# The Lax-Oleinik semi-group: a Hamiltonian point of view.

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The Weak KAM theory was developped by Fathi in order to study the dynamics of convex Hamiltonian systems. It somehow makes a bridge between viscosity solutions of the Hamilton-Jacobi equation and Mather invariant sets of Hamiltonian systems, although this was fully understood only a posteriori. These theories converge under the hypothesis of convexity, and the richness of applications mostly comes from this remarkable convergence. In the present course, we provide an elementary exposition of some of the basic concepts of weak KAM theory. In a companion lecture, Albert Fathi exposes the aspects of his theory which are more directly related to viscosity solutions. Here on the contrary, we focus on dynamical applications, even if we also discuss some viscosity aspects to underline the connections with Fathi's lecture. The fundamental reference on Weak KAM theory is the still unpublished book of Albert Fathi *Weak KAM theorem in Lagrangian dynamics*. Although we do not offer new results, our exposition is original in several aspects. We only work with the Hamiltonian and do not rely on the Lagrangian, even if some proofs are directly inspired from the classical Lagrangian proofs. This approach is made easier by the choice of a somewhat specific setting. We work on $\mathbb{R}^d$ and make uniform hypotheses on the Hamiltonian. This allows us to replace some compactness arguments by explicit estimates. For the most interesting dynamical applications however, the compactness of the configuration space remains a useful hypothesis and we retrieve it by considering periodic (in space) Hamiltonians. Our exposition is centered on the Cauchy problem for the Hamilton-Jacobi equation and the Lax-Oleinik evolution operators associated to it. Dynamical applications are reached by considering fixed points of these evolution operators, the Weak KAM solutions. The evolution operators can also be used for their regularizing properties, this opens a second way to dynamical applications.

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# 1 The method of characteristics, existence and uniqueness of regular solutions.

We consider a $C^2$ Hamiltonian

$$H(t,q,p) : \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d)^* \longrightarrow \mathbb{R}$$

and study the associated Hamiltonian system

$$\dot{q}(t) = \partial_p H(t, q(t), p(t)) \quad , \quad \dot{p}(t) = -\partial_q H(t, q(t), p(t)) \tag{HS}$$

and Hamilton-Jacobi equation

$$\partial_t u + H(t, q, \partial_q u(t,q)) = 0. \tag{HJ}$$

We denote by $X_H(x) = X_H(q,p)$ the Hamiltonian vector field $X_H = JdH$, where $J$ is the matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The Hamiltonian system can be written in condensed terms $\dot{x}(t) = X_H(t, x(t))$. We will always assume that that the solutions extend to $\mathbb{R}$. We denote by

$$\varphi_\tau^t = (Q_\tau^t, P_\tau^t) : T^*\mathbb{R}^d \longrightarrow T^*\mathbb{R}^d$$

the flow map which, to a point $x \in T^*\mathbb{R}^d$, associate the value at time $t$ of the solution $x(s)$ of (HS) which satisfies $x(\tau) = x$.

If $u(t,q)$ solves (HJ), and if $q(s)$ is a curve in $\mathbb{R}^d$, then the formula

$$u(t_1, q(t_1)) - u(t_0, q(t_0)) = \int_{t_0}^{t_1} \partial_q u(s, q(s)) \cdot \dot{q}(s)) - H(s, \partial_q u(s, q(s)))ds \tag{1}$$

follows from an obvious computation. The integral on the right hand side is the **Hamiltonian action** of the curve $s \longmapsto (q(s), \partial_q u(s, q(s)))$. The **Hamiltonian action** of the curve $(q(s), p(s))$ on the interval $[t_0, t_1]$ is the quantity

$$\int_{t_0}^{t_1} p(s) \cdot \dot{q}(s) - H(s, q(s), p(s))ds.$$

A classical and important property of the Hamiltonian actions is that orbits are critical points of this functional. More precisely, we have:

**Proposition 1.** *The $C^1$ curve $(q(t), p(t)) : [t_0, t_1] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ is a Hamiltonian trajectory if and only if the equality*

$$\frac{d}{ds}\Big|_{s=0} \left( \int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t, q(t,s), p(t,s))dt \right) = 0,$$

*where the dot is the derivative with respect to $t$, holds for each $C^1$ variation $(q(t,s), p(t,s)) : [t_0, t_1] \times \mathbb{R} \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ satisfying $q(t,0) = q(t)$ for each $t$, $(q(t_0,s), p(t_0,s)) = (q(t_0), p(t_0))$ and $(q(t_1,s), p(t_1,s)) = (q(t_1), p(t_1))$ for each $s$.*

PROOF. We set $\theta(t) = dq/ds|_{s=0}$, $\zeta(t) = dp/ds|_{s=0}$ and compute:

$$
\begin{aligned}
&\frac{d}{ds}\Big|_{s=0}\left(\int_{t_0}^{t_1} p(t,s)\dot q(t,s) - H(t,q(t,s),p(t,s))dt\right)\\
&= \int_{t_0}^{t_1} p(t)\dot\theta(t) + \zeta(t)\dot q(t) - \partial_q H(t,q(t),p(t))\theta(t) - \partial_p H(t,q(t),p(t))\zeta(t)dt\\
&= p(t_1)\theta(t_1) - p(t_0)\theta(t_0) + \int_{t_0}^{t_1}\big(\dot q(t) - \partial_p H(t,q(t),p(t))\big)\zeta(t)dt\\
&- \int_{t_0}^{t_1}\big(\dot p(t) + \partial_q H(t,q(t),p(t))\big)\theta(t)dt.
\end{aligned}
$$

The result then follows from the classical fact that the integral $\int_{t_0}^{t_1} f(t)\theta(t)dt$ vanishes for each continuous function $\theta(t)$ null on the boundaries if and only if the function $f$ (assumed continuous) is identically null. □

Let us state our first connection between (HS) and (HJ).

**Theorem 1.** *Let $\Omega \subset \mathbb{R}\times\mathbb{R}^d$ be an open set, and let $u(t,q) : \Omega \longrightarrow \mathbb{R}$ be a $C^2$ solution of the Hamilton Jacobi equation (HJ). Let $q(t) : [t_0,t_1] \longrightarrow \mathbb{R}^d$ be a $C^1$ curve such that $(t,q(t)) \in \Omega$ and*

$$\dot q(t) = \partial_p H\big(q(t), \partial_q u(t,q(t))\big)$$

*for each $t \in [t_0,t_1]$. Then, setting $p(t) = \partial_q u(t,q(t))$, the curve $(q(t),p(t))$ solves (HS).*

The curves $q(t)$ satisfying the hypothesis of the Theorem, as well as the associated trajectories $(q(t),p(t))$ are called the **characteristics** of $u$.

PROOF. Let $q(t,s)$ be a $C^1$ variation of $q(t)$ which fixes the end points, which means that $q(t_i,s) = q(t_i)$ for $i=0$ and $i=1$, and that $q(t,0) = q(t)$ for each $t \in [t_0,t_1]$. Let us define $p(t,s)$ by

$$p(t,s) := \partial_q u(t,q(t,s))).$$

Our hypothesis is that $\dot q(t) = \partial_p H(t,q(t),p(t))$, which is the first part of (HS). For each $s$, we have

$$u(t_1,q(t_1)) - u(t_0,q(t_0)) = \int_{t_0}^{t_1} p(t,s)\cdot\dot q(t,s) - H(t,q(t,s),p(t,s))ds.$$

Setting $\theta(t) = dq/ds|_{s=0}$ and $\zeta(t) = dp/ds|_{s=0}$ as above, we deduce that

$$
\begin{aligned}
0 &= \frac{d}{ds}\Big|_{s=0}\left(\int_{t_0}^{t_1} p(t,s)\dot q(t,s) - H(t,q(t,s),p(t,s))dt\right)\\
&= \int_{t_0}^{t_1}\big(\dot q(t) - \partial_p H(t,q(t),p(t))\big)\zeta(t)dt - \int_{t_0}^{t_1}\big(\dot p(t) + \partial_q H(t,q(t),p(t))\big)\theta(t)dt\\
&= -\int_{t_0}^{t_1}\big(\dot p(t) + \partial_q H(t,q(t),p(t))\big)\theta(t)dt.
\end{aligned}
$$

Since this equalilty holds for each $\theta$ vanishing on the endpoints, we deduce that

$$\dot p(t) + \partial_q H(t,q(t),p(t)) = 0,$$

hence both equations (HS) hold. □

The following restatement of Theorem 1 has a more geometric flavor:

**Theorem 2.** *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t,q) : \Omega \longrightarrow \mathbb{R}$ be a $C^2$ solution of the Hamilton Jacobi equation (HJ). Then the extended Hamiltonian vectorfield $Y_H = (1, X_H)$ is tangent to the graph*

$$G := \{(t, q, \partial_q u) : (t,q) \in \Omega\}.$$

The following corollary is espacially important:

**Corollary 2.** *Let $u(t,q)$ be a $C^2$ solution of (HJ) defined on the open set $\Omega = ]t_0, t_1[ \times \mathbb{R}^d$. Then, for each $s$ and $t$ in $]t_0, t_1[$ we have*

$$\Gamma_t = \varphi_s^t(\Gamma_s),$$

*where $\Gamma_t$ is defined by*

$$\Gamma_t := \{(q, du_t(q)) : q \in \mathbb{R}^d\}.$$

Proof. Although Theorem 2 is a restatement of Theorem 1, we give another, more direct, proof. We first differentiate the Hamilton-Jacobi equation with respect to $q$:

$$\begin{aligned} 0 &= \partial^2_{tq} u(t,q) + \partial_q H(t, q, \partial_q u(t,q)) + \partial_p H(t, q, \partial_q u(t,q)) \circ \partial^2_{qq} u(t,q) \\ &= \partial^2_{tq} u(t,q) + \partial_q H(t, q, \partial_q u(t,q)) + \partial^2_{qq} u(t,q) \cdot \partial_p H(t, q, \partial_q u(t,q)) \end{aligned}$$

In the first line, we see $\partial_p H$ as a linear map from $(\mathbb{R}^d)^*$ to $\mathbb{R}$ and $\partial^2_{qq} u$ as a linear map from $\mathbb{R}^d$ to $(\mathbb{R}^d)^*$ so that the composition $\partial_p H \circ \partial^2_{qq} u$ is a linear form on $\mathbb{R}^d$. In the second expression, we see $\partial_p H$ as an element of $\mathbb{R}^d$, and $\partial^2_{qq} u \cdot \partial_p H$ is the linear form obtained by applying the map $\partial^2_{qq} u$ to the vector $\partial_p H$. The equality between these expressions follows from the symmetry of $\partial^2_{qq} u$ as shown by the following calculation, where $v \in \mathbb{R}^d$:

$$\partial_p H \circ \partial^2_{qq} u(v) = \partial_p H(\partial^2_{qq} u(v)) = \partial^2_{qq} u(v)(\partial_p H) = \partial^2_{qq} u(\partial_p H)(v) = \partial^2_{qq} u \cdot \partial_p H(v).$$

The sceptical reader should write everything explicitely in coordinates to check these formula. We have proved that

$$-\partial_q H(t, q, \partial_q u(t,q)) = \partial^2_{tq} u(t,q) + \partial^2_{qq} u(t,q) \cdot \partial_p H(t, q, \partial_q u(t,q)),$$

which implies that the extended Hamiltonian vectorfield

$$Y(t, q, \partial_q u(t,q)) = \big(1, \partial_p H(t, q, \partial_q u(t,q)), -\partial_q H(t, q, \partial_q u(t,q))\big)$$

belongs to the tangent space of $G$ at point $(t, q, \partial_q u(t,q))$, which is the graph of the linear map

$$(\tau, \xi) \longmapsto \partial^2_{tq} u(t,q) \cdot \tau + \partial^2_{qq} u(t,q) \cdot \xi.$$

□

Let us consider the projection $\pi : (t, q, p) \longmapsto (t, q)$. It is useful to lift the function $u$ to the surface $G$ by defining $w = u \circ \pi$. On the surface $G$, we have

$$dw = pdq - Hdt. \tag{2}$$

We say that the function $w$ is a primitive of the canonical form $pdq - Hdt$. This formula is essentially equivalent to (1).

Let us now consider an initial condition $u_0(q)$ and study the Cauchy problem consisting of finding a $C^2$ solution $u(t,q)$ of (HJ) such that $u(0,q) = u_0(q)$. If such a solution $u(t,q)$ exists on the open set $]t_0, t_1[ \times \mathbb{R}^d$ (with $0 \in ]t_0, t_1[$), then the equality

$$\Gamma_t = \varphi_0^t(\Gamma_0)$$

determines $u_t$ up to a constant (which may depend on $t$). We conclude

**Proposition 3.** *Given a time interval* $]t_0, t_1[$ *an initial time* $\tau \in ]t_0, t_1[$ *and a* $C^2$ *initial condition* $u_\tau$*, there is at most one solution* $u(t,q) : ]t_0, t_1[ \times \mathbb{R}^d$ *of (HJ) such that* $u(\tau, q) = u_\tau(q)$ *for all* $q \in \mathbb{R}^d$*.*

Proof. If $u$ and $v$ are two solutions of this Cauchy problem, then $\partial_q u = \partial_q v$ hence there exists a $C^2$ function $f(t)$ such that $u(t,q) = v(t,q) + f(t)$. Plugging this expression in (HJ) gives that $f'(t) = 0$ for each $t$, while the initial condition at $t = \tau$ implies $f(\tau) = 0$. We conlude that $f = 0$ on $]t_0, t_1[$. □

In order to discuss the existence problem, we first define the set

$$G := \bigcup_{t \in ]t_0, t_1[} \{t\} \times \varphi_0^t(\Gamma_0) \tag{G}$$

and, denoting by $\dot{Q}_t^s(x)$ the derivative with respect to $s$, the function

$$\begin{aligned} w : \quad & G \longrightarrow \mathbb{R} \\ & (t,x) \longmapsto u_\tau(Q_t^\tau(x)) + \int_\tau^t P_t^s(x) \dot{Q}_t^s(x) - H(s, \varphi_t^s(x)) ds. \end{aligned} \tag{w}$$

More generally, we say that $(\Gamma_0, w_0)$ is a geometric initial condition if $\Gamma_0$ is a $C^1$ submanifold of $\mathbb{R}^d \times \mathbb{R}^{d*}$ of dimension $d$ and if $w_0$ is a $C^1$ function on $\Gamma_0$ such that the relation $dw_0 = pdq$ holds on $\Gamma_0$. A geometric initial condition can be associated to a genuine initial condition $u_0$ by setting $\Gamma_0 = \{(q, du_0), q \in \mathbb{R}^d\}$ and $w_0 = u_0 \circ \pi$. The other example of geometric initial condition that will be relevant in the sequel is $\Gamma_0 = \{q\} \times \mathbb{R}^{d*}$ (for any fixed $q$) and $w_0 \equiv 0$.

**Proposition 4.** *Let* $(\Gamma_0, w_0)$ *be a geometric initial condition. Then, the function* $w$ *defined by (w) satisfies* $dw = pdq - Hdt$ *on* $G$ *(as defined in (G)).*

The pair $(G, w)$ is the geometric solution associated to the geometric initial condition $(\Gamma_0, w_0)$.

