(1.1)

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g,$$

where $a = a(x,y), \dots, g = g(x,y)$ are given functions.

Elliptic if $b^2 - ac < 0$,
hyperbolic if $b^2 - ac > 0$,
parabolic if $b^2 - ac = 0$.

Γ is a characteristic if for any given values of u_x, u_y on Γ it is <u>not</u> possible to determine u_{xx}, u_{xy}, u_{yy} uniquely on Γ.

Elliptic : there are <u>no</u> characteristics.
Hyperbolic: there are <u>two</u> families of characteristics.
Parabolic: there is <u>one</u> family of characteristics.

ODE for characteristics: suppose we are given u_x, u_y on the graph of $y = y(x)$, say

$$u_x(x, y(x)) = h(x), \quad u_y(x, y(x)) = k(x),$$

where $h(x), k(x)$ are given C^1 functions. If we differentiate these identities with respect to x it follows from the chain rule that

$$\begin{aligned} u_{xx} + u_{xy}\frac{dy}{dx} &= h'(x) \\ u_{xy} + u_{yy}\frac{dy}{dx} &= k'(x) \\ a u_{xx} + 2b u_{xy} + c u_{yy} &= \ell(x), \end{aligned}$$

where u_{xx}, u_{xy}, u_{yy} are evaluated at $(x, y(x))$ and

$\ell(x) = g - d u_x - e u_y - f u$ evaluated at $(x, y(x))$.

A unique solution (u_{xx}, u_{xy}, u_{yy}) exists regardless of the values of $h'(x), k'(x), \ell(x)$ if and only if

$$\begin{vmatrix} 1 & \frac{dy}{dx} & 0 \\ 0 & 1 & \frac{dy}{dx} \\ a & 2b & c \end{vmatrix} \neq 0,$$

i.e.

$$c + a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} \neq 0.$$

$\therefore$ the graph of $y = y(x)$ is a characteristic exactly when

$$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0.$$

When $b^2 - ac \geq 0$ and $a \neq 0$ this is the same as

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

(1.2) (1) $a=1$, $b=\frac{1}{2}$, $c=1$, so $b^2-ac=\frac{1}{4}-1=-\frac{3}{4}<0$ $\therefore$ elliptic.

(2) $a=4$, $b=2$, $c=1$, so $b^2-ac=4-4=0$ $\therefore$ parabolic.

(3) $a=1$, $b=-2$, $c=1$, so $b^2-ac=4-1=3>0$ $\therefore$ hyperbolic.

(1.3) ODE for characteristics: $a(y')^2-2by'+c=0$

i.e. $(y')^2-5y'+6=0$ i.e.

$y'=3$ or $y'=2$, so char's are

$y-3x=c_1$, $y-2x=c_2$ $(c_1,c_2\in\mathbb{R})$.

Let $\begin{cases}\varphi = y-3x\\ \psi = y-2x\end{cases}$, $\tilde{u}(\varphi,\psi)=u(x,y)$.

$$u_x = \tilde{u}_\varphi\varphi_x+\tilde{u}_\psi\psi_x = -3\tilde{u}_\varphi-2\tilde{u}_\psi$$

$$u_{xx} = -3(\tilde{u}_{\varphi\varphi}\varphi_x+\tilde{u}_{\varphi\psi}\psi_x)-2(\tilde{u}_{\psi\varphi}\varphi_x+\tilde{u}_{\psi\psi}\psi_x)$$

$$= -3(-3\tilde{u}_{\varphi\varphi}-2\tilde{u}_{\varphi\psi})-2(-3\tilde{u}_{\psi\varphi}-2\tilde{u}_{\psi\psi})$$

$$= 9\tilde{u}_{\varphi\varphi}+12\tilde{u}_{\varphi\psi}+4\tilde{u}_{\psi\psi}$$

$$u_{xy} = -3(\tilde{u}_{\varphi\varphi}\varphi_y+\tilde{u}_{\varphi\psi}\psi_y)-2(\tilde{u}_{\psi\varphi}\varphi_y+\tilde{u}_{\psi\psi}\psi_y)$$

$$= -3\tilde{u}_{\varphi\varphi}-5\tilde{u}_{\varphi\psi}-2\tilde{u}_{\psi\psi}$$

$$u_y = \tilde{u}_\varphi\varphi_y+\tilde{u}_\psi\psi_y = \tilde{u}_\varphi+\tilde{u}_\psi$$

$$u_{yy} = \tilde{u}_{\varphi\varphi}+2\tilde{u}_{\varphi\psi}+\tilde{u}_{\psi\psi}.$$

Thus: $0 = u_{xx}+5u_{xy}+6u_{yy} =$

$$(9-3\cdot5+6)\tilde{u}_{\varphi\varphi}+(12-5\cdot5+6\cdot2)\tilde{u}_{\varphi\psi}+(4-5\cdot2+6)\tilde{u}_{\psi\psi}$$

so $0=\tilde{u}_{\varphi\psi}$. GS is $\tilde{u}=F(\varphi)+G(\psi)$,

so

$$u(x,y) = F(y-3x)+G(y-2x),$$

where $F,G\in C^2(\mathbb{R})$ arbitrary.

