

1.1. Write down the general second order linear PDE for a function $u = u(x, y)$ of two variables.

Define what it means for the equation to be

(i) elliptic, (ii) hyperbolic, (iii) parabolic.

What does it mean that a smooth curve Γ is a characteristic for the equation?

What is the connection between characteristics and the classification of the equation into (i), (ii) and (iii)?

Derive the differential equation for characteristics.

1.2. Classify the following equations considered on the whole plane \mathbb{R}^2 :

$$u_{xx} + u_{xy} + u_{yy} = 0 \quad (1)$$

$$4u_{xx} + 4u_{xy} + u_{yy} = 0 \quad (2)$$

$$u_{xx} - 4u_{xy} + u_{yy} = 0 \quad (3)$$

1.3. Find the characteristic curves to the PDE

$$\frac{\partial^2 u}{\partial x^2} + 5 \frac{\partial^2 u}{\partial x \partial y} + 6 \frac{\partial^2 u}{\partial y^2} = 0$$

considered on the plane \mathbb{R}^2 . Show that the general solution is

$$u(x, y) = F(y - 3x) + G(y - 2x),$$

where $F, G \in C^2(\mathbb{R})$ are arbitrary.

1.4. Find the characteristics for the PDE

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$$

considered in the half plane $x < y$. Show that the general solution is

$$u(x, y) = \frac{F(y^2 - x^2) + G(y - x)}{y - x},$$

where $F, G \in C^2((0, \infty))$ are arbitrary.

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1.5. Consider the Cauchy problem

$$u_{xy} = 0, \quad (x, y) \in \mathbb{R}^2,$$
$$\begin{cases} u(x, 0) = h(x) & x \in \mathbb{R} \\ u_y(x, 0) = k(x) & x \in \mathbb{R}, \end{cases}$$

where $h, k \in C^2(\mathbb{R})$ are given functions.

Show that no solution exists unless the function $k(x)$ is constant. Next show that if $k(x) = c$ (constant), then there are infinitely many solutions.

1.6. Reduce the equation

$$u_{xx} - y^2 u_{yy} - y u_y = 0, \quad y > 0, \tag{4}$$

to canonical form and show that its general solution is

$$u(x, y) = F(ye^x) + G(ye^{-x}),$$

where $F, G \in C^2((0, \infty))$ are arbitrary.

Let $f \in C^2((0, \infty))$. Solve the Cauchy problem for (4) with Cauchy data

$$\begin{cases} u(x, 1) = f(x) & x \in \mathbb{R} \\ u_y(x, 1) = 0 & x \in \mathbb{R}. \end{cases}$$

2.1. Consider the 1-D wave equation

$$u_{tt} - u_{xx} = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (1)$$

with the Cauchy data

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x) \quad \text{for} \quad x \in \mathbb{R}, \quad (2)$$

where $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ are given functions.

(a) Find the general solution to (1).

(b) Use the result in (a) to derive D'Alembert's solution formula for the Cauchy problem (1), (2).

(c) Solve the Cauchy problem (1), (2) when $f(x) = x^2$ and $g(x) = 4x^3$.

2.2. Reduce the equation

$$x(y-x)u_{xx} - (y^2-x^2)u_{xy} + y(y-x)u_{yy} + (y+x)(u_x - u_y) = 0 \quad (3)$$

considered in the half plane $x > y$ to canonical form and find its general solution.

Let $g \in C^1(\mathbb{R})$ be a given function. Show that the Cauchy problem for (3) with data

$$u(x, 1) = 0 \quad \text{and} \quad u_y(x, 1) = g(x) \quad \text{for} \quad x \geq 1, \quad (4)$$

does not admit a solution when $g(1) \neq 0$.

Suppose that $g(x) = (x-1)h(x)$, where $h \in C^1(\mathbb{R})$ is a given function. Find the solution(s) to the Cauchy problem (3), (4).