Proof. Let $(T(\epsilon), \theta(\epsilon), \zeta(\epsilon))$ be a $C^1$ curve on $G$. We have to prove that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \big(w(T(\epsilon), \theta(\epsilon), \zeta(\epsilon))\big) = \zeta(0) \left.\frac{d\theta}{d\epsilon}\right|_{\epsilon=0} - H(T(0), \theta(0), \zeta(0)) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0}.$$

Let us set $Q(s,\epsilon) = Q^s_{T(\epsilon)}(\theta(\epsilon), \zeta(\epsilon))$ and $P(s,\epsilon) = P^s_{T(\epsilon)}(\theta(\epsilon), \zeta(\epsilon))$. We have

$$w(T(\epsilon), \theta(\epsilon), \zeta(\epsilon)) = w_0(Q(0,\epsilon), P(0,\epsilon)) + \int_0^{T(\epsilon)} P(s,\epsilon)\dot{Q}(s,\epsilon) - H(s, Q(s,\epsilon), P(s,\epsilon)) ds.$$

Since $dw_0 = pdq$ on $\Gamma_0$, the calculations in the proof of Proposition 1 imply that

$$\begin{aligned} &\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \big(w(T(\epsilon), \theta(\epsilon), \zeta(\epsilon))\big) \\ =& P(0,0)\partial_\epsilon Q(0,0) + \Big(P(T(0),0)\dot{Q}(T(0),0) - H(T(0), Q(T(0),0), P(T(0),0))\Big) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0} \\ &+ P(T(0),0)\partial_\epsilon Q(T(0),0) - P(0,0)\partial_\epsilon Q(0,0) \\ =& \zeta(0) \left(\partial_\epsilon Q(T(0),0) + \dot{Q}(T(0),0) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0}\right) + H(T(0), \theta(0), \zeta(0)) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0}. \end{aligned}$$

The desired equality follows from the observation that

$$\left.\frac{d\theta}{d\epsilon}\right|_{\epsilon=0} = \partial_\epsilon Q(T(0),0) + \dot{Q}(T(0),0) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0}.$$

□

**Corollary 5.** *If there exists a $C^1$ map $\chi : \Omega \longrightarrow \mathbb{R}^{d*}$ on some open subset $\Omega$ of $]t_0, t_1[\times \mathbb{R}^d$ such that $(t, q, \chi(t, q)) \subset G$ for all $(t, q) \in \Omega$, then the function*

$$u(t, q) := w(t, q, \chi(t, q))$$

*solves (HJ) on $\Omega$.*

Proof. The relation $dw = pdq - Hdt$ is equivalent to the equations

$$\chi(t, q) = \partial_q u \quad , \quad \partial_t u + H(t, q, \chi(t, q)) = 0.$$

□

The following hypothesis, which it is stronger than is really necessary, will allow us to rest on simple estimates in this course.

**Hypothesis 1.** *There exists a constant $M$ such that*

$$\|d^2 H(t, q, p)\| \leqslant M$$

*for each $(t, q, p)$.*

This hypothesis implies that the Hamiltonian vectorfield is Lipschitz, hence that the Hamiltonian flow is complete. The hypothesis can be exploited further to estimate the differential

$$d\varphi_0^t = \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}$$

using the variational equation

$$\begin{bmatrix} \partial_q \dot{Q}_0^t(x) & \partial_p \dot{Q}_0^t(x) \\ \partial_q \dot{P}_0^t(x) & \partial_p \dot{P}_0^t(x) \end{bmatrix} = \begin{bmatrix} \partial_{qp} H(t, x) & \partial_{pp} H(t, x) \\ -\partial_{qp} H(t, x) & -\partial_{pp} H(t, x) \end{bmatrix} \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}.$$

We obtain the following estimates:

$$\|d\varphi_\tau^t - I\| \leqslant M|t| \tag{M}$$

or componentwise

$$\|\partial_q Q_0^t - I\| \leqslant M|t| \quad , \quad \|\partial_p P_0^t - I\| \leqslant M|t| \quad , \quad \|\partial_q P_0^t\| \leqslant M|t| \quad , \quad \|\partial_p Q_0^t\| \leqslant M|t|.$$

We can now prove:

**Theorem 3.** *Let $H : \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d)^*$ be a $C^2$ Hamiltonian satisfying Hypothesis 1. Let $u_0$ be an initial condition which is $C^2$ with bounded second differential. There exists a time $T > 0$ and a $C^2$ solution $u(t, q) : ]-T, T[\times \mathbb{R}^d \longrightarrow \mathbb{R}$ of (HJ) such that $u(0, q) = u_0(q)$. Moreover, one can take*

$$T = \Big(2M(1 + \|d^2 u_0\|_\infty)\Big)^{-1},$$

*and we have*

$$\|d^2 u_t\|_\infty \leqslant \|d^2 u\|_\infty + 5tM(1 + \|d^2 u\|_\infty)$$

*when $|t| \leqslant T$, with a constant $C$ which depends only on $M$.*

PROOF. It is enough to prove that the map

$$F(t,q) := (t, Q_0^t(q, du(q)))$$

is a $C^1$ diffeomorphism on $]-T,T[\times\mathbb{R}^d$. Indeed, denoting by $(t, G(t,q))$ the inverse diffeomorphism, we can set

$$\chi(t,q) = P_0^t\big(G(t,q), du(G(t,q))\big),$$

and observe that the geometric solution is the graph of $\chi$. We have

$$dF(t,q) = \big(1, \partial_q Q_0^t(q, du(q)) + \partial_p Q_0^t(q, du(q)) \cdot d^2u(q)\big).$$

By the estimates (M), we have

$$\|dF(t,q) - Id\| \leqslant tM(1 + |d^2u_0(q)|),$$

so that $\|dF(t,q) - Id\| \leqslant 1/2$ provided $|t| \leqslant \Big(2M(1+\|d^2u_0\|_\infty)\Big)^{-1}$. We conclude using the classical Proposition 45 of the Appendix that $F$ is a diffeomorphism. Given a point $q$, let us set $(q_0, p_0) = (G(t,q), du(G(q,q)))$. Since $\Gamma_{u_t} = \varphi_0^t(\Gamma_u)$, we have,

$$d^2u_t(q) = \big(\partial_q P_0^t(q_0,p_0) + \partial_p P_0^t(q_0,p_0)\cdot d^2u(q_0)\big)\big(\partial_q Q_0^t(q_0,p_0) + \partial_p Q_0^t(q_0,p_0)\cdot d^2u(q_0)\big)^{-1}.$$

We now estimate

$$|\partial_q Q_0^t(q_0,p_0) + \partial_p Q_0^t(q_0,p_0)\cdot d^2u(q_0) - I| \leqslant tM(1 + |d^2u(q_0)|)$$

hence

$$\big|\big(\partial_q Q_0^t(q_0,p_0) + \partial_p Q_0^t(q_0,p_0)\cdot d^2u(q_0)\big)^{-1} - I\big) \leqslant 2tM(1 + |d^2u(q_0)|)$$

provided $|t| \leqslant \Big(2M(1+\|d^2u\|_\infty)\Big)^{-1}$. The first factor can be estimated as follows

$$\big|\partial_q P_0^t(q_0,p_0) + \partial_p P_0^t(q_0,p_0)\cdot d^2u(q_0) - d^2u(q_0)\big| \leqslant tM(1 + |d^2u(q_0)|).$$

We conclude that

$$\big|d^2u_t(q) - d^2u(q_0)\big| \leqslant 5tM(1 + |d^2u(q_0)|)$$

provided $|t| \leqslant \Big(2M(1+\|d^2u\|_\infty)\Big)^{-1}$. □

### 1.1 Exercise :

Take $d = 1$, $H(t,q,p) = (1/2)p^2$, and $u_0(q) = -q^2$, and prove that the $C^2$ solution can't be extended beyond $t = 1/2$.

## 2 Convexity, the twist property, and the generating function.

We make an additional assumption on $H$. Once again, we make the assumption in a stronger form than what would be necessary, this allows to obtain the simplest statements:

**Hypothesis 2.** *There exists $m > 0$ such that*

$$\partial^2_{pp} H \geqslant mId$$

*for each $(t,q,p)$, in the sense of quadratic forms.*

Let us first study the consequences of this hypothesis on the structure of the flow.

**Proposition 6.** *There exists $\sigma > 0$ such that the map $p \longmapsto Q_0^t(q,p)$ is $(mt/2)$-monotone when $t \in ]0,\sigma]$, in the sense that the inequality*

$$(Q_0^t(q,p') - Q_0^t(q,p)) \cdot (p'-p) \geqslant mt|p'-p|^2/2$$

*holds for each $q \in \mathbb{R}^d$, each $t \in [0,\sigma]$. As a consequence, it is a $C^1$ diffeomorphism onto $\mathbb{R}^d$.*

We say that the flow has the Twist property.
Proof. Fix a point $q$ and denoite by $F^t$ the map $p \longmapsto Q_0^t(q,p)$. We have $dF^t(p) = \partial_p Q_0^t(q,p)$. In order to estimate this linear map, we recall the variational equation

$$\partial_p \dot{Q}_0^t(x) = \partial_{qp} H(t, \varphi_0^t(x)) \partial_p Q_0^t(x) + \partial_{pp} H(t, \varphi_0^t(x)) \partial_p P_0^t(x).$$

We deduce that

$$\partial_p \dot{Q}_0^t(x) - \partial_{pp}^2 H(t, \varphi_0^t(x)) = \partial_{qp}^2 H(t, \varphi_0^t(x)) \partial_p Q_0^t(x) + \partial_{pp}^2 H(t, \varphi_0^t(x))(\partial_p P_0^t(x) - Id)$$

and then that

$$\|\partial_p \dot{Q}_0^t(x) - \partial_{pp}^2 H(t, \varphi_0^t(x))\| \leqslant 2M^2 t$$

As a consequence, for $t \leqslant \sigma = m/(4M^2)$, we have

$$\partial_p \dot{Q}_0^t \geqslant (m - 2M^2 t) I \geqslant (m/2) I$$

in the sense of quadratic forms (note that the matrix $\partial_p \dot{Q}_0^t$ is not necessarily symmetric). Since

$$\partial_p Q_0^t(x) = \int_0^t \partial_p \dot{Q}_0^s(x) ds,$$

we conclude that

$$dF^t(p) = \partial_p Q_0^t(q,p) \geqslant (m/2) Id,$$

which means that $(dF^t(p)z, z) \geqslant (m/2)|z|^2$ for each $z \in \mathbb{R}^{d*}$. This estimate can be integrated, and implies the monotony of the map $F^t$:

$$\begin{aligned}
(Q^t(q,p') - Q^t(q,p)) \cdot (p'-p) &= \left( \int_0^1 \partial_p Q^t(q, p + s(p'-p)) \cdot (p'-p) ds \right) \cdot (p'-p) \\
&= \int_0^1 \big( \partial_p Q^t(q, p + s(p'-p)) \cdot (p'-p) \big) \, ds \\
&\geqslant \int_0^1 (m/2) t (p'-p) \cdot (p'-p) ds \geqslant (m/2) t (p'-p) \cdot (p'-p).
\end{aligned}$$

It is then a classical result that the map $F^t$ is a $C^1$ diffeomorphism, see Proposition 46 in the appendix.
□

**Corollary 7.** *The map $(t,q,p) \longmapsto (t,q,Q_0^t(q,p))$ is a $C^1$ diffeomorphism from $]0,\sigma[ \times \mathbb{R}^d \times \mathbb{R}^{d*}$ onto its image $]0,\sigma[ \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof. We have already proved that this map is a bijection. In order to prove that it is a diffeomorphism, it is enough to observe that its differential $(\tau, \theta, \xi) \longmapsto (\tau, \theta, \partial_p Q_0^t(\xi))$ is an invertible linear map.
□
We denote by $\rho_0(t, q_0, q_1)$ the unique momentum $p$ such that $Q_0^t(t, q_0, \rho_0(t, q_0, q_1)) = q_1$. In other

words, $\rho_0(t,q_0,q_1)$ is the initial momentum $p(0)$ of the unique orbit $(q(s),p(s)):[0,t]\longrightarrow \mathbb{R}^d\times\mathbb{R}^{d*}$ of (HS) which satisfies $q(0)=q_0$ and $q(t)=q_1$. By the Corollary we just proved, the map $\rho_0$ is $C^1$. We denote by $S^t(q_0,q_1)$ the action of the unique trajectory $(q(s),p(s)):[0,t]\longrightarrow\mathbb{R}^d\times\mathbb{R}^{d*}$ of (HS) which satisfies $q(0)=q_0$ and $q(t)=q_1$:

$$S^t(q_0,q_1)=\int_0^t P_0^s(q_0,\rho_0(t,q_0,q_1))\dot{Q}_0^s(q_0,\rho_0(t,q_0,q_1))-H(s,\varphi_0^s(q_0,\rho_0(t,q_0,q_1))ds.$$

The function $(t,q_0,q_1)\longmapsto S^t(q_0,q_1)$ is $C^1$. This function is closely related to the Hamilton-Jacobi equation. More precisely, if we consider the "geometric" initial condition ($\Gamma_0=\{q_0\}\times\mathbb{R}^{d*},w_0=0$), and the associated geometric solution $(G,w)$, then Proposition 6 implies that $G$ is a $C^1$ graph above $]0,\sigma[\times\mathbb{R}^d$, and that

$$S^t(q_0,q)=w\circ\left(\pi_{|G}\right)^{-1}(t,q).$$

Proposition 4 and its corollary then imply that the function $(t,q)\longmapsto S^t(q_0,q)$ is a solution of (HJ). It is the genuine solution associated to the generalized initial condition $(\Gamma_0,w_0)$. Finally, if we denote by $\rho_1(t,q_0,q_1)$ the final value $p(t)$ of the momentum of the unique trajectory $(q(s),p(s)):[0,t]\longrightarrow\mathbb{R}^d\times\mathbb{R}^{d*}$ of (HS) which satisfies $q(0)=q_0$ and $q(t)=q_1$, then we see that

$$G=\{(t,q,\rho_1(t,q_0,q))),\quad (t,q)\in]0,\sigma[\times\mathbb{R}^d\}$$

and that

$$\rho_1(t,q_0,q_1)=\partial_1 S^t(q_0,q_1),$$

where $\partial_1$ is the differential with respect to the second variable.

