Tutorial 4 — div, grad and curl: applications and 2nd order operators

1. Show that $\underline{f}(x,y,z) = (yz, xz, xy)$ is irrotational, and find a scalar field $\phi(x,y,z)$ such that that $\underline{f} = \operatorname{grad}\phi$. Can you find another $\phi(x,y,z)$ such that $\underline{f} = \operatorname{grad}\phi$ still holds?

2. Show that if the scalar field $\phi(x,y,z)$ is a solution of Laplaces equation $\nabla^2\phi = 0$, then $\operatorname{grad}\phi$ is both irrotational and solenoidal.

3. Suppose $\phi(x,y,z)$ is a scalar field, and $\underline{f}(x,y,z)$ is a vector field. Prove the following:

(i) $\operatorname{curl}(\operatorname{grad}(\phi)) = 0$

(ii) $\operatorname{curl}(\operatorname{curl}(\underline{f})) = \operatorname{grad}(\operatorname{div}(\underline{f})) - \nabla^2\underline{f}$

(iii) $\nabla \times \underline{f} = \underline{i} \times \frac{\partial \underline{f}}{\partial x} + \underline{j} \times \frac{\partial \underline{f}}{\partial y} + \underline{k} \times \frac{\partial \underline{f}}{\partial z}$

4. Maxwell's equations in a vacuum are

$$\operatorname{div}\underline{E} = \rho/\varepsilon_0 \qquad \operatorname{div}\underline{B} = 0$$

$$\operatorname{curl}\underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \operatorname{curl}\underline{B} = \mu_0\underline{J} + \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t}$$

It is possible to write $\underline{B} = \operatorname{curl}\underline{A}$ and $\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \operatorname{grad}V$. $\underline{A}$ and V are called the vector and scalar potentials, and are related via $\operatorname{div}\underline{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}$. Show that these definitions are consistent with Maxwell's equations if we have

$$-\nabla^2\underline{A} + \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = \mu_0\underline{J} \quad \text{and} \quad -\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \rho/\varepsilon_0$$

where $c^2 = (\mu_0\varepsilon_0)^{-1}$.