2.3. Consider the initial value problem:

$$u_{tt} - u_{xx} = 0, \quad (1)$$

$$u(x, 0) = f(x), \quad u_x(x, 0) = g(x). \quad (2)$$

a) Let $f(x) = 0$ and $g(x)$ be a smooth function, very close to the step-function $\Pi(x)$, equal 1 for $0 \leq x \leq 2$ and equal 0 otherwise. Draw graphs of solutions $x \rightarrow u(x, t)$ for $t = \frac{1}{2}, 1$ and 2. (HINT: To DRAW a graph of the solution you can replace in the D'Alembert formula the function $g(x)$ by $\Pi(x)$.)

b) Same question when $g = 0$ and f is very close to the Heaviside function $H(x)$, equal 0 for $x < 0$ and equal 1 for $x \geq 0$.

3.1. Consider the homogeneous 1-D heat equation:

$$u_t - u_{xx} = 0, \quad x \in \mathbb{R}, t > 0, \quad (1)$$

and let $k(x, t) = (4\pi t)^{-1/2} \exp(-x^2/(4t))$ denote the 1-D heat kernel.

(a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any bounded piecewise continuous function. Show that

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) f(y) dy \quad (2)$$

is a solution to (1). Next show that when $f(x)$ is continuous at $x_0 \in \mathbb{R}$, then

$$u(x_0, t) \rightarrow f(x_0) \text{ as } t \rightarrow 0^+.$$

(b) Assume that

$$f(x) = \begin{cases} 1/a & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise,} \end{cases}$$

where $a > 0$ is a positive constant. If $u(x, t)$ is given by (2) and

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

show that

$$u(x, t) = \frac{1}{2a} \left(\Phi\left(\frac{x}{\sqrt{4t}}\right) - \Phi\left(\frac{x-a}{\sqrt{4t}}\right) \right).$$

(c) Use the results in (a) and (b) to find the limits

$$\lim_{t \rightarrow 0^+} u(x, t) \quad \text{and} \quad \lim_{t \rightarrow \infty} u(x, t)$$

for each $x \in \mathbb{R}$.

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3.2. Let $f(x)$ be a bounded continuous function and consider the initial value problem

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = f(x) & x \in \mathbb{R}. \end{cases} \quad (3)$$

We have shown (e.g. in 6.1(a)) that

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) f(y) dy$$

is a solution to (3).

(a) Show that

$$|u_x(x, t)| \leq \frac{1}{\sqrt{\pi t}} \sup\{|f(y)| : y \in \mathbb{R}\}$$

for $x \in \mathbb{R}$ and $t > 0$.

(b) Show that

$$\int_{-\infty}^{\infty} k(x, t)^2 dx = \frac{1}{2\sqrt{2\pi t}}$$

for $t > 0$.

(c) Assume that $\int_{-\infty}^{\infty} f(y)^2 dy < \infty$. Use the Cauchy-Schwartz inequality to show that $|u(x, t)| = O(t^{-\frac{1}{4}})$ as $t \rightarrow \infty$ uniformly in $x \in \mathbb{R}$ (i.e., show that there exist constants $C > 0$ and $T > 0$, such that

$$|u(x, t)| \leq Ct^{-\frac{1}{4}}$$

holds for all $x \in \mathbb{R}$ and all $t \geq T$.)

4.1. State the maximum principle for the 1D heat equation. State the minimum principle, and explain how it follows from the maximum principle.

Next assume that $u(x, t)$ is continuous on $[-1, 1] \times [0, 2]$, C^2 on $(-1, 1) \times (0, 2)$ and $u_t - u_{xx} = 0$ on $(-1, 1) \times (0, 2)$. If $u(x, 0) = x^2 - 1$ for $|x| \leq 1$, $u(-1, t) = t/3$ and $u(1, t) = -t/4$ for $t \in [0, 2]$ find the maximum and the minimum of $u(x, t)$ on $[-1, 1] \times [0, 2]$.