We have defined the function $S^t(q_0,q_1)$ as the action of the unique orbit joining $q_0$ and $q_1$ between time 0 and $t$. We can define similarly the function $S^t_\tau(q_0,q_1)$ as the action of the unique orbit joining $q_0$ to $q_1$ between time $\tau$ and time $t$, all this being well-defined provided $0<t-\tau<\sigma$. It is possible to prove as above that the function $(s,q)\longmapsto S_s^t(q,q_1)$ solves the Hamilton-Jacobi equation

$$\partial_s u+H(t,q,-\partial_q u)=0,$$

on $s<t$, and that

$$\partial_0 S^t(q_0,q_1)=\partial_0 S_0^t(q_0,q_1)=-\rho_0(t,q_0,q_1).$$

We have proved the relation

$$\varphi_0^t(q_0,-\partial_0 S(q_0,q_1))=(q_1,\partial_1 S^t(q_0,q_1)).$$

We say that $S^t$ is a **generating function** of the flow map $\varphi_0^t$. See [13], chapter 9, for more material on generating functions. The relations $\partial_0 S=-\rho_0$, $\partial_1 S=\rho_1$, $\partial_t S=-H(t,q_1,\rho_1)=-H(0,q_0,\rho_0)$ that we have proved imply that the function $S$ is actually $C^2$. It is useful to estimate its second differentials:

**Lemma 8.** *The function $S$ is $C^2$ on $]0,\sigma[\times\mathbb{R}^d\times\mathbb{R}^d$, and the estimates*

$$\partial^2_{00}S^t\geqslant\frac{c}{t}I\quad,\quad\partial^2_{11}S^t\geqslant\frac{c}{t}I$$
$$\|\partial^2_{00}S^t\|+\|\partial^2_{01}S^t\|+\|\partial^2_{01}S^t\|\leqslant\frac{C}{t}$$

*hold, with constants $c$ and $C$ which depend only on $m$ and $M$.*

Proof. Let us first observe that

$$\partial^2_{11} S^t(q_0,q_1) = \big(\partial_p P_0^t(q_0,\rho_0(t,q_0,q_1))\big)\big(\partial_p Q_0^t(q_0,\rho_0(t,q_0,q_1))\big)^{-1},$$

and recall the estimates:

$$\|\partial_p P_0^t - Id\| \leqslant Mt, \quad \|\partial_p Q_0^t\| \leqslant Mt, \quad \partial_p Q_0^t \geqslant (mt/2)Id.$$

We conclude that (see Lemma 47)

$$(\partial_p Q_0^t)^{-1} \geqslant \frac{m}{2M^2 t} Id \quad , \quad \|(\partial_p Q_0^t)^{-1}\| \leqslant 2/(mt).$$

Finally, we obtain that

$$\partial^2_{11} S(q_0,q_1) \geqslant \left(\frac{m}{2M^2 t} - \frac{2M}{m}\right) Id \geqslant \frac{m}{4M^2 t} Id$$

provided $t \leqslant m^2/(8M^3)$. The other estimates can be proved similarly, using the expressions

$$\begin{aligned}
\partial^2_{00} S^t(q_0,q_1) &= -\big(\partial_p P_t^0(q_1,\rho_1(t,q_0,q_1))\big)\big(\partial_p Q_t^0(q_1,\rho_1(t,q_0,q_1))\big)^{-1},\\
\partial^2_{10} S^t(q_0,q_1) &= \big(\partial_p Q_0^t(q_0,\rho_0(t,q_0,p_0))\big)^{-1}.
\end{aligned}$$

□

We have defined the function $S^t(q_0,q_1)$ as the action of the unique orbit joining $q_0$ and $q_1$ between time 0 and $t$. We can define similarly the function $S^t_\tau(q_0,q_1)$ as the action of the unique orbit joining $q_0$ to $q_1$ between time $\tau$ and time $t$, all this being well-defined provided $0 < t - \tau < \sigma$.

**Proposition 9.** *We have the triangle inequality*

$$S_0^t(q_0,q_t) + S_t^\tau(q_t,q_\tau) \geqslant S_0^\tau(q_0,q_\tau).$$

*Moreover,* $S_0^\tau(q_0,q_\tau) = \min_q \big(S_0^t(q_0,q) + S_t^\tau(q,q_\tau)\big)$.

Proof. Let us consider the map

$$q \longmapsto f(q) = S_0^t(q_0,q) + S_t^\tau(q,q_\tau).$$

We have $d^2 f \geqslant 2c$ hence the map $f$ is convex. Now let us denote by $(q(s),p(s)) : [0,\tau] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ the unique orbits which satisfies $q(0) = q_0$ and $q(\tau) = q_\tau$. We can compute

$$df(q(s)) = \partial_1 S_0^t(q_0,q(s)) + \partial_0 S_t^\tau(q(s),q_\tau) = p(s) - p(s) = 0.$$

The point $q(s)$ is thus a critical point of the convex function $f$, hence it is a minimum of this function. We conclude that

$$S_0^t(q_0,q) + S_t^\tau(q,q_\tau) \geqslant S_0^t(q_0,q(s)) + S_t^\tau(q(s),q_\tau) = S_0^\tau(q_0,q_\tau)$$

for all $q$. □

Under the convexity hypothesis 2, Theorem 1 can be extended to $C^1$ solutions:

**Theorem 4.** *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t,q) : \Omega \longrightarrow \mathbb{R}$ be a $C^1$ solution of the Hamilton Jacobi equation (HJ). Let $q(t) : [t_0,t_1] \longrightarrow \mathbb{R}^d$ be a $C^1$ curve such that $(t,q(t)) \in \Omega$ and*

$$\dot q(t) = \partial_p H\big(q(t), \partial_q u(t,q(t))\big)$$

*for each $t \in [t_0,t_1]$. Then, setting $p(t) = \partial_q u(t,q(t))$, the curve $(q(t),p(t))$ solves (HS).*

PROOF. As in the proof of Theorem 1, we consider a variation $q(t,s) = q(t) + s\theta(t)$ of $q(t)$, where $\theta$ is smooth and vanishes on the endpoints. We choose the vertical variation in such a way that the equation

$$\dot{q}(t,s) = \partial_p H(t, q(t,s), p(t,s))$$

holds. The map $p(t,s)$ defined by this relation is differentiable in $s$, because $q$ and $\dot{q}$ are. It is also useful to consider

$$P(t,s) := \partial_q u(t, q(t,s))).$$

Our hypothesis is that $\dot{q}(t) = \partial_p H(t, q(t), p(t))$, which is the first part of (HS). We can conclude that the second equation holds as in the proof of Theorem 1, provided we establish that

$$\frac{d}{ds}\Big|_{s=0} \left( \int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t, q(t,s), p(t,s))dt \right) = 0.$$

We deduce this equality from the observation that $s = 0$ is a local minimum of the function

$$s \longmapsto F(s) := \int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t, q(t,s), p(t,s))dt.$$

This claim follows from the equality

$$F(0) = u(t_1, q(t_1)) - u(t_0, q(t_0)) = \int_{t_0}^{t_1} P(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), P(t,s))ds,$$

which holds for all small $s$, and from the inequality

$$F(s) \geqslant \int_{t_0}^{t_1} P(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), P(t,s))ds$$

which results, in view of the convexity of $H$, from the computation

$$\begin{aligned} H(t, q(t,s), P(t,s)) &\geqslant (P(t,s) - p(t,s)) \cdot \partial_p H(t, q(t,s), p(t,s)) + H(t, q(t,s), p(t,s)) \\ &\geqslant (P(t,s) - p(t,s)) \cdot \dot{q}(t,s) + H(t, q(t,s), p(t,s)). \end{aligned}$$

□

A last property of the functions $S$ will be useful. Assume that we are considering a family $H_\mu, \mu \in I$ of Hamiltonians, where $I \subset \mathbb{R}$ is an interval, such that the whole function $H(\mu, t, q, p)$ is $C^2$ and such that each of the Hamiltonians $H_\mu$ satisfy our hypotheses 1 and 2, with uniform constants $m$ and $M$. Then, for each value of $\mu$, we have the function $S^t(\mu; q_0, q_1)$, which is defined for $t \in ]0, \sigma]$, the bound $\sigma > 0$ being independent of $\mu$. Since everything we have done so far was based on the local inversion theorem, the function $S^t(\mu; q_0, q_1)$ is $C^1$ in $\mu$, or more precisely the function $(\mu, t, q_0, q_1) \longmapsto S^t(\mu; q_0, q_1)$ is $C^1$. Moreover, a computation similar to the proof of Proposition 1 yields

$$\partial_\mu S^t(\mu; q_0, q_1) = -\int_0^t \partial_\mu H_\mu(s, q(\mu, s), p(\mu, s))ds,$$

where $s \longmapsto (q(\mu, s), p(\mu, s))$ is the only $H_\mu$-trajectory satisfying $q(\mu, 0) = q_0$ and $q(\mu, t) = q_1$. We can exploit this remark when $H_\mu$ is the linear interpolation $H_\mu = H_0 + \mu(H_1 - H_0)$ between two Hamiltonians $H_0$ and $H_1$, and conclude the important monotony property:

$$H_0 \leqslant H_1 \quad \Rightarrow \quad S^t(0; q, q') \geqslant S^t(1; q, q'). \qquad \text{(Monotone)}$$

### 2.1 Exercise :

If $H(t,q,p)=h(p)$ is a function of $p$, then

$$S^t(q_0,q_1)=th^*\left(\frac{q_1-q_0}{t}\right),$$

where $h^*$ is the Legendre transform of $h$. As an example, when $H(t,q,p)=a|p|^2/2$, then

$$S^t(q_0,q_1)=\frac{1}{2ta}|q_1-q_0|^2.$$

## 3 Extension of the generating function: The minimal action.

We now assume, in addition to Hypotheses 1 and 2, that the Hamiltonian satisfies:

**Hypothesis 3.**

$$\frac{m}{2}|p|^2-M\leqslant H(t,q,p)\leqslant\frac{M}{2}|p|^2+M$$

By exploiting the monotony property (Monotone), we conclude immediately that

$$\frac{1}{2tM}|q_1-q_0|^2-Mt\leqslant S^t(q_0,q_1)\leqslant\frac{1}{2tm}|q_1-q_0|^2+Mt.$$

Th following estimate will also be useful:

**Lemma 10.**

$$p\cdot\partial_pH(t,q,p)-H(t,q,p)\geqslant\frac{m}{M}H(t,q,p)-(m+M).$$

PROOF. We deduce from Hypothesis 2 that

$$H(t,q,0)\geqslant H(t,q,p)-p\cdot\partial_pH(t,q,p)+\frac{m}{2}|p|^2.$$

We deduce that

$$p\cdot\partial_pH(t,q,p)-H(t,q,p)\geqslant\frac{m}{2}|p|^2-H(t,q,0)\geqslant\frac{m}{M}(H(t,q,p)-M)-M$$

□

It is useful to extend the function $S$ to all $t>0$. We can do this by setting

$$A^t(q_0,q_1)=\min_{\theta_1,\theta_2,\ldots\theta_{n-1}}\left(S_0^{t/n}(q_0,\theta_1)+S_{t/n}^{2t/n}(\theta_1,\theta_2)+S_{(n-1)t/n}^{t}(\theta_{n-1},q_1)\right)\tag{A}$$

where $n$ is any integer such that $t/n<\sigma$. It is an easy consequence of our estimates on $S$ that this minimum is achieved.

**Lemma 11.** *The minimum is well defined in the definition of $A^t$, and its value does not depend on $n$ provided $t/n<\sigma$. Moreover, we have*

$$\frac{1}{2Mt}|q_1-q_0|^2-Mt\leqslant A^t(q_0,q_1)\leqslant\frac{1}{2tm}|q_1-q_0|^2+Mt.$$

Proof. Since we have not yet proved the independance of $n$, we temporarily denote by $A^t_n(q_0, q_1)$ the value of the minimum. We have

$$S^{t/n}(q_0, \theta_1) + \cdots + S^{t/n}(\theta_{n-1}, q_1) \geqslant \frac{n}{2Mt}\big(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + \cdots + |q_1 - \theta_{n-1}|^2\big) - Mt$$

hence the minimum is achieved. Moreover

$$\begin{aligned} A^t_n(q_0, q_1) &\geqslant \min_{\theta_1,\theta_2,\ldots\theta_{n-1}} \Big(\frac{n}{2Mt}\big(|\theta_1 - q_0|^2 + \cdots + |q_1 - \theta_{n-1}|^2\big) - Mt\Big) \\ &= \frac{1}{2Mt}|q_1 - q_0|^2 - Mt. \end{aligned}$$

If $t < \sigma$, then the equality $S^t(q_0, q_1) = A^t_n(q_0, q_1)$ can be proved by recurrence for each $n$ using Proposition 9. For general $t$, let us prove that $A^t_n$ is independent of $n$. We take two integers $n$ and $m$ such that $t/n < \sigma, t/m < \sigma$ and want to prove that $A^t_n = A^t_m$. We will prove that $A^t_n = A^t_{nm} = A^t_m$. Since $t/m < \sigma$, we have

$$A^{t/m}_n(q_0, q_1) = S^{t/n}(q_0, q_1)$$

$$\begin{aligned} A^t_{nm}(q_0, q_1) = & \min_{\theta_1,\theta_2,\ldots\theta_{nm-1}} \big[S^{t/nm}(q_0, \theta_1) + S^{t/nm}(\theta_1, \theta_2) + S^{t/nm}(\theta_{n-1}, \theta_n) + \\ & S^{t/nm}(\theta_n, \theta_{n+1}) + S^{t/nm}(\theta_{n+1}, \theta_{n+2}) + S^{t/nm}(\theta_{2n-1}, \theta_{2n}) + \\ & \cdots + \\ & S^{t/nm}(\theta_{(m-1)n}, \theta_{(m-1)n+1}) + S^{t/nm}(\theta_{(m-1)n+1}, \theta_{(m-1)n+2}) + S^{t/nm}(\theta_{mn-1}, q_1)\big] \\ = & \min_{\theta_{2n},\theta_{3n},\ldots,\theta_{(m-1)n}} \big[S^{t/m}(q_0, \theta_n) + S^{t/m}(\theta_n, \theta_{2n}) + \cdots + S^{t/m}(\theta_{(m-1)n}, q_1)\big] \\ = & A^t_m(q_0, q_1). \end{aligned}$$

We have proved that $A^t_{nm} = A^t_m$, by symmetry we also have $A^t_{nm} = A^t_n$ hence $A^t_n = A^t_m$. Finally, we have

$$S^{t/n}(q_0, \theta_1) + S^{t/n}(\theta_1, \theta_2) + \cdots + S^{t/n}(\theta_{n-1}, q_1) \leqslant \frac{n}{2mt}\big(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2\big) + Mt$$

hence

$$A^t_n(q_0, q_1) \leqslant \min_{\theta_1,\theta_2,\ldots\theta_{n-1}} \frac{n}{2mt}\big(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2\big) + Mt = \frac{1}{2mt}|q_1 - q_0|^2 + Mt.$$

□

The following remark is called the method of broken geodesics, it appeared in Riemannian geometry, and was first used in the context of Hamiltonian dynamics by Marc Chaperon, see [6]. It allows to prove the existence of various kinds of orbit of the Hamiltonian using finite dimensional critical point theory.

**Lemma 12.** *If $(\Theta_1, \cdots, \Theta_{n-1})$ is a critical point of the function*

$$(\theta_1, \ldots, \theta_{n-1}) \longmapsto S^{t/n}_0(q_0, \theta_1) + S^{2t/n}_{t/n}(\theta_1, \theta_2) + \cdots + S^t_{(n-1)t/n}(\theta_{n-1}, q_1),$$

*then there exists a unique orbit $(q(s), p(s)) : [0, t] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(0) = q_0$, $q(t) = q_1$, and $q(it/n) = \Theta_i$ for $i = 1, \ldots, n-1$. The action of this orbit is $S^{t/n}(q_0, \Theta_1) + S^{t/n}_0(\Theta_1, \Theta_2) + \cdots + S^t_{(n-1)t/n}(\Theta_{n-1}, q_1)$.*

Proof. Let $(q(s), p(s))$ be the piecewise orbit defined on $[it/n, (i+1)t/n]$ by the constraints $q(it/n) = \Theta_i$ and $q((i+1)t/n) = \Theta_{i+1}$. The action of this piecewise orbit is $S^{t/n}(q_0, \Theta_1) + S_0^{t/n}(\Theta_1, \Theta_2) + \cdots + S^t_{(n-1)t/n}(\Theta_{n-1}, q_1)$. We have to prove that, if $(\Theta_1, \ldots, \Theta_{n-1})$ is a critical point, then this piecewise orbit is actually an orbit, or in other words that $p^-(it/n) = p^+(it/n)$. At a critical point, we have

$$0 = \partial_1 S^{t/n}(\Theta_{i-1}, \Theta_i) + \partial_0 S^{t/n}(\Theta_i, \Theta_{i+1}) = p^-(it/n) - p^+(it/n)$$

as desired. □

Since the function in the statement of the lemma is coercive, it has a minimum hence there exists an orbit joining points $q_0$ and $q_1$ in time $t$. Note however that, although the function

$$(\theta_1, \cdots, \theta_{n-1}) \longmapsto S_0^{t/n}(q_0, \theta_1) + S_{t/n}^{2t/n}(\theta_1, \theta_2) + \cdots + S^t_{(n-1)t/n}(\theta_{n-1}, q_1)$$

is convex separately in each of its variable, it is not necessarily jointly convex when $n > 2$. As a consequence, it can have several critical points. This reflects the possible existence of more than one Hamiltonian orbit joining $q_0$ to $q_1$ in time $t$ when $t \geqslant 2\sigma$. The Function $A^t(q_0, q_1)$ is then defined as the minimum of the actions of such orbits. There may exist more than one minimal orbit.