(1.4) ODE for char.'s: $a\left(\frac{dy}{dx}\right)^2-2b\frac{dy}{dx}+c=0$,

i.e. $y\left(\frac{dy}{dx}\right)^2-(x+y)\frac{dy}{dx}+x=0$ i.e.

$$\frac{dy}{dx} = \frac{x+y\pm\sqrt{(x+y)^2-4xy}}{2y} = \frac{x+y\pm|x-y|}{2y} \overset{x<y}{=}$$

$$\frac{x+y\pm(x-y)}{2y},\ \text{so}\ \frac{dy}{dx}=\frac{x}{y}\ \text{or}\ \frac{dy}{dx}=1.$$

$$\frac{dy}{dx}=1 \iff y-x=c_1 \quad (c_1\in\mathbb{R})$$

$$\frac{dy}{dx}=\frac{x}{y} \overset{c\in\mathbb{R}}{\iff} \int y\,dy=\int x\,dx+c \iff$$

$y^2-x^2=\frac{1}{2}c$ $\therefore$ Char.'s: $y-x=c_1$, $y^2-x^2=c_2$ $(c_1,c_2\in\mathbb{R})$

Let $\begin{cases}\varphi = y-x\\ \psi = y^2-x^2\end{cases}$, $\tilde{u}(\varphi,\psi)=u(x,y)$.

Then $u_x=-\tilde{u}_\varphi-2x\tilde{u}_\psi$, $u_y=\tilde{u}_\varphi+2y\tilde{u}_\psi$

$$u_{xx} = \tilde{u}_{\varphi\varphi}+4x\tilde{u}_{\varphi\psi}+4x^2\tilde{u}_{\psi\psi}-2\tilde{u}_\psi$$

$$u_{yy} = \tilde{u}_{\varphi\varphi}+4y\tilde{u}_{\varphi\psi}+4y^2\tilde{u}_{\psi\psi}+2\tilde{u}_\psi$$

$$u_{xy} = -\tilde{u}_{\varphi\varphi}-2(x+y)\tilde{u}_{\varphi\psi}-4xy\tilde{u}_{\psi\psi}$$

$$yu_{xx}+(x+y)u_{xy}+xu_{yy} =$$

$$(y-(x+y)+x)\tilde{u}_{\varphi\varphi}+(4xy+4xy-2(x+y)^2)\tilde{u}_{\varphi\psi}$$

$$+(4x^2y-4xy(x+y)+4xy^2)\tilde{u}_{\psi\psi}-2y\tilde{u}_\psi+2x\tilde{u}_\psi=$$

$$-2(x^2+y^2-2xy)\,\tilde{u}_{\varphi\psi} + 2(x-y)\tilde{u}_\psi = -2(x-y)^2\,\tilde{u}_{\varphi\psi} + 2(x-y)\tilde{u}_\psi ,$$

hence: $\varphi\,\tilde{u}_{\varphi\psi} + \tilde{u}_\psi = 0$

$0 = \varphi\,\tilde{u}_{\varphi\psi} + \tilde{u}_\psi = (\varphi\,\tilde{u}_\varphi + \tilde{u})_\psi$, so

$\varphi\,\tilde{u}_\varphi + \tilde{u} = F_1(\varphi)$, where $F_1(\varphi)$ arbitrary.

$F_1(\varphi) = \varphi\,\tilde{u}_\varphi + \tilde{u} = (\varphi\,\tilde{u})_\varphi$, so

$\varphi\,\tilde{u} = \int F_1(\varphi)\,d\varphi + G_1(\psi)$ where $G_1(\psi)$ arbitrary.

GS: $\tilde{u} = F(\varphi) + \frac{1}{\varphi}G(\psi)$, $F, G \in C^2((0,\infty))$,

i.e. $u = F(y-x) + \dfrac{G(y^2-x^2)}{y-x}$, $F, G \in C^2((0,\infty))$.

(Because F, G are arbitrary this is the same GS as the one on problem sheet: for $\bar{F}, \bar{G} \in C^2((0,\infty))$ let $F(t) = \frac{\bar{G}(t)}{t}$, $G(t) = \bar{F}(t)$.)

(1.5) From lectures: GS $u = F(x) + G(y)$,

so $u_y = G'(y)$, i.e. $u_y(x,0) = G'(0)$.

If u satisfies the Cauchy conditions, then in particular

$$G'(0) = u_y(x,0) = k(x)$$

for all x, so k has to be constant.

Suppose $k(x) \equiv c$. Let $F(x) = h(x) - G(0)$, where G is any C^2 function such that $G'(0) = c$.

Then $u = h(x) - G(0) + G(y)$ is a solution,

$\therefore$ infinitely many solutions.

(1.6) ODE for char.'s: $a(y')^2 - 2by' + c = 0$,

i.e. $(y')^2 - y^2 = 0$, so $y' = y$ or $y' = -y$, i.e. $y = c_1 e^x$, $y = c_2 e^{-x}$ $(c_1, c_2 \in \mathbb{R})$

Let $\begin{cases} \varphi = ye^x \\ \psi = ye^{-x} \end{cases}$, $\tilde{u}(\varphi,\psi) = u(x,y)$.