4.2. Suppose that $u(x, t)$ is C^2 on $(0, a) \times (0, \infty)$ and continuous on $[0, a] \times [0, \infty)$, where $a > 0$. Show that if

$$u_t - u_{xx} = 0 \quad \text{in} \quad (0, a) \times (0, \infty)$$

and

$$A \leq u(x, t) \leq B \quad \text{on} \quad \{x = 0, t > 0\} \cup \{x = a, t > 0\} \cup \{0 \leq x \leq a, t = 0\},$$

then

$$A \leq u(x, t) \leq B \quad \text{in} \quad [0, a] \times [0, \infty).$$

4.3. Assume that $u(x, t)$ is continuous on $[0, 2] \times [0, \infty)$, C^2 on $(0, 2) \times (0, \infty)$ and that

$$\begin{cases} u_t = u_{xx} & \text{on } (0, 2) \times (0, \infty) \\ u(0, t) = f(t) \text{ and } u(2, t) = 0 & \text{for } t > 0 \\ u(x, 0) = 0 & \text{for } x \in [0, 2], \end{cases}$$

where $f = f(t) : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function with $f(0) = 0$.

(i) Show that if $0 \leq f(t) \leq 1$ for all $t \geq 0$, then $0 \leq u(x, t) \leq 1 - x/2$ on $[0, 2] \times [0, \infty)$.

(ii) Show that if $f(t) \geq 1 - \exp(-\frac{\pi^2 t}{16})$ for all $t \geq 0$, then

$$u(x, t) \geq 1 - \frac{x}{2} - \exp(-\frac{\pi^2 t}{16}) \cos(\frac{\pi x}{4})$$

on $[0, 2] \times [0, \infty)$.

5.1. Let Ω be a bounded and open subset of \mathbb{R}^n . A C^2 function $f: \Omega \rightarrow \mathbb{R}$ is called subharmonic if $\Delta f(x) \geq 0$ for all $x \in \Omega$.

(a) State and prove the maximum principle for subharmonic functions.

(b) State the minimum principle for Δ , and explain how it follows from the maximum principle. (A C^2 function $f: \Omega \rightarrow \mathbb{R}$ is called superharmonic if $\Delta f(x) \leq 0$ for all $x \in \Omega$.)

(c) Suppose that $\Omega = (-1, 1) \times (0, 3)$, that $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is harmonic in Ω and that $f(x, 0) = f(x, 3) = x^2 - 1$ for $x \in [-1, 1]$ and $f(-1, y) = y(y - 3)$, $f(1, y) = y^2(y - 3)^2$ for $y \in [0, 3]$.

Find maximum and minimum for f on $\bar{\Omega}$.

(d) Let Ω be an open and bounded subset of \mathbb{R}^n , and assume that $u, v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Show that if $\Delta v \geq 0$, $\Delta u = 0$ in Ω and $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω . (This is the reason for saying that a function v satisfying $\Delta v \geq 0$ is subharmonic.)

5.2. Let

$$u(x, y) = \frac{1}{4\pi} \ln(x^2 + y^2), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Show that $u \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$ and that $\Delta u = 0$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

5.3. Let Ω be a bounded and open subset of \mathbb{R}^n and let $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ satisfy $\Delta u = \alpha$ in Ω , where $\alpha \in \mathbb{R}$ is a constant. Show that if $w = |\nabla u|^2$, then

$$\Delta w = 2 \sum_{j,k=1}^n \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)^2.$$

Deduce that the maximum of $|\nabla u|$ is attained on $\partial\Omega$.

5.4. Let $f, g \in C^2(\Omega)$, where Ω is an open and connected subset of \mathbb{R}^n .

Show that

$$\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g.$$

Hence show that if $\Delta f = \Delta(f^2) = 0$ in Ω , then f is constant on Ω .

5.5 Let Ω be an open and bounded subset of \mathbb{R}^n with a smooth boundary (e.g., Ω could be a ball). Let $f: \Omega \rightarrow \mathbb{R}$ and $g: \partial\Omega \rightarrow \mathbb{R}$ be two continuous functions.