The minimal action $A^t(q_0, q_1)$ is not necessarily $C^1$, we need some definitions before we can study its regularity. The linear form $l$ is called a $K$-superdifferential of the function $u$ at point $q$ if the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds in a neighborhood of $q$. The linear form $l$ is a proximal superdifferential of $u$ at point $q$ if it is a $K$-superdifferential for some $K$. The form $l$ is a proximal superdifferential of $u$ at $q$ if and only if there exists a $C^2$ function $v$ such that $dv(q) = l$ and such that the difference $v - u$ has a minimum at $q$. More generally, we will say that $l$ is a superdifferential of $u$ at $q$ if there exists a $C^1$ function $v$ such that $dv(q) = l$ and such that the difference $v - u$ has a minimum at $q$. A superdifferential is not necessarily a proximal superdifferential.

A function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ is called $K$-semi-concave if it admits a $K$-superdifferential at each point. It is equivalent to require that the function $\theta \longmapsto u(\theta) + K|\theta|^2$ is convex. A function is called semi-concave if it is $K$-semi-concave for some $K$. If $u$ is a $K$-semi-concave function, and if $l$ is a superdifferential at $u$, then the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds for each $\theta$. In particular, $l$ is a $K$-superdifferential.

**Lemma 13.** *The function $A_0^t$ is $C(1 + 1/t)$-semi-concave, with some constant $C$ which depends only on $m$ and $M$.*

Proof. Let us first assume that $t \in ]0, \sigma[$. In this case, $A_0^t = S_0^t$, this function is $C^2$ and its second derivative was estimated in Lemma 8. Let us now assume that $t \geqslant \sigma$. Then, there exists $n \in \mathbb{N}$ such that $t/n \in [\sigma/3, \sigma/2[$. We have

$$A_0^t(q, q') = \min_{\theta, \theta'} \big(S_0^{t/n}(q, \theta) + A_{t/n}^{t-t/n}(\theta, \theta') + S^t_{t-t/n}(\theta', q')\big).$$

Considering a minimizing pair $(\theta_0, \theta_1)$ in the expression above at $(q_0, q_1)$, we see that the $C^2$ function

$$(q, q') \longmapsto S_0^{t/n}(q, \theta_0) + A_{t/n}^{t-t/n}(\theta_0, \theta_1) + S^t_{t-t/n}(\theta_1, q')$$

is touching from above the function $A_0^t$ at point $(q_0, q_1)$. In view of Lemma 8, this provides a uniform (for $t \geqslant \sigma$) semi-concavity constant for $A_0^t$. □

## 4 The Lax-Oleinik operators.

We define the Lax-Oleinik operator $T^t$ which, to each function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ associates the function

$$T^t u(q) := \min_{\theta \in \mathbb{R}^d} \big(u(\theta) + A^t(\theta, q)\big).$$

More generally, for each $\tau \in \mathbb{R}$ and $t \geqslant \tau$, we define the analogous operator $T^t_\tau$ where the time origin is $\tau$ instead of 0. We have the Markov (or semi-group) property:

$$T^s_t \circ T^t_\tau = T^s_\tau.$$

At some occasions, we will need to use the dual operators

$$\check{T}^t u(q) := \max_{\theta \in \mathbb{R}^d} \big(u(\theta) - A^t(\theta, q)\big).$$

Each property concerning the Lax-Oleinik operator has a counterpart for the dual operator, that we will not bother to state, but not hesitate to use. The Lax-Oleinik operators solve (HJ) in various important ways. Let us first establish the relation with regular solutions.

**Proposition 14.** *Let $u(t,x) : ]t_0, t_1[ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $C^1$ solution of HJ, then $T^t_\tau u_\tau = u_t$ for each $\tau \leqslant t$ in $]t_0, t_1[$.*

This property is one of the main motivations to introduce the Lax-Oleinik operators.
Proof. It is enough to prove the result for $0 < t - \tau < \sigma$. Given $q$ and $\theta$ in $\mathbb{T}^d$, we consider the unique orbit $(q(s), p(s))$ such that $q(\tau) = \theta$ and $q(t) = q$. By the convexity of $H$, we have

$$H(q(s), \partial_q u(s, q(s))) \geqslant H(q(s), p(s)) + (\partial_q u(s, q(s)) - p(s)) \cdot \partial_p H(s, q(s), p(s)).$$

Noticing that $\dot{q}(s) = \partial_p H(s, q(s), p(s))$ and integrating gives:

$$\begin{aligned} S^t_\tau(\theta, q) &= \int_\tau^t p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds \\ &\geqslant \int_\tau^t \partial_q u(s, q(s)) \cdot \dot{q}(s) - H(s, q(s), \partial_q u(s, q(s))) ds \\ &= u(t, q) - u(\tau, \theta), \end{aligned}$$

with equality if $p(s) = \partial_q u(s, q(s))$ for each $s$. We conclude that

$$T^t_\tau u_\tau(q) \geqslant u_t(q),$$

with equality if there exists an orbit $(q(s), p(s)) : [\tau, t] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $p(s) = \partial_q u(s, q(s))$ and $q(t) = q$. In order to prove that such an orbit exists, we consider the continuous vectorfield $V(s, q) = \partial_p H(s, q, \partial_q u(s, q))$. By the Cauchy-Peano Theorem, there exists a solution $q(s)$ of the equation $\dot{q}(s) = V(s, q(s))$, defined in a neighborhood of $s = t$, and satisfying $q(t) = q$. By Theorem 4, this solution lifts to a Hamiltonian trajectory $(q(s), p(s))$, where $p(s) = \partial_q u(s, q)$. By the completeness of the Hamiltonian flow, this solution can be extended to the whole interval $]t_0, t_1[$. We obtain that $u(t, q) = u(\tau, q(\tau)) + S^t_\tau(q(\tau), q)$, and therefore that $T^t_\tau u_\tau(q) = u(t, q)$. □

The Lax-Oleinik operators solve the Cauchy problem for (HJ) in the viscosity sense:

**Proposition 15.** *For each Lipschitz initial condition $u_0$, the function*

$$(t, q) \longmapsto u(t, q) = T^t_0 u_0(q)$$

*is a viscosity solution of (HJ) on $]0, \infty) \times \mathbb{R}^d$ (hence a solution almost everywhere).*

PROOF. We consider a point $(T, Q)$, set $s = \min(T, \sigma/2)$ and $\tau = T - s$, and let $\Theta$ be a minimizer in the expression

$$u(T, Q) = \min_{\theta} \big(u_\tau(\theta) + S_\tau^T(\theta, q)\big).$$

We have

$$u(t, q) \leqslant u_\tau(\Theta) + S_\tau^t(\Theta, q)$$

for all $(t, q)$ with $t \in [\tau, \tau + \sigma[$ hence the $C^2$ function $(t, q) \longmapsto u_\tau(\Theta) + S_\tau^t(\Theta, q)$ is touching from above the function $u$ at point $(T, Q)$. We conclude that

$$\big(-H(T, Q, \partial_1 S_\tau^T(\Theta, Q)), \partial_1 S_\tau^T(\Theta, Q)\big)$$

is a superdifferential of the function $u$ at point $(T, Q)$. If the function $u$ is differentiable at point $(T, Q)$, we must have

$$\partial_t u(T, Q) = -H(T, Q, \partial_1 S_\tau^T(\Theta, Q)) \quad , \quad \partial_q u(T, Q) = \partial_1 S_\tau^T(\Theta, Q)$$

which implies that

$$\partial_t u(T, Q) + H(T, Q, \partial_Q u(T, Q)) = 0.$$

We have proved that the function $u$ solves (HJ) at all its points of differentiability. Let us now prove that it is a viscosity supersolution. We consider a subdifferential $(h, p)$ of $u$ at $(T, Q)$ (if no subdifferential exist, then there is nothing to prove). Since the function $u$ has a superdifferential and a subdifferential at $(T, Q)$, it is differentiable at this point, with $(\partial_t u, \partial_q u) = (h, p)$. We conclude that $h + H(T, Q, p) = 0$, hence that $h + H(T, Q, p) \geqslant 0$, as desired.

It is slightly more difficult to prove that $u$ is also a viscosity subsolution. We could invoque here the general fact, proved in Fathi's joined lecture, that almost everywhere subsolutions of the Hamilton-Jacobi equation are viscosity sub-solutions when the Hamiltonian is convex in the momentum. We rather present an independent proof. Let $v(t, q) : ]\tau, \tau + \sigma[ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $C^2$ function touching $u$ from above at $(T, Q)$. Let $w_T$ be a $C^2$ function, with bounded second differential, such that $w_T = u_T$ near $Q$. Let $w$ be the $C^2$ solution of (HJ) satisfying $w(T, q) = w_T$, defined on some small time interval containing $T$. The existence of such a solution follows from Theorem 3. Considering the characteristic

$$s \longmapsto Q(s) := Q_T^s(Q, \partial_q w(T, Q))$$

through $(T, Q)$, we have

$$w(T, Q) = w(s, Q(s)) + S_s^T(Q(s), Q),$$

for $s \leqslant T$ while, on the other hand,

$$u(T, Q) \leqslant u(s, Q(s)) + S_s^T(Q(s), Q).$$

We conclude that

$$w(s, Q(s)) \leqslant u(s, Q(s)) \leqslant v(s, Q(s))$$

for $s \leqslant T$. Since $w_T = v_T$ near $Q$, this implies that that $\partial_t w(T, Q) \geqslant \partial_t v(T, Q)$. Recalling that $w$ solves (HJ), we conclude that $\partial_t v(T, Q) + H(T, Q, \partial_q v(T, Q)) \leqslant 0$, as desired. □

The Lax-Oleinik operators moreover solve the Cauchy problem for (HJ) in the following geometric sense, where we denote by $\Gamma_u$ the graph of the differential of $u$ on its domain of definition,

$$\Gamma_u := \{(q, du(q)) : \quad q \in \mathbb{R}^d, \quad du(q) \text{ exists}\}.$$

**Proposition 16.** *Let $u$ be a semi-concave and Lipschitz function. The set*

$$\varphi_t^0\left(\bar{\Gamma}_{T_0^t u}\right)$$

*is contained in $\Gamma_u$ for each $t > 0$, and it is a Lipschitz graph.*

Proof. Let $(q,p)$ be a point of $\Gamma_{T_0^t u}$, which means that the function $T_0^t u$ is differentiable at $q$ and that $d(T_0^t u)(q) = p$. Let $\Theta$ be a minimizing point in the expression $T_0^t u(q) = \min_\theta u(\theta) + A_0^t(\theta, q)$. Since each of the functions $u$ and $A_0^t(., q)$ are semiconcave, this implies that they are both differentiable at $\Theta$, and that $du(\Theta) + \partial_0 A_0^t(\Theta, q) = 0$. Moreover, this implies that the function $u(\Theta) + A_0^t(\Theta, .)$ touches the function $T_0^t u$ from above at point $q$, hence that $A_0^t(\Theta, .)$ is differentiable at $q$, with a differential equal to $p$. We then have

$$\varphi_t^0(q,p) = \varphi_t^0(q, \partial_1 A_0^t(\Theta, q)) = (\Theta, -\partial_0 A_0^t(\Theta, q) = (\Theta, du(\Theta)) \subset \Gamma_u.$$

We have proved that $\varphi_t^0(\Gamma_{T_0^t u}) \subset \Gamma_u$. Moreover, we have $Q_t^0(\Gamma_{T_0^t u}) \subset \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}^d$ is the set of points $\theta$ which are minimizing in the definition of $T_0^t u(q)$ for some point $q$.

We claim that $u$ is $C^{1,1}$ on $\mathcal{I}$, meaning that $u$ is differentiable at each point of $\mathcal{I}$, and that the map $\theta \longmapsto du(\theta)$ is Lipschitz on $\mathcal{I}$. In other words, the projection of $\Gamma_u$ to $\mathbb{R}^d$ contains $\mathcal{I}$, and the set

$$\Gamma_{u|\mathcal{I}} := \{(\theta, du(\theta), \theta \in \mathcal{I}\}$$

is a Lipschitz graph.

We prove the claim by observing that $u$ has $C$-superdifferentials and $C$-subdifferentials at each point of $\mathcal{I}$, where $C$ is the common semiconcavity constants of all the functions $A_0^t(., q)$ and of the function $u$. The claim then follows from a result of Fathi, see Proposition 48 in the Appendix.

Let now $(q,p)$ be a point in the closure $\bar{\Gamma}_{T_0^t u}$ of $\Gamma_{T_0^t u}$. There exists a sequence $(q_n, p_n)$ of points of $\Gamma_{T_0^t u}$ which converges to $(q,p)$. By definition, the function $T_0^t u$ is differentiable at $q_n$, and $p_n = d(T_0^t u)(q_n)$. Let $\Theta_n = Q_t^0(q_n, p_n)$ be the sequence of points such that

$$T_0^t u(q_n) = u(\Theta_n) + A_0^t(\Theta_n, q_n).$$

The sequence $\Theta_n$ is converging to $\Theta = Q_t^0(q,p)$, and, at the limit, we see that

$$T_0^t u(q) = u(\Theta) + A_0^t(\Theta, q).$$

We conclude that $\Theta \in \mathcal{I}$. Since we have already proved the Lipschitz reguilarity of $du$ on $\mathcal{I}$, we deduce that

$$\varphi_t^0(q,p) = \lim(\varphi_t^0(q_n,p_n)) = \lim(\Theta_n, du(\Theta_n)) = (\Theta, du(\Theta)) \in \Gamma_{u|\mathcal{I}} \subset \Gamma_u.$$

□

The action of the Lax-Oleinik operators on semi-convex functions also has a remarkable property, see [3]:

**Proposition 17.** *If $u$ is $K$-semi-convex, then there exists $T \in ]0, \sigma[$ such that $T^t u$ is $(K+1)$-semi-convex, hence $C^{1,1}$, for each $t \in ]0, T[$. One can take*

$$T = \min\left(\sigma/2, (5M(1+2K))^{-1}\right)$$

PROOF. Let us denote by $L$ the set of points $(Q, P)$ such that $P$ is a proximal subdifferential of $u$ at $Q$. Note that $\Gamma_u \subset L$. For each $(Q, P) \in L$, we have

$$u(q) \geqslant u(Q) + P(q - Q) - K|q - Q|^2,$$

we denote by $l_{Q,P}(q)$ the function on the right in this inequality, so that

$$u = \max_{(Q,P)\in L} l_{Q,P}.$$

Taking $T$ as in the statement, it follows from Theorem 3 that the functions $T^t(l_{Q,P}), t \in [-T, T]$ are $C^2$ with a second derivative bounded by $2K + 1$. We claim that

$$T^t u = \max_{(Q,P)\in L} T^t(l_{Q,P}),$$

for $t \in [0, T]$, which implies that $T^t u$ is $K + 1$-semi-convex. We prove the claim in two steps. First, the inequality

$$T^t u \geqslant \max_{(Q,P)\in L} T^t(l_{Q,P}),$$

follows immediately from the fact that $u \geqslant l_{Q,P}$ for each $(Q, P) \in L$. In order to prove the other inequality, we consider a point $q$ and a point $\theta$ such that $T^t u(q) = u(\theta) + S^t(\theta, q)$. This implies that the point $(\theta, \zeta) = (\theta, -\partial_1 S(\theta, q))$ belongs to $L$, and that $q = Q^t(\theta, \zeta)$. Then, we have

$$T^t(l_{\theta,\zeta})(q) = l_{\theta,\zeta}(\theta) + S^t(\theta, q) = u(\theta) + S^t(\theta, q) = T^t u(q)$$

hence

$$T^t u(q) \leqslant \max_{(Q,P)\in L} T^t(l_{Q,P})(q).$$

□

The following was first stated by Marie-Claude Arnaud in [1].

