$

(2) Show that for any $n \in \mathbb{N}$,

$$\|\sum_{j=0}^{2^n-2} g_{n,j}(t)\|_{L^q(I;Y)} \leq \Big(\sum_n \|g_{n,j}(t)\|^q_{L^q(I;Y)}\Big)^{1/q}.$$

(3) By noticing that

$$g_{n,j} = 1_{F(t)\in I_{n,j}} R 1_{F(s)\in J_{n,j}},$$

deduce that for any $n \in \mathbb{N}$

$$\|\sum_{j=0}^{2^n-2} g_{n,j}(t)\|_{L^q(I;Y)} \leq \|R\|_{L^p(I;X)\to L^q(I;Y)}\Big(\sum_{j=0}^{2^n-2} 2^{-nq/p}\Big)^{1/q} \leq \|R\|_{L^p(I;X)\to L^q(I;Y)} 2^{\frac{n}{q}-\frac{n}{p}}. \tag{A.3}$$

(4) Conclude

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Université Paris Sud, Mathématiques, Bât 425, 91405 Orsay Cedex et Institut universitaire de France

E-mail address: Nicolas.burq@math.u-psud.fr