(a) Show that if $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial N} = g & \text{on } \partial\Omega \end{cases}$$

then

$$\int_{\Omega} f(x) dx = \int_{\partial\Omega} g(x) dS(x).$$

Hint: Use Green's first identity: for $u, v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$

$$\int_{\Omega} (v\Delta u + \nabla v \cdot \nabla u) dx = \int_{\partial\Omega} v \frac{\partial u}{\partial N} dS.$$

(b) If $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $\alpha \in \mathbb{R}$ is a constant, show that

$$\begin{cases} \Delta u = \alpha & \text{in } \Omega \\ \frac{\partial u}{\partial N} = 2 & \text{on } \partial\Omega \end{cases} \quad (1)$$

has no solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ unless $\alpha = 4$.

(c) By considering $u(x) = \lambda(x_1^2 + x_2^2)$, find a solution to (1) when $\alpha = 4$.

6.1. Calculate the Fourier series for the square wave

$$f(x) = 0 \text{ if } -2 < x < -1,$$

$$f(x) = k \text{ if } -1 < x < 1,$$

$$f(x) = 0 \text{ if } 1 < x < 2,$$

with $f(x+4) = f(x)$.

6.2. Calculate the Fourier series for the “sawtooth wave”

$$f(x) = x, \quad -\pi < x < \pi$$

with $f(x+2\pi) = f(x)$.

6.3. Calculate the odd half-range Fourier series for the function

$$f(x) = x, \quad 0 < x < L.$$

6.4. a) Find the general solution for the wave equation with zero boundary conditions:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad u(0, t) = u(L, t) = 0. \quad (1)$$

b) Find a solution $u(x, t)$ of (1) such that

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

where $f(x)$ is the function from Question 6.3.

7.1. Use separation of variables to solve the 1D heat equation

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad (1)$$

with boundary conditions

$$(a) \quad u(0, t) = 0, \quad u_x(1, t) = 0,$$

$$(b) \quad u_x(0, t) = 0, \quad u(1, t) = 0.$$

Give the general form of the solution in both cases.

7.2. Use separation of variables to solve the 1D wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1,$$

with boundary conditions

$$(a) \quad u(0, t) = 0, \quad u_x(1, t) = 0,$$

$$(b) \quad u_x(0, t) = 0, \quad u(1, t) = 0.$$

Give the general form of the solution in both cases.

7.3. Equation (1) with zero boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0 \quad t > 0,$$

has general solution

$$u(x, t) = \sum_{k=1}^{\infty} A_k e^{-k^2 \pi^2 \alpha^2 t} \sin(k\pi x).$$

Find the constants A_k for $u(x, t)$ to satisfy the initial conditions

$$u(x, 0) = \sin(12\pi x).$$

PDE Tutorial 8

8.1. Consider

$$u_t + f(u)_x = 0, \quad u(x, 0) = h(x), \quad x \in \mathbb{R},$$

where $f''(u) > 0$ for all u . Suppose that $y_1 < y_2$ and $h(y_1) > h(y_2)$. In lectures it was shown that the characteristics through $(y_1, 0)$ and $(y_2, 0)$ intersect at a point (\bar{x}, \bar{t}) . Find \bar{x} and \bar{t} .

8.2. Consider $u_t + uu_x = 0$ with $u(x, 0) = 1$ for $x \leq 0$, $u(x, 0) = 1 - x$ for $0 \leq x \leq 1$ and $u(x, 0) = 0$ for $x \geq 1$. Find the explicit solution for $0 < t < 1$.

8.3. Consider $u_t + uu_x = 0$. Show that the piecewise smooth solution $u(x, t) = -\frac{2}{3}(t + (3x + t^2)^{1/2})$ for $4x + t^2 > 0$ and $u(x, t) = 0$ for $4x + t^2 < 0$ satisfies the jump condition.