**Addendum 1.** *We have $L = \varphi^{-t}(\Gamma_{T^t u})$ for each $t \in ]0, T[$. Moreover, for each $q$, we have $T^t u(q) = u(\theta) + S^t(\theta, q)$, with $\theta = Q^{-t}(q, d(T^t u)(q))$.*

PROOF. For each $q \in \mathbb{R}^d$, we have seen that there exists a $(\theta, \zeta) \in L$ such that $T^t u(q) = u(\theta) + S^t(\theta, q)$ and $\zeta = -\partial_1 S^t(\theta, q)$. Since we know that $T^t u$ is $C^1$, the first of these equalities implies that $d(T^t u)(q) = \partial_2 S^t(\theta, q)$, while the second implies that $\varphi^t(\theta, \zeta) = (q, \partial_2 S^t(\theta, q))$. We conclude that $\varphi^{-t}(\Gamma_{T^t u}) \subset L$. Moreover, since $T^t u$ is $C^1$, we have $\partial_2(\theta, q) = d(T^t u)(q)$, hence $\theta = Q^{-t}(q, d(T^t u)(q))$.

Conversly, let us consider a point $(\theta, \zeta) \in L$, and denote by $l$ the associated function $l_{\theta,\zeta}$. By Proposition 14, the function $(t, q) \longmapsto T^t l(q)$ is the $C^2$ solution of (HJ) defined on the interval $] - T, T[$. As a consequence, we have

$$T^t l(Q^t(\theta, \zeta)) = l(\theta) + S^t(\theta, Q^t(\theta, \zeta)) = u(\theta) + S^t(\theta, Q^t(\theta, \zeta)) \geqslant T^t u(Q^t(\theta, \zeta)).$$

Since we know that $T^t l \leqslant T^t u$, we conclude that this last inequality is actually an equality. Setting $q_1 = Q^t(\theta, \zeta)$, this implies that

$$(\theta, \zeta) = (\theta, -\partial_1 S(\theta, q_1)) = \varphi^{-t}(q_1, \partial_2 S^t(\theta, q_1)) = \varphi^{-t}(q_1, dT^t u(q_1)) \subset \varphi^{-t}(\Gamma_{T^t u}).$$

□

## 5 Subsolutions of the stationary Hamilton-Jacobi equation.

We assume from now on that the Hamiltonian does not explicitly depend on time. Then, in addition to (HJ), we can consider the stationary Hamilton-Jacobi equation

$$H(q, du(q)) = a, \tag{HJ$a$}$$

for each real parameter $a$. This stationary equation is the main character of Fathi's joined lecture. Formally, a function $u(q)$ solves (HJ$a$) if and only if the function $(t, q) \longmapsto u(q) - at$ solves (HJ). It is not hard to check that this also holds in the sense of viscosity solutions: The function $u(q)$ is a viscosity solution of (HJ$a$) if and only if the function $(t, q) \longmapsto u(q) - at$ is a viscosity solution of (HJ).

In this autonomous context, the Lax Oleinik operators form a semi-group, the famous Lax-Oleinik semi-group. More precisely, we have $T^t \circ T^s = T^{t+s}$, as follows easily from the Markov property of the operators and from the equality $T^t = T_s^{t+s}$. Another important specificity of the autonomous context is that the Hamiltonian $H$ is constant along Hamiltonian orbits, as can be checked by an easy computation.

**Proposition 18.** *The following properties are equivalent for a function $u$:*

1. *The function $u$ is Lipschitz and it solves the inequation $H(q, du(q)) \leqslant a$ almost everywhere.*
2. *The inequality $u(q_1) - u(q_0) \leqslant A^t(q_0, q_1) + at$ holds for each $q_0 \in \mathbb{R}^d, q_1 \in \mathbb{R}^d, t > 0$.*
3. *The inequality $u \leqslant T^t u + ta$ holds for each $t \geqslant 0$.*
4. *The function $u$ is a viscosity subsolution of the Hamilton-Jacobi equation $H(q, du(q)) = a$.*
5. *The function $u$ is Lipschitz and the inequation $H(q, du(q)) \leqslant a$ holds at each point of differentiability $q$ of $u$ (by Rademacher Theorem, the set of points of differentiability has full measure).*

Proof. It is tautological that $5 \Rightarrow 1$ and easy that $2 \Leftrightarrow 3$. Let us prove that $1 \Rightarrow 2$, following Fathi. If 1 holds, then there exists a set $M \subset \mathbb{R}^d$ of full measure composed of points of differentiability $q$ of $u$ such that $H(q, du(q)) \leqslant a$. We first assume that $t < \sigma$ and prove 2 (recall that $A^t = S^t$). Let us consider the map

$$(q_0, q_1, \tau) \longmapsto (q(\tau), q_1, \tau),$$

where $q(\tau)$ is the value at time $\tau$ of the unique orbit $(q(s), p(s))$ which satisfies $q(0) = q_0$ and $q(t) = q_1$. This map is a diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^d \times ]0, t[$, the inverse diffeomorphism being

$$(\theta, q_1, \tau) \longmapsto (q(0), q_1, \tau),$$

where $(q(s), p(s))$ is the unique orbit such that $q(\tau) = \theta$ and $q(t) = q_1$. As a consequence, for almost each pair $(q_0, q_1)$, the function $u$ is differentiable at the point $q(s)$ for almost every $s \in ]0, t[$. If $(q_0, q_1)$ is such a pair, we have, using the convexity of $H$ in $p$,

$$\begin{aligned} u(q_1) - u(q_0) = u(q(t)) - u(q(0)) &= \int_0^t du_{q(s)} \cdot \dot{q}(s) ds = \int_0^t du_{q(s)} \cdot \partial_p H(q(s), p(s)) ds \\ &\leqslant \int_0^t H(q(s), du_{q(s)}) + \partial_p H(q(s), p(s)) \cdot p(s) - H(q(s), p(s)) ds \\ &\leqslant at + S_0^t(q(0), q(t)) = at + A^t(q_0, q_1). \end{aligned}$$

We have proved the desired inequality for almost every pair $(q_0, q_1)$, hence on a dense subset of pairs. Since both sides of the inequality are continuous, we deduce that the inequality holds for all pairs $(q_0, q_1)$, provided $t < \sigma$. In order to deduce the inequality when $t \geqslant \sigma$, we write, for $n$ large enough,

$$\begin{aligned}A^t(q_0, q_1) + at &= \min_{\theta_1,\dots,\theta_{n-1}} \big(S^{t/n}(q_0, \theta_1) + at/n + \cdots + S^{t/n}(q_{n-1}, q_1) + at/n\big)\\ &\geqslant \min_{\theta_1,\dots,\theta_{n-1}} \big(u(\theta_1) - u(q_0) + \cdots + u(q_1) - u(\theta_{n-1})\big) = u(q_1) - u(q_0).\end{aligned}$$

Let us now prove that 3 ⇒ 4. Let $u$ be a function satisfying 3. This function then satisfies 2, hence it is Lipschitz. We consider a $C^2$ function $v(q)$ which touches $u$ from above at some point $\theta$, which means that $v - u$ has a global minimum at $\theta$. Since the function $u$ is Lipschitz, we can modify $v$ at infinity and assume that it has bounded second differential. Then, there exists a $C^2$ solution $V(t, q)$ of (HJ) defined on $]-T, T[\times \mathbb{R}^d$ with $T > 0$, and such that $V(0, q) = v(q)$. For $t \geqslant 0$, we have $V_t = T^t v$, by Proposition 14. Since $v \geqslant u$, we obtain that

$$V(t, q) = T^t v(q) \geqslant T^t u(q) \geqslant u(q) - at$$

for $t \in ]0, T[$, hence $\partial_t V(\theta) \geqslant -a$ (recall that $\theta$ is the point of contact between $u$ and $v$). Since we know that $V$ solves (HJ), we conclude that

$$H(\theta, \partial_q V(0, \theta)) = H(\theta, dv(\theta)) \leqslant a.$$

The proof that 4 ⇒ 5 is very classical and can be found in Fathi's lecture, but we recall it here for completeness. If $q$ is a point of differentiability of $u$, then $du(q)$ is a superdifferential (but not necessarily a proximal superdifferential) of $u$ at $q$, hence $H(q, du(q)) \leqslant a$. We will now prove that the function $u$ is locally Lipschitz. The estimate $H(q, du(q)) \leqslant a$ , which holds at each point of differentiability of $u$, then implies that it is globally Lipschitz.

Let $B(Q, 1)$ be a closed ball, of radius one. Let us set $r = \max_{\theta\in B(Q,2), q\in B(Q,1)}(u(\theta) - u(q))$. Let $k$ be a positive number greater that $4r$ and such that $|p| \geqslant k \Rightarrow H(q, p) > a$ for each $q$. Given $q$ in $B(Q, 1)$, the function

$$\theta \longmapsto k|\theta - q| - u(\theta)$$

has a local minimum in the interior of the ball $B(Q, 2)$. If this minimum is reached at a point $q_1$ different from $q$, then the function $v(\theta) := k|\theta - q|$ is smooth at $q_1$, and, since $u$ is a viscosity subsolution, we have $H(q_1, dv(q_1)) \leqslant a$, which is in contradiction with the fact that $|dv(q_1)| = k$. Hence the minimum must be reached at $q$, which implies that $k|\theta - q| - u(\theta) \geqslant -u(q)$ or equivalently that

$$u(\theta) - u(q) \leqslant k|\theta - q|$$

for each $\theta \in B(Q, 2)$ and all $q \in B(Q, 1)$. We conclude that $u$ is $k$-Lipschitz on $B(Q, 1)$. □

**Corollary 19.** *If $u$ is a subsolution of (HJa), then, for each $t \geqslant 0$, $T^t u$ is a subsolution of (HJa), and so is $\breve{T}^t u$.*

Proof. The function $u$ is a subsolution if and only if $T^s u + as \geqslant u$ for each $t \geqslant 0$. Applying $T^t$, we obtain $T^t T^s u + as = T^s T^t u + as \geqslant T^t u$. Since this inequality holds for each $s \geqslant 0$, we conclude that $T^t u$ is a subsolution. □

**Corollary 20.** *If the function $u$ is Lischitz, and if the Hamiltonian is autonomous, then the functions $T^t u, t \geqslant 0$ are equi-Lischitz.*

PROOF. If the function $u$ is $k$-Lipschitz, then $du(q) \leqslant k$ almost everywhere, hence $u$ is a subsolution to (HJa) for some $a$ (one can take $a = \sup_{|p| \leqslant k} H(q,p)$). As a consequence, the functions $T^t u, t \geqslant 0$ are all subsolutions to (HJa), hence they are $K$-Lipschitz, with $K = \sup\{|p|, H(q,p) \leqslant a\}$.
□

# 6 Weak KAM solutions and invariant sets.

We derive here the first dynamical consequences from the theory.

**Definition 21.** *The function $u$ is called a Weak KAM Solution at level $a$ if $T^t u + ta = u$ for each $t \geqslant 0$. Weak KAM solutions at level $a$ are viscosity solutions of (HJa). We say that the function $u$ is a Weak KAM Solution if it is a Weak KAM solution at some level $a$.*

If $u$ is a weak KAM solution, then it is semiconcave (with a semiconcavity constant which depends only on $M$ and $m$). By Theorem 16, for $t > 0$, we have the inclusion

$$\varphi^{-t}(\bar{\Gamma}_u) \subset \Gamma_u$$

and this set is a Lipschitz graph. The set

$$\mathcal{I}^*(u) := \cap_{n \in \mathbb{N}} \varphi^{-n}(\bar{\Gamma}_u)$$

is a closed invariant set contained in a Lipschitz graph. It would be a very nice result to have obtained a distinguished closed invariant subsets of our Hamiltonian system contained in a Lipschitz graph. Unfortunately, at this point, we can't prove (because it is not necessarily true) that the set $\mathcal{I}^*(u)$ is not empty. In order to obtain interesting dynamical consequences from this theory, it is thus useful to make an additional assumption.

**Hypothesis 4.** *We say that the Hamiltonian $H$ is periodic if $H(q+w,p) = H(q,p)$ for each $w \in \mathbb{Z}^d$, $q \in \mathbb{R}^d$ and $p \in \mathbb{R}^{d*}$.*

Under this hypothesis, we should see the Hamiltonian system as defined on the phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$, with $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Indeed, the flow $\varphi^t$ commutes with the translations $(q,p) \longmapsto (q+w,p)$, $w \in \mathbb{Z}^d$. The compactness of this new configuration space has remarkable consequences, sumed up in the following Theorem:

**Theorem 5.** *If the Hamiltonian is autonomous and periodic, then there exists a periodic Weak KAM solution. The corresponding set $\mathcal{I}^*(u)$ is a non-empty closed invariant set which is contained in a Lipschitz graph and which is invariant under the translations $(q,p) \longmapsto (q+w,p)$, $w \in \mathbb{Z}^d$.*

This last property on the invariance under translations means that $\mathcal{I}^*(u)$ naturally gives rise to an invariant space on the quotient phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$.
PROOF. Let us first prove the second part of the Theorem. If $u$ is a periodic Weak KAM solution, then the set $\bar{\Gamma}_u$ is containd in $\{|p| \leqslant C\}$ for some constant $C$, and it is invariant under the integral translations, hence it descends to a compact subset of $\mathbb{T}^d \times \mathbb{R}^{d*}$, that we still denote by $\bar{\Gamma}_u$. Then the sets $\varphi^{-n}(\bar{\Gamma}_u)$ form a decreasing sequence of compact sets, hence their intersection is a non-empty compact set.

Let us now prove that there exists a periodic Weak KAM solution. We follow the proof of [5], which is slightly different from the original proof of Fathi. Observe first that The functions $A^t(q_0,q_1)$ are periodic in the sense that $A^t(q_0+w,q_1+w) = A^t(q_0,q_1)$ for each $w \in \mathbb{Z}^d$. This implies that $T^t u$ is periodic when $u$ is periodic. Considering the Cauchy problem for (HJ) with initial condition equal to zero, we define $v(t,q) := T^t 0(q)$. The quantities $a^+(t) = \max_q v_t(q)$ and $a^-(t) = \min_q v_t(q)$ will be useful. Since the functions $v_t, t \geqslant 0$ are equi-Lipschitz, there exists a constant $K$ such that $a^+(t) - a^-(t) \leqslant K$ for all $t \geqslant 0$. We have

$$a^+(t+s) = \max T^{t+s}0 = \max T^t(T^s 0) \leqslant T^t(a^+(s)) = a^+(s) + T^t(0) \leqslant a^+(s) + a^+(t),$$

and similarly

$$a^-(t+s) \geqslant a^-(t) + a^-(s).$$

By standard results on subadditive functions, we conlude that $a^+(t)/t$ and $a^-(t)/t$ converge respictively to $\inf_{t\geqslant 0} a^+(t)/t$ and $\sup_{t\geqslant 0} a^-(t)/t$. Since $a^+ - a^-$ is bounded, these two limits have the same value, let us call it $-a$. We have

$$K - ta \geqslant a^-(t) + K \geqslant a^+(t) \geqslant -ta \geqslant a^-(t) \geqslant a^+(t) - K \geqslant K - at$$

for all $t \geqslant 0$, hence

$$K \geqslant v(t,q) + ta \geqslant -K.$$

We can now define

$$u(q) := \liminf_{t \longrightarrow \infty} (v(t,q) + ta).$$

We claim that $u$ is a Weak KAM solution at level $a$. Since the functions $v_t + ta$ are equi-Lipschitz and equi-bounded, the function $u$ is well-defined and Lipschitz. We have to prove that $T^t u + ta = u$ for all $t \geqslant 0$.

We have

$$v(t+s, q_1) + (t+s)a \leqslant v(s, q_0) + ta + A^t(q_0, q_1) + ta$$

for each $q_0, q_1$ and $t \geqslant 0, s \geqslant 0$. Taking the $\liminf$ in $s$ yields

$$u(q_1) \leqslant u(q_0) + A^t(q_0, q_1) + ta.$$

We have proved that $u$ is a subsolution to (HJ$a$).

Conversely, we have to prove that $T^t u + ta \geqslant u$. Let us pick a point $q$ and consider a sequence $t_n$ such that $v(t_n, q) + t_n a \longrightarrow u(q)$. Fixing $t > 0$, we consider a sequence $q_n$ in $\mathbb{R}^d$ such that

$$v(t_n, q) + t_n a = v(t_n - t, q_n) + (t_n - t)a + A^t(q_n, q) + ta.$$

This equality implies that the sequence $q_n$ is bounded, and we assume by taking a subsequence that it has a limit $q'$. We can also assume that the sequence $v(t_n - t, q') + (t_n - t)a$ has a limit, that we denote by $l$. Note that $l \geqslant u(q')$. Since the functions $v_t$ are equi-Lipschitz, we have $v(t_n - t, q_k) + (t_n - t)a \longrightarrow l$ hence, taking the limit in the equality above,

$$u(q) = l + A^t(q', q) + at \geqslant u(q') + A^t(q', q) + at \geqslant T^t u(q) + at.$$

We have proved that $u$ is a periodic Weak KAM solution at level $a$. □

The periodic Weak KAM solutions at level $a$ are the periodic viscosity solutions of (HJ$a$), as is proved in Fathi's joined lecture. The existence of periodic viscosity solutions was first obtained by Lions, Papanicolaou and Varadhan in a famous unpublished preprint, [11]. The most important aspect of Fathi's weak KAM theorem that we just exposed is that these viscosity solutions have a dynamical relevance and give rise to invariant sets.

Let us comment a bit further in that direction, and explain the name "Weak KAM". Consider a periodic Lipschitz function $u$, and the associated set $\Gamma_u$, that we consider here as a subspace of $\mathbb{T}^d \times \mathbb{R}^{d*}$.

Assume first that $u$ is $C^2$, so that $\Gamma_u$ is a $C^1$ graph. This graph is invariant if and only if there exists $a$ such that $u$ solves (HJ$a$). This follows from Section 1: If $u$ solves (HJ$a$), then the function $U(t,q) = u(q) - at$ solves $HJ$, hence

$$\varphi^t(\Gamma_u) = \Gamma_{U_t} = \Gamma_u.$$

Conversely, if $\Gamma_u$ is invariant, then $T^t u$ is equal to $u$ up to an additive constant $a(t)$. Since $T^t$ is a semi-group, it is easy to deduce that $a(t) = at$ for some $a \in \mathbb{R}$. As a consequence, $u$ is a $C^2$ Weak KAM solution, hence a classical solution of (HJ$a$).

The classical KAM theorem gives the existence, in certain very specific settings, of some invariant $C^1$ graphs of the form $\Gamma_u$. From what we just explained, it can be interpreted as giving the existence of $C^2$ solutions of (HJ$a$), although this point of view is not the right one to obtain its proof. It is natural to expect that the Hamilton-Jacobi equation could be used to produce invariant sets in more general situations. Since we do not know any direct method to prove the existence of $C^2$ solutions of (HJ$a$), we should deal with some kind of weak solutions. However, if $u$ is just a Lipschitz solution almost everywhere, we can't say much about the dynamical properties of $\Gamma_u$. It is remarkable that the inclusion $\varphi^t(\Gamma_u) \supset \bar{\Gamma}_u$ holds for viscosity solutions (or, equivalentely Weak KAM solutions) in the convex case. This is the starting point of the construction of the invariant set $\mathcal{I}^*(u)$ we exposed in the present Section.

# 7 Regular subsolutions and the Aubry set.

We abandon for a moment the hypothesis 4 and describe a new construction of invariant sets based on the study of regular subsolutions. We define the Aubry set. We mostly follow [3] in this section. The following result is at the base of our constructions, see [3, 1, 9].

**Theorem 6.** *If (HJa) admits a subsolution, then it admits a $C^{1,1}$ subsolution. Moreover, the set of $C^{1,1}$ subsolutions is dense in the set of all subsolutions for the uniform topology.*

Proof. Let $u$ be a subsolution at level $a$. Then, for each $t \geqslant 0$ and $s \geqslant 0$, the function $T^s \check{T}^t u$ is a subsolution at level $a$. We claim that

$$u - Ct \leqslant T^s \check{T}^t u \leqslant u + Cs$$

for some constant $C$ depending only on $M$ and $m$. We prove this inequlity by writing first

$$u(q) - Ct \leqslant u(q) - A^t(q,q) \leqslant \check{T}^t u(q) \leqslant u(q),$$

and then

$$u(q) - Ct \leqslant \check{T}^t u(q) \leqslant T^s \check{T}^t u(q) \leqslant \check{T}^t u(q) + A^s(q,q) \leqslant \check{T}^t u(q) + Cs \leqslant u(q) + Cs.$$

The function $\check{T}^t u$ is $C(1+1/t)$-semiconvex hence, by Proposition 17, there exists $\delta > 0$ such that the subsolution $T^{\delta t} \check{T}^t u$ is $C^{1,1}$. This subsolution converges uniformly to $u$ when $t > 0$ converges to zero.

In some occasions, it can be better to rather use the formula $T^{\delta t} \check{T}^t T^{(1-\delta)t} u$ to regularize $u$. It is not hard to check that this subsolution is $C^{1,1}$ and that it converges uniformly to $u$ as $t \longrightarrow 0$. A nice feature of this formula is that $T^{\delta t} \check{T}^t T^{(1-\delta)t} u = u$ when $t$ is small enough if $u$ is already $C^{1,1}$. □

**Definition 22.** *The critical value of $H$ is the real number $\alpha$ (or $\alpha(H)$) defined as the infimum of all real numbers $a$ such that (HJa) has a subsolution. The subsolutions of (HJ$\alpha$) are called critical subsolutions.*

**Lemma 23.** *We have the estimate $-M \leqslant \alpha \leqslant M$.*

PROOF. The function $u = 0$ is a subsolution at level $M$, hence $\alpha \leqslant M$. Conversely, since $H \geqslant -M$ there exist no subsolution at level $a$ when $a < -M$. □

**Proposition 24.** *There exists a $C^{1,1}$ subsolution of (HJ$\alpha$).*

PROOF. Let $a_n$ be a sequence decreasing to $\alpha$. Since $a_n > \alpha$, the Hamilton-Jacobi equation at level $a_n$ has a subsolution $u_n$. The sequence $u_n$ is equi-Lipschitz, and we can assume by adding constants that it is also equi-bounded. Taking a susequence, we can also assume that it converges locally uniformly to a limit $u$. Taking the limit $n \longrightarrow \infty$ in the inequalities $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1) + ta_n$ gives $u(q_1) - u(q_0) \leqslant A^t(q_0, q_1) + ta_n$. This holds for all $q_0, q_1$ and $t > 0$, hence $u$ is a subsolution at level $\alpha$, or in other words a critical subsolution. Since there exists a critical subsolution, Theorem 6 implies that there exists a $C^{1,1}$ critical subsolution.

□

**Definition 25.** *The projected Aubry set is the set $\mathcal{A} \subset \mathbb{R}^d$ of points $q$ such that the equality $H(q, du(q)) = \alpha$ holds for all $C^1$ critical subsolutions $u$.*

**Lemma 26.** *If $q \in \mathcal{A}$, then all $C^1$ critical subsolutions $u$ have the same differential at $q$. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the $C^1$ critical subsolution $u$.*

PROOF. Let $u$ and $v$ be two critical subsolutions, and $q$ a point in $\mathcal{A}$. We have to prove that $du(q) = dv(q)$. Assume, by contradiction, that this equality does not hold and consider the subsolution $w = (u + v)/2$. Since $H(q, du(q)) = H(q, dv(q)) = \alpha$, the strict convexity of $H(q, .)$ implies that $H(q, dw(q)) < \alpha$, which contradicts the definition of $\mathcal{A}$. □

**Lemma 27.** *There exists a $C^{1,1}$ subsolution $u_0$ such that $\mathcal{A} = \Gamma_{u_0} \cap \{H = \alpha\}$. The subsolution $u_0$ satisfies the strict inequality $H(q, du_0(q)) < \alpha$ for all $q$ in the complement of $\mathcal{A}$.*

PROOF. The set of $C^1$ functions is separable for the topology of uniform $C^1$ convergence on compact sets. This topology can be defined for example by the distance

$$d(u, v) = \sum_n \frac{\sup_{|q| \leqslant n} \arctan(|u(q)| + |du(q)|)}{2^n}.$$

Since a subset of a separable space is separable, there exists a sequence $u_n$ of $C^1$ critical subsolutions which is dense for this topology in the set of all $C^1$ critical subsolutions. Let us set

$$a_n = \frac{a_0}{2^n \sup_{k \leqslant n, |q| \leqslant n}(1 + |u_k(q)| + |du_k(q)|)}$$

and choose $a_0$ such that $\sum_{n \geqslant 1} a_n = 1$. The sum $\sum_{n \geqslant 1} a_n u_n$ converges uniformly with its differentials on each compact sets to a $C^1$ limit $v_0$. The function $v_0$ is a critical subsolution, and we claim that $H(q, dv_0(q)) = \alpha$ if and only if $q$ belongs to $\mathcal{A}$. Indeed, this equality holds only if all the inequalities $H(q, du_n(q)) \leqslant \alpha$ are equalities, which, in view of the density of the sequence $u_n$, implies that $H(q, du(q)) = \alpha$ for all $C^1$ subsolutions $u$. By definition, this implies that $q$ belongs to $\mathcal{A}$. We have constructed a $C^1$ subsolution $v_0$ such that

$$H(q, dv_0(q)) < \alpha$$

outside of $\mathcal{A}$. We have to prove the existence of a $C^{1,1}$ critical subsolution with the same property. We consider a smooth function $V(q)$ which is bounded in $C^2$, which is positive outside of $\mathcal{A}$, and such that

$$0 \leqslant V(q) \leqslant \alpha - H(q, dv_0(q))$$

for all $q \in \mathbb{R}^n$. The modified Hamiltonian $\tilde{H}(q,p) = H(q,p) + V(q)$ satisfies all our hypotheses. Since $\tilde{H} \geqslant H$, the corresponding critical value $\tilde{\alpha}$ satisfies $\tilde{\alpha} \geqslant \alpha$. Since $v_0$ is a subsolution of the inequation

$$\tilde{H}(q, dv_0(q)) \leqslant \alpha,$$

we can apply Theorem 6 to $\tilde{H}$ at level $\alpha$, and obtain the existence of a $C^{1,1}$ subsolution $u_0$ to the same inequation. The inequality

$$H(q, du_0(q)) \leqslant \alpha - V(q)$$

implies that $u_0$ is a critical subsolution for $H$ which is strict on the set $\{V > 0\}$ which, from our construction of $V$, is the complement of $\mathcal{A}$. □

**Definition 28.** *The Aubry set $\mathcal{A}^*$ is defined as:*

$$\mathcal{A}^* = \cap_u \Gamma_{u|\mathcal{A}} = \cap_u \Gamma_u,$$

*where the intersections are taken on the set of $C^1$ critical subsolutions.*

Note that $\mathcal{A}^* = \Gamma_{u|\mathcal{A}}$ for each $C^1$ subsolution $u$, hence $\pi(\mathcal{A}^*) = \mathcal{A}$, where $\pi : \mathbb{R}^d \times \mathbb{R}^{d*} \longrightarrow \mathbb{R}^d$ is the projection on the first factor. The second equality comes from the existence of a solution $u_0$ such that $H(q, du_0(q)) < \alpha$ on the complement of $\mathcal{A}$. The set $\mathcal{A}^*$ is obviously contained in a the Lipschitz graph $\Gamma_{u_0}$. As in Section 6, we have obtained an invariant set contained in a Lischitz graph, but which may be empty in general:

**Proposition 29.** *The Aubry set is a closed invariant set.*

Proof. Let $u_0$ be a $C^{1,1}$ critical solution such that $H(q, du_0(q)) < \alpha$ outside of $\mathcal{A}$. By Proposition 17, there exists $T > 0$ such that $T^t u_0$ is still $C^{1,1}$ for $t \in [-T, T]$. Given $(q,p) \in \mathcal{A}^*$, we conclude that, for $t \in [0, T]$, we have $p = d(T^t u_0)(q)$. Setting $\theta = Q^{-t}(q,p)$, the addendum to Proposition 17 implies that $T^t u_0(q) = u_0(\theta) + S^t(\theta, q)$, and that

$$\varphi^t(\theta, du_0(\theta)) = (q,p).$$

Since the flow preserves the Hamiltonian, we get that $H(\theta, du_0(\theta)) = \alpha$, hence the point $\theta$ belongs to $\mathcal{A}$, and then

$$\varphi^{-t}(q,p) = \big(\theta, du_0(\theta)\big) \in \mathcal{A}^*.$$

We have proved that $\varphi^{-t}(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0, T]$. We can prove in a similar way, using the $C^{1,1}$ subsolution $\check{T}^t u_0$ instead of $T^t u_0$, that $\varphi^t(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0, T]$, and hence that

$$\varphi^t(\mathcal{A}^*) = \mathcal{A}^*$$

for each $t \in [-T, T]$, which clearly implies that this equality holds for all $t$. We have proved the invariance of $\mathcal{A}^*$. □