8.4. Consider

$$u_t + f(u)_x = 0. \tag{*}$$

If $f'(g(v)) = v$, show that the change of variables $u = g(v)$ reduces (*) to

$$v_t + vv_x = 0. \tag{**}$$

Why do we not study the simpler equation (**) instead of (*)?

8.5. Consider

$$u_t + uu_x = 0, \quad u(x, 0) = \sin(x), \quad x \in \mathbb{R}.$$

By considering $y_1 = \pi$ and $y_2 = \pi + \epsilon$ for small $\epsilon > 0$, show that there cannot be a smooth solution, defined for all x and all $0 < t < T$, where $T > 1$. [Hint: use Question 8.1].

8.6. Consider

$$u_t + uu_x = 0, \quad \text{where } u(x, 0) = -1 \text{ for } x < 0 \text{ and } u(x, 0) = 1 \text{ for } x > 0.$$

- Find a solution with a discontinuity. Is this solution admissible?
- Find a continuous solution. [Hint: use $\frac{x}{t}$].

9.1. Let $f \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Consider the initial value problem

$$\begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R}, t > 0, & (1) \\ u(x, 0) = h(x), & x \in \mathbb{R}. & (2) \end{cases}$$

(a) Suppose that $u = u(x, t)$ is a C^1 solution of (1)-(2) for $x \in \mathbb{R}$ and $0 < t < T$, where $0 < T \leq \infty$. Show that the level curves for $u = u(x, t)$ are line segments (half-lines when $T = \infty$).

(b) Consider the initial value problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x, 0) = h_i(x) \end{cases}$$

where

$$h_1(x) = \begin{cases} 2 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0, \end{cases} \quad h_2(x) = x \quad \text{and} \quad h_3(x) = -x \quad (x \in \mathbb{R}).$$

(i=1) Show that there exists no C^1 solution.

(i=2) Find the C^1 solution $u = u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$.

(i=3) Find the C^1 solution $u = u(x, t)$, $x \in \mathbb{R}$, $0 \leq t < 1$, and show that there exists no C^1 solution defined for $t \geq 1$.

9.2. Corresponding to the nonlinear conservation law (1) we have the integrated nonlinear conservation law

$$\frac{d}{dt} \int_a^b u(x, t) dx + f(u(b, t)) - f(u(a, t)) = 0 \quad (2)$$

where $a, b \in \mathbb{R}$ and $t > 0$.

(a) Let $\Omega \subseteq \mathbb{R} \times (0, \infty)$ be an open set. Show that if $u \in C^1(\Omega)$, then u satisfies (1) in Ω if and only if u satisfies (2) for $a, b \in \mathbb{R}$, $t > 0$, such that $[a, b] \times \{t\} \subset \Omega$.

(b) What does it mean that a function $u = u(x, t)$ is admissible? State the definition of a weak solution of (1).

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(c) Let $\Gamma : x = \gamma(t)$ be a smooth curve in the upper half-plane and suppose that $u \in C^1(\mathbb{R} \times (0, \infty) \setminus \Gamma)$ is admissible. Put

$$u^+(t) = \lim_{x \rightarrow \gamma(t)^+} u(x, t) \quad \text{and} \quad u^-(t) = \lim_{x \rightarrow \gamma(t)^-} u(x, t).$$

Prove that $u = u(x, t)$ is a weak solution to (1) if and only if

- (i) u satisfies (1) in $\mathbb{R} \times (0, \infty) \setminus \Gamma$, and
- (ii) on Γ ,

$$\frac{d\gamma}{dt} = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

holds.

(d) Find a weak solution to the initial value problem

$$\begin{cases} u_t + u^3 u_x = 0 & \text{for } x \in \mathbb{R}, t > 0 \\ u(x, 0) = h(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where

$$h(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0. \end{cases}$$

Hint: $f(u) = \frac{1}{4}u^4$.