**Proposition 30.** *The equality*

$$\check{T}^t u(q) - t\alpha = u(q) = T^t u(q) + t\alpha$$

*holds for each critical subsolution $u$, each $t \geqslant 0$ and each $q \in \mathcal{A}$. The inclusion $\mathcal{A}^* \subset \Gamma_u$ holds for each critical subsolution, hence the inclusion $\mathcal{A}^* \subset \mathcal{I}^*(u)$ holds for each weak KAM solution at level $\alpha$.*

PROOF. Let $(q(s), p(s))$ be a trajectory contained in $\mathcal{A}^*$, and $t \geqslant 0$ be given. For each $C^1$ critical subsolution $u$, we have $p(s) = du_{q(s)}$, and

$$\begin{aligned} u(q(t)) - u(q(0)) = &\int_0^t du_{q(s)}\dot{q}(s)ds = t\alpha + \int_0^t du_{q(s)}\dot{q}(s) - H(q, du_{q(s)})ds \\ \geqslant &A^t(q(0), q(t)) + t\alpha. \end{aligned}$$

Since $u$ is a critical sub-solution, the last inequality must be an equality, hence

$$u(q(t)) - u(q(s)) = A^{t-s}(q(s), q(t)) + (t-s)\alpha$$

for each $t \geqslant s$. In the terminology of Fathi, we have proved that the curve $q(s)$ is calibrated by the subsolution $u$. We can now write

$$u(q(t)) \leqslant T^t u(q(t)) + t\alpha \leqslant u(q(0)) + A^t(q(0), q(t)) + t\alpha = u(q(t)).$$

This implies that $T^t u + t\alpha = u$ on $\mathcal{A}$, and, similarly, $\breve{T}^t u - t\alpha = u$ on $\mathcal{A}$. Let us now fix $t \in ]0, \sigma[$. Given an orbit $(q(s), p(s))$ in $\mathcal{A}^*$, we have

$$u(q(0)) \leqslant u(\theta) + S^t(\theta, q(0)) + t\alpha$$

for each subsolution $u$ and each $\theta$, with equality at $\theta = q(-t)$. This implies that $\partial_2 S(q(-t), q(0))$ is a superdifferential of $u$ at $q(0)$. This holds for $C^1$ subsolutions, which satisfy $du(q(0)) = p(0)$, hence $\partial_2 S(q(-t), q(0)) = p(0)$. We have proved that $p(0)$ is a superdifferential of $u$ at $q(0)$. Similarly, using the inequality

$$u(q(0)) \geqslant u(\theta) - S^t(q(0), \theta) - t\alpha,$$

with equality at $\theta = q(t)$, we conclude that $p(0)$ is a subdifferential of $u$ at $q(0)$. This implies that $u$ is differentiable at $q(0)$, and that $du(q(0)) = p(0)$. As a consequence, $\mathcal{A}^* \subset \Gamma_u$ for each subsolution $u$. □

In the course of the above proof, we have established the following Lemma, which will be needed later:

**Lemma 31.** *Let $u$ be a subsolution at level $a$, and let $(q(s), p(s))$ be a Hamiltonian trajectory contained in $\Gamma_u \cap \{H = a\}$ (note that this set is not necessarily invariant in general), then, the equality $\breve{T}^t u(q(s)) - ta = u(q(s)) = T^t u(q(s)) + ta$ holds, for each $t \geqslant 0$ and each $s \in \mathbb{R}$.*

# 8 The Mañé Potential.

The Mañé Potential at level $a$ is the function

$$\Phi^a(q_0, q_1) := \inf_{t>0} \big(A^t(q_0, q_1) + at\big).$$

This function was first introduced by Ricardo Mañé, see [12]. We leave as an easy exercise for the reader to prove the triangle inequality

$$\Phi^a(q_0, \theta) + \Phi^a(\theta, q_1) \geqslant \Phi^a(q_0, q_1).$$

In view of Proposition 18, Each subsolution $u$ at level $a$ satisfies

$$u(q_1) - u(q_0) \leqslant \Phi^a(q_0, q_1)$$

for each $q_0$ and $q_1$. We conclude that $\Phi^a$ is finite if there exists a subsolution at level $a$, which holds if and only if $a \geqslant \alpha$. Conversely, If the function $\Phi^a$ is finite, then we see from the triangle inequality that the function $q \longmapsto \Phi^a(q_0, q)$ is a subsolution at level $a$, which implies that $a \geqslant \alpha$. The estimates of Lemma 11 imply that

$$\Phi^a(q_0, q_1) \leqslant 2\sqrt{2m(M+a)}|q_1 - q_0|$$

provided $a \geqslant \alpha$ (note that $\alpha \geqslant -M$ ). We have proved that the Mañé Potential is the function called the viscosity semi-distance in Fathi's lecture:

**Proposition 32.** *If $a \geqslant \alpha$, then the function $q \longmapsto \Phi^a(q_0, q)$ is the maximum of all subsolutions $u$ at level $a$ which vanish at $q_0$. If $a < \alpha$, then there is no such subsolution and $\Phi^a$ is identically equal to $-\infty$.*

This statemet also implies that the Mañé Potential at level $a$ only depends on the energy level $\{H = a\}$. More precisely, if $G$ is another Hamiltonian satisfying our hypotheses and such that $H = a \Leftrightarrow G = a$, then $G$ and $H$ have the same Mañé potential at level $a$, because $G$ and $H$ have the same subsolutions at level $a$, as follows easily from the first characterization of subsolutions in Proposition 18. This is also reflected in the following Proposition by the fact that the involved orbits are contained in the set $\{H = a\}$.

**Proposition 33.** *Given $q_0 \neq q_1$ , there exists $\tau \in ]0, \infty]$ and an orbit $(q(s), p(s)) : (-\tau, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(0) = q_1$, $A^0_s(q_0, q(s)) - as = \Phi^a(q(s), q_1)$,*

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1)$$

*and $H(q(s), p(s)) = a$ for each $s \in (-\tau, 0]$. If moreover $\tau$ is finite, then $q(-\tau) = q_0$.*

Proof. If $q_0 \neq q_1$, then either the functions $t \longmapsto A^t(q_0, q_1) + at$ reaches its minimum at some finite time $\tau > 0$, or it has a minimizing sequence $\tau_n \longrightarrow \infty$.

In the first case, there exists an orbit $(q(t), p(t)) : [-\tau, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(-\tau) = q_0$, $q(0) = q_1$, and

$$\int_{-\tau}^{0} p \cdot \dot{q} - H(q, p)dt = A^\tau(q_0, q_1) = \Phi^a(q_0, q_1) - \tau a.$$

We obtain, for each $s \in [-\tau, 0]$, that

$$\begin{aligned}
\Phi^a(q_0, q_1) + a\tau = \int_{-\tau}^{0} p \cdot \dot{q} - H(q, p)dt &= \int_{-\tau}^{s} p \cdot \dot{q} - H(q, p)dt + \int_{s}^{0} p \cdot \dot{q} - H(q, p)dt \\
&\geqslant A^s_{-\tau}(q_0, q(s)) + A^0_s(q(s), q_1) \\
&\geqslant \Phi^a(q_0, q(s)) - a(s + \tau) + \Phi^a(q(s), q_1) + as \\
&\geqslant \Phi^a(q_0, q_1) + a\tau.
\end{aligned}$$

We conclude that all these inequalities are equalities, hence

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1).$$

We also deduce from the above chain of inequalities that $A^0_s(q(s), q_1) - as = \Phi^a(q(s), q_1)$, which implies that the function $t \longmapsto A^t(q(s), q_1) + at$ is minimal for $t = -s$. Taking $s \in ] - \sigma, 0[$, we can differentiate with respect to $t$ at $t = -s$ and get

$$\partial_{t|t=-s} S^t(q(s), q_1) + a = 0.$$

Recalling the equality

$$\partial_t S^t(q(s), q_1) + H(q_1, p(0)) = 0$$

(because $p(0) = \rho_1(t, q(s), q_1)$ in the notations of Section 2), we conclude that $H(q_1, p(0)) = a$, and, since the Hamiltonian is constant on Hamiltonian orbits, $H(q(t), p(t)) = a$ for each $t$.

In the second case, there exists a sequence of orbits $(q_n(t), p_n(t))$ on $[-\tau_n, 0]$ such that

$$\int_{-\tau_n}^{0} p_n \cdot \dot{q}_n - H(q_n, p_n)dt + a\tau_n = A^{\tau_n}(q_0, q_1) + a\tau_n \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

where $\delta_n \longrightarrow 0$. By Lemma 10, this implies that the sequence $h_n := H(q_n(s), p_n(s))$ is bounded. As a consequence, the curves $p_n(s)$ are uniformly bounded, hence so is $\dot{q}_n(s) = \partial_p H(q_n(s), p_n(s))$. On each compact interval of time $[t, 0]$, the curves $x_n(t) = (q_n(t), p_n(t))$ is thus uniformly bounded, hence uniformly Lipschitz. Up to taking a subsequence, we can thus assume that the curves $x_n(t)$ converge, uniformly on compact time intervals, to a Hamiltonian orbit $x(t) = (q(t), p(t)) : (-\infty, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$. Passing at the limit in the inequality

$$\Phi^a(q_0, q_n(s)) + \Phi^a(q_n(s), q_1) \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

which holds for each $s \in (-\infty, 0]$, yields

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) \leqslant \Phi^a(q_0, q_1),$$

which must actually be an equality. We prove as in the first case that $H(q_1, p(0)) = a$, thus $H(q(s), p(s)) \equiv a$. □

The projected Aubry set $\mathcal{A}$ can be characterized in terms of the Mañé potential (see also Fathi's lecture):

**Proposition 34.** *The following statements are equivalent for a point $q_0$ and a real number $a$, where we denote by $u$ the function $\Phi^a(q_0, .)$:*

1. *$q_0 \in \mathcal{A}$ and $a = \alpha$.*
2. *$T^t u(q_0) + ta = u(q_0) = 0$ for each $t \geqslant 0$.*
3. *The function $u$ is a Weak KAM solution at level $a$.*
4. *$u$ is differentiable at $q_0$.*

Proof. $1 \Rightarrow 2$. This follows from Proposition 30 since $u$ is a subsolution at level $a = \alpha$.

$2 \Rightarrow 3$. Let us fix $t > 0$ and $q_1$. We have to prove that there exists $\theta$ such that $u(q_1) \geqslant u(\theta) + A^t(\theta, q_1) + ta$ (this inequality is then an equality). If $q_1 = q_0$, the existence of this point follows from the equality $T^t u(q_0) + ta = u(q_0)$.

If $q_1 \neq q_0$, we can apply Proposition 33. With the notations of Proposition 33, if $\tau \geqslant t$, then the point $\theta = q(-t)$ fulfills our demand. If $\tau < t$, then we set $s = t - \tau$. We have $q(-\tau) = q_0$ and $A^\tau(q_0, q_1) + a\tau = u(q_1)$. Since $T^s u(q_0) + sa = u(q_0)$, there exists $\theta$ such that $u(\theta) + A^s(\theta, q_0) + sa = u(q_0) = 0$. We conclude that

$$u(\theta) + A^t(\theta, q_1) + at \leqslant u(\theta) + A^s(\theta, q_0) + sa + A^\tau(q_0, q_1) + a\tau = u(q_1).$$

$3 \Rightarrow 4$. If $u$ is a Weak KAM solution, then it has a proximal superdifferential at each point. Conversely, if $v$ is a $C^1$ subsolution, then $u - v$ has a minimum at $q_0$ hence $dv(q_0)$ is a subdifferential of $u$ at $q_0$. The function $u$ both has a superdifferential and a subdifferential at $q_0$, hence it is differentiable at $q_0$.

$4 \Rightarrow 1$. If $a > \alpha$ or if $q_0$ does not belong to $\mathcal{A}$, then there exists a $C^1$ subsolution $v$ at level $a$ which is strict near $q_0$. We can then slightly perturb the function $v$ near $q_0$ and build a subsolution $w$ such that $dw(q_0) \neq dv(q_0)$. In view of the characterization of $u$ as the largest subsolution vanishing at $q_0$, we conclude that $dv(q_0)$ as well as $dw(q_0)$ are subdifferentials of $u$ at $q_0$, hence $u$ is not differentiable at this point. □

The Mañé potential also allows to build Weak KAM solutions in the non periodic case by the Busemann method, see [8] and Fathi's Lecture. Let $q_n$ be a sequence of points of $\mathbb{R}^d$ such that $|q_n| \geqslant n$. We consider the sequence of functions

$$u_n(q) = \Phi^a(q_n, q) - \Phi^a(q_n, q_0).$$

By construction, $u_n(q_0) = 0$, and it follows from the triangle inequality that the functions $u_n$ are equi-Lipschitz. We can then assume, without loss of generality, that the functions $u_n$ converge, uniformly on compact sets, to a Lipschitz limit $u(q)$.

**Proposition 35.** *The limit function $u(q)$ is a Weak KAM solution at level $a$.*

PROOF. The functions $u_n$ are all subsolutions at level $a$, which means that $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1)$ for each $t \geqslant 0$, $q_0$, $q_1$. At the limit $n \longrightarrow \infty$, we obtain that that $T^t u + ta \geqslant u$ for each $t \geqslant 0$.

We have to prove that $T^t u + ta \leqslant u$ for all $t \geqslant 0$. Let us fix a point $q$ and a time $t \geqslant 0$, and consider a sequence $t_n$ such that

$$A^{t_n}(q_n, q) + at_n \leqslant \Phi^a(q_n, q) + 1/n.$$

This inequality implies that

$$\frac{1}{2Mt_n}|q_n - q|^2 \leqslant 1 + (M - a)t_n + 2\sqrt{2m(M + a)}|q_n - q|$$

and, since $|q_n - q| \longrightarrow \infty$, that $t_n \longrightarrow \infty$. When $n$ is large enough, we have $t_n \geqslant t$ and there exists $\theta_n \in \mathbb{T}^d$ such that $A^t_n(q_n, q) = A^{t_n - t}(q_n, \theta_n) + A^t(\theta_n, q)$. This implies that

$$\begin{aligned}
\Phi^a(q_n, q) &\geqslant A^{t_n}(q_n, q) + at_n - 1/n \\
&\geqslant A^{t_n - t}(q_n, \theta_n) + a(t_n - t) + A^t(\theta_n, q) + at - 1/n \\
&\geqslant \Phi^a(q_n, \theta_n) + A^t(\theta_n, q) + at - 1/n.
\end{aligned}$$

This inequality implies that

$$u_n(q) \geqslant u_n(\theta_n) + A^t(\theta_n, q) + at - 1/n.$$

Since the functions $u_n$ are equi-Lipschitz, this implies that the sequence $\theta_n$ is bounded. By taking a subsequence, we assume that $\theta_n$ has a limit $\theta$, and, at the limit, we obtain

$$u(q) \geqslant u(\theta) + A^t(\theta, q) + at,$$

which implies that $u(q) \geqslant T^t u(q) + ta$. □

## 9 Return to the periodic case.

A more precise link can be established between the contents of Sections 6 and 7 under the assumption that $H$ is periodic (see Hypothesis 4). It is useful first to expose a variant of Section 7 adapted to the periodic case. We leave as exercises the proofs which are direct adaptations of the ones given above. From now on, we assume Hypothesis 4.

**Theorem 7.** *If (HJa) admits a periodic subsolution, then it admits a periodic $C^{1,1}$ subsolution. Moreover, the set of periodic $C^{1,1}$ subsolutions is dense in the set of all periodic subsolutions for the uniform topology.*

**Definition 36.** *The periodic critical value of $H$ is the real number $\alpha(0)$ defined as the infimum of all real numbers $a$ such that (HJa) has a periodic subsolution. The periodic subsolutions at level $\alpha(0)$ are called critical periodic subsolutions.*

**Definition 37.** *The projected periodic Aubry set is the set $\mathcal{A}(0) \subset \mathbb{T}^d$ of points $q$ such that the equality $H(q, du(q)) = \alpha(0)$ holds for all $C^1$ periodic critical subsolutions $u$.*

**Lemma 38.** *If $q \in \mathcal{A}(0)$, then all $C^1$ critical periodic subsolutions $u$ have the same differential at $q$. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the $C^1$ critical periodic subsolution $u$.*

**Proposition 39.** *There exists a $C^{1,1}$ periodic critical subsolution $u_0$ such that $H(q, du_0(q)) < \alpha(0)$ outside of $\mathcal{A}(0)$.*

Without suprise, we define the periodic Aubry set $\mathcal{A}^*(0)$ as

$$\mathcal{A}^*(0) := \Gamma_{u_0|\mathcal{A}},$$

whith $u_0$ given by the proposition (there is not a single $u_0$, but the Aubry set is well defined).

**Proposition 40.** *The set $\mathcal{A}^*(0) \subset \mathbb{T}^d \times \mathbb{R}^{d*}$ is compact, non empty, and invariant.*

Proof. Let us prove that $\mathcal{A}(0)$, hence $\mathcal{A}^*(0)$ is not empty. Assuming by contradiction that it was empty, then the equality $H(q, du_0(q)) < \alpha(0)$ would hold for all $q \in \mathbb{R}^d$. Since the function $q \longmapsto H(q, du_0(q))$ is periodic, we could conclude that $\sup_q H(q, du_0(q)) < \alpha(0)$, which is in contradiction with the defnition of $\alpha(0)$. □

We are now in a position to specify the connection with the invariant sets introduced in Section 6:

**Proposition 41.** *In the periodic case, we have the equality*

$$\mathcal{A}^*(0) = \cap_u \mathcal{I}^*(u),$$

*where the intersection is taken on all periodic weak KAM solutions.*

Proof. The inclusion $\mathcal{A}^*(0) \subset \cap_u \mathcal{I}^*(u)$ is proved as in Section 7. Our goal is to prove the other inclusion. Let $u_0$ be a $C^{1,1}$ periodic subsolution which is strict outside of $\mathcal{A}(0)$. The map $t \longmapsto T^t u_0 + t\alpha(0)$ is non-decreasing. In addition, the functions $T^t u + t\alpha(0)$ are equiLipschitz, and they coincide with $u_0$ on $\mathcal{A}$, hence they are equibounded. As a consequence, $T^t u_0 + t\alpha \longrightarrow u_\infty$ uniformly as $t \longrightarrow \infty$.

*We claim that the limit $u_\infty$ is a periodic weak KAM solution, and that the inequality $u_0 \leqslant u_\infty$ is strict outside of $\mathcal{A}(0)$.*

In order to prove that $u_\infty$ is a weak KAM solution, it is enough to notice that the function $T^{t+s}u_0 + (t+s)\alpha(0)$ converges both to $u_\infty$ and to $T^s u_\infty + s\alpha(0)$ when $t \longrightarrow \infty$. This implies, as desired, that $T^s u_\infty + s\alpha(0) = u_\infty$ for each $s \geqslant 0$.

We know that $u_\infty \geqslant u_0$, with equality on $\mathcal{A}(0)$. Conversely, let us consider a point $q$ such that $u_\infty(q) = u_0(q)$. The point $q$ is minimizing the difference $u_\infty - u_0$. Since $u_\infty$ is semi-concave and $u_0$ is $C^1$, the function $u_\infty$ must be differentiable at $q$ with $du_\infty(q) = du_0(q)$. Since $u_\infty$ solves the Hamilton-Jacobi equation at its points of differentiability, we conclude that $H(q, du_0(q)) = H(q, du_\infty(q)) = \alpha(0)$, hence $q \in \mathcal{A}(0)$. We have proved the claim.

Let us now establish that $\mathcal{I}(u_\infty) = \mathcal{A}$, which implies the proposition. By Lemma 31, we have $\check{T}^t u_\infty - t\alpha = u_\infty$ on $\mathcal{I}(u_\infty)$ for each $t \geqslant 0$. Setting $\epsilon(t) = \sup(u_\infty - T^t u_0 - t\alpha(0))$, we have

$$u_\infty \geqslant u_0 \geqslant \check{T}^t \circ T^t u_0 \geqslant \check{T}^t(u_\infty - \epsilon(t) - t\alpha(0)) \geqslant \check{T}^t u_\infty - \epsilon(t) - t\alpha(0) = u_\infty - \epsilon(t)$$

on $\mathcal{I}(u_\infty)$. Since this holds for all $t \geqslant 0$, and since $\lim_{t \longrightarrow \infty} \epsilon(t) = 0$, we conclude that $u_0 = u_\infty$ on $\mathcal{I}(u_\infty)$. On the other hand, we have seen that $u_0 < u_\infty$ outside of $\mathcal{A}(0)$, hence $\mathcal{I}(u_\infty) \subset \mathcal{A}(0)$.

□

We finish with an easy remark which is specific to the periodic case:

**Proposition 42.** *All periodic weak KAM solutions have level $\alpha(0)$.*

Proof. Let $u_0$ be a critical periodic subsolution, and let $u$ be a periodic weak KAM solution at level $a$. Since $u$ is a periodic subsolution at level $a$, the definition of $\alpha(0)$ implies that $a \geqslant \alpha(0)$. On the other hand, there exists a constant $C$ such that $u - C \leqslant u_0 \leqslant u + C$, which implies

$$u = T^t u + ta \geqslant T^t u_0 - C + ta \geqslant u_0 + t(a - \alpha(0)) - C \geqslant u + t(a - \alpha(0)) - 2C.$$

We obtain that $t(a - \alpha(0)) \leqslant 2C$ for each $t \geqslant 0$, hence $a - \alpha(0) \leqslant 0$. □

## 10 The Lagrangian.

In most expositions of weak KAM theory, the Lagrangian plays an important role. In the present section, we relate it to our main objects in order to facilitate the connection with the core of the literature, where where what we state here as properties is usually taken as definitions. We define the Lagrangian as

$$\begin{aligned} L : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d &\longrightarrow \mathbb{R} \\ (t, q, v) &\longmapsto \sup_{p \in (\mathbb{R}^d)^*} \big(p \cdot v - H(t, q, p)\big). \end{aligned}$$

By standard results on convex Analysis we then have

$$H(t, q, p) = \sup_{v \in \mathbb{R}^d} \big(p \cdot v - L(t, q, v)\big).$$

We obviously have the Legendre inequality

$$H(t, q, p) + L(t, q, v) \geqslant p \cdot v$$

for all $t, q, p, v$. This inequality is an equality if and only if

$$p = \partial_v L(t, q, v) \text{ or equivalently } v = \partial_p H(t, q, p).$$

Let $q(t) : ]t_0, t_1[ \longrightarrow M$ be a curve, The **action** of $q$ is the number

$$\int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) dt.$$

We can also call it Lagrangian action if we want to distinguish from the previously defined Hamiltonian action. The Lagrangian and Hamiltonian actions are related as follows:

The Hamiltonian action of a curve $(q(t), p(t))$ is smaller than the Lagrangian action of its projection $q(t)$, with equality if and only if $p(t) \equiv \partial_v L(t, q(t), \dot{q}(t))$. In particular, the Hamiltonian action of an orbit is equal to the Lagrangian action of its projection.

**Lemma 43.** *let $q_0$ and $q_1$ be two points of $\mathbb{R}^d$, and $t_0, t_1$ be two times, with $0 < t_1 - t_0 < \min(\sigma, 1/2M)$. If $(q(s), p(s))$ is the orbit satisfying $q(t_0) = q_0, q(t_1) = q_1$, we have*

$$S_{t_0}^{t_1}(q_0, q_1) = \int_{t_0}^{t_1} L(s, q(s), \dot{q}(s))ds = \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s))ds,$$

*where the minimum is taken on the set of Lipschitz curves $\theta : [t_0, t_1] \longrightarrow \mathbb{R}^d$ which satisfy $\theta(t_0) = q_0$ and $\theta(t_1) = q_1$.*

Proof. Since $S_{t_0}^{t_1}(q_0, q_1)$ is the Hamiltonian action of the unique orbit $(q(t), p(t))$, hence the Lagrangian action of the curve $q(t)$:

$$S_{t_0}^{t_1}(q_0, q_1) = \int_{t_0}^{t_1} L(s, q(s), \dot{q}(s))ds.$$

The function $u(t, q) := S_{t_0}^{t}(q_0, q)$ solves (HJ) on $]0, \sigma[\times \mathbb{R}^d$. Let us now consider any Lipschitz curve $\theta(s) : [t_0, t_1] \longrightarrow \mathbb{R}^d$ satisfying $\theta(t_0) = q_0$ and $\theta(t_1) = q_1$, and write

$$\begin{aligned}
\int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s))ds &\geqslant \int_{t_0}^{t_1} du(\theta(s)) \cdot \dot{\theta}(s) - H(s, \theta(s), \dot{\theta}(s))ds \\
&= \int_{t_0}^{t_1} du(\theta(s)) \cdot \dot{\theta}(s) - \partial_t u(s, \theta(s))ds \\
&= S_{t_0}^{t_1}(q_0, q_1).
\end{aligned}$$

□

The following proposition is usually taken as the definition of $A$:

**Proposition 44.** *Given two points $q_0$ and $q_1$ and two times $t_0 < t_1$, we have*

$$A_{t_0}^{t_1}(q_0, q_1) = \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s))ds,$$

*where the minimum is taken on the set of Lipschitz curves $\theta : [t_0, t_1] \longrightarrow \mathbb{R}^d$ which satisfy $\theta(t_0) = q_0$ and $\theta(t_1) = q_1$.*

It is part of the statement that the minimum is achieved. This is usually called the Theorem of Tonelli. The statement can be extended to absolutely continuous curves instead of Lipschitz curves, but this setting is not useful for our discussion.

Proof. For $n$ large enough, we have $(t_1 - t_0)/n \leqslant \min(\sigma, 1/2M)$, hence, setting $\tau_i = t_0 + i(t_1 - t_0)/n$,

$$\begin{aligned}
A_{t_0}^{t_1}(q_0, q_1) &= \min_{(\theta_1, \dots, \theta_{n-1})} \big( S_{t_0}^{\tau_1}(q_0, \theta_1) + S_{\tau_1}^{\tau_2}(\theta_1, \theta_2) + \dots + S_{\tau_{n-1}}^{t_1}(\theta_{n-1}, q_1) \big) \\
&= \min_{(\theta_1, \dots, \theta_{n-1})} \big( \min_{\theta(s)} \int_{t_0}^{\tau_1} L(s, \theta(s), \dot{\theta}(s))ds + \ldots + \min_{\theta(s)} \int_{\tau_{n-1}}^{t_1} L(s, \theta(s), \dot{\theta}(s))ds \big) \\
&= \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s))ds.
\end{aligned}$$

□

# A Some technical results.

**Proposition 45.** *A $C^1$ map $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ which satisfies $|dF(q) - Id| \leqslant 1/2$ for each $q$ is a global diffeomorphism of $\mathbb{R}^d$.*

PROOF. The equation $F(q) = \theta$ can be rewritten

$$\theta - (F(q) - q) = q$$

And the function on the left is contracting. □

**Proposition 46.** *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a $C^1$, $c$-monotone map on $\mathbb{R}^d$, with $c > 0$. Then $F$ is a diffeomorphism from $\mathbb{R}^d$ onto itself.*

PROOF. Let us consider a point $\theta \in \mathbb{R}^d$, and the line $\theta(s) = F(0) + s(\theta - F(0))$. Since $F$ is a local diffeomorphism around 0, the points $\theta(s)$ for small $s$ have a unique preimage $p(s)$. Let $S$ be the infimum of the positive real numbers $s$ such that the equation $F(p) = \theta(s)$ does not have a solution in $\mathbb{R}^d$. The curve $p(s)$ is well-defined, $C^1$, and Lipschitz on $[0, S[$, hence, if $S$ is finite, it extends at $S$ with with $F(p(S)) = \theta(S)$. Since $F$ is a local diffeomorphism at $p(S)$, the points near $\theta(S)$ have preimages, which contradicts the definition of $S$. Hence $S$ can't be finite. □

**Lemma 47.** *Let $A$ be a $d \times d$ matrix, such that $A \geqslant aId$ in the sense of quadratic forms, and $\|A\| \leqslant b$. Then $A^{-1} \geqslant (a/b^2)I$ in the sense of quadratic forms.*

PROOF. We have

$$(A^{-1}v, v) = (AA^{-1}v, A^{-1}v) \geqslant a|A^{-1}v|^2 \geqslant a(|v|/b)^2.$$

□

The following important result appears in Fathi's book on Weak KAM theory (the proof is also his):

**Proposition 48.** *Let $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function and $K$ be a positive number. Let $\mathcal{I} \in \mathbb{R}^d$ be the set of points where $u$ has bothe a $K$-superdifferential and a $K$-subdifferential. Then, the function $u$ is differentiable at each point of $\mathcal{I}$ and the function $q \longmapsto du(q)$ is $6K$-Lipschitz on $\mathcal{I}$.*

PROOF. For each $q \in \mathcal{I}$, there exists a unique $l(q) \in \mathbb{R}^{d*}$ such that

$$|u(q + \theta) - u(q) - l(q) \cdot \theta| \leqslant K\|\theta\|^2.$$

We conclude that $l(q)$ is the differential of $u$ at $q$, and we have to prove that the map $q \longmapsto l(q)$ is Lipschitz on $\mathcal{I}$. We have, for each $q$, $\theta$ and $y$ in $H$:

$$l(q) \cdot (y + \theta) - K\|y + \theta\|^2 \leqslant u(q + y + \theta) - u(q) \leqslant l(q) \cdot (y + \theta) + K\|y + \theta\|^2$$

$$l(q + y) \cdot (-y) - K\|y\|^2 \leqslant u(q) - u(q + y) \leqslant l(q + y) \cdot (-y) + K\|y\|^2$$

$$l(q + y) \cdot (-\theta) - K\|\theta\|^2 \leqslant u(q + y) - u(q + y + \theta) \leqslant l(q + y) \cdot (-\theta) + K\|\theta\|^2.$$

Taking the sum, we obtain

$$\big|(l(q + y) - l(q)) \cdot (y + \theta)\big| \leqslant K\|y + \theta\|^2 + K\|y\|^2 + K\|z\|^2.$$

By a change of variables, we get

$$\left|(l(q+y)-l(q))\cdot\theta\right| \leqslant K\|\theta\|^2 + K\|y\|^2 + K\|\theta - y\|^2.$$

Taking $\|\theta\| = \|y\|$, we obtain

$$\left|(l(q+y)-l_(q))\cdot(\theta)\right| \leqslant 6K\|\theta\|\|y\|$$

for each $\theta$ such that $\|\theta\| = \|y\|$, we conclude that

$$\|l(q+y)-l(q)\| \leqslant 6K\|y\|.$$

□

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