$u_x = ye^x\,\tilde{u}_\varphi - ye^{-x}\,\tilde{u}_\psi$, $u_y = e^x\,\tilde{u}_\varphi + e^{-x}\,\tilde{u}_\psi$

$u_{xx} = ye^x\,\tilde{u}_\varphi + y^2e^{2x}\,\tilde{u}_{\varphi\varphi} - 2y^2\,\tilde{u}_{\varphi\psi} + y^2e^{-2x}\,\tilde{u}_{\psi\psi} + ye^{-x}\,\tilde{u}_\psi$

$u_{yy} = e^{2x}\,\tilde{u}_{\varphi\varphi} + 2\tilde{u}_{\varphi\psi} + e^{-2x}\,\tilde{u}_{\psi\psi}$,

so $0 = u_{xx} - y^2u_{yy} - yu_y =$

$(y^2e^{2x} - y^2e^{2x})\tilde{u}_{\varphi\varphi} + (-2y^2 - 2y^2)\tilde{u}_{\varphi\psi} + (y^2e^{-2x} - y^2e^{-2x})\tilde{u}_{\psi\psi} + (ye^x - ye^x)\tilde{u}_\varphi + (ye^{-x} - ye^{-x})\tilde{u}_\psi = -4y^2\,\tilde{u}_{\varphi\psi}$, i.e.

$\tilde{u}_{\varphi\psi} = 0$. GS $\tilde{u} = F(\varphi) + G(\psi)$, so

$u = F(ye^x) + G(ye^{-x})$, $F, G \in C^2((0,\infty))$

$u_y = F'(ye^x)e^x + G'(ye^{-x})e^{-x}$, so

$$\left.\begin{aligned} f(x) &= u(x,1) = F(e^x) + G(e^{-x}) \\ 0 &= u_y(x,1) = F'(e^x)e^x + G'(e^{-x})e^{-x} \end{aligned}\right\}$$

Now $F'(e^x)e^x + G'(e^{-x})e^{-x} = \frac{d}{dx}(F(e^x) - G(e^{-x}))$

so $F(e^x) - G(e^{-x}) = F(1) - G(1)$, i.e.

$F(e^x) = \frac{1}{2}f(x) + \frac{1}{2}(F(1) - G(1))$, or: $F(t) = \frac{1}{2}f(\log t) + \frac{F(1)-G(1)}{2}$

Similarly: $G(t) = \frac{1}{2}f(-\log t) - \frac{F(1)-G(1)}{2}$, so

$$u(x,y) = \frac{1}{2}\big(f(x + \log y) + f(x - \log y)\big)$$

PDE TUTORIAL 2: SOLUTIONS

(2.1) (a) & (b): see lecture notes.

(c): insert in D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left((x+t)^2 + (x-t)^2 + (x+t)^4 - (x-t)^4\right).$$

(2.2) $a = x(y-x)$, $b = -\frac{1}{2}(y^2 - x^2)$,

$c = y(y-x)$, so $b^2 - ac = \frac{1}{4}(y-x)^4 > 0$

for $x > y$.

ODE's for char.'s: $\dfrac{dy}{dx} = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$,

i.e. (i) $\dfrac{dy}{dx} = \dfrac{x^2 - xy}{x(y-x)}$ or (ii) $\dfrac{dy}{dx} = \dfrac{-y^2 + xy}{x(y-x)}$.

(i) $\Leftrightarrow \dfrac{dy}{dx} = -1 \Leftrightarrow y = -x + c_1$

(ii) $\underset{x \neq 0}{\Leftrightarrow} \dfrac{dy}{dx} = -\dfrac{y}{x} \underset{x \neq 0}{\Leftrightarrow} \displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \Leftrightarrow$

$\log|y| = -\log|x| + c \Leftrightarrow \log|xy| = c \Leftrightarrow$

$|xy| = e^c \Leftrightarrow xy = c_2$, $c_2 = \pm e^c$.

$y = 0$ is also a solution, so $c_2 = 0$ admissible.
(note that $x = 0$ is also a char.)

Put $\begin{cases} \varphi = x + y \\ \psi = xy \end{cases}$, $\tilde{u}(\varphi, \psi) = u(x,y)$.

$u_x = \tilde{u}_\varphi + y\tilde{u}_\psi$, $u_y = \tilde{u}_\varphi + x\tilde{u}_\psi$,

$u_{xx} = \tilde{u}_{\varphi\varphi} + 2y\tilde{u}_{\varphi\psi} + y^2\tilde{u}_{\psi\psi}$,

$u_{xy} = \tilde{u}_{\varphi\varphi} + (x+y)\tilde{u}_{\varphi\psi} + xy\tilde{u}_{\psi\psi} + \tilde{u}_\psi$,

$u_{yy} = \tilde{u}_{\varphi\varphi} + 2x\tilde{u}_{\varphi\psi} + x^2\tilde{u}_{\psi\psi}$

Inserting in (3) yields: $\tilde{u}_{\varphi\psi} = 0$

which is the canonical form. Its GS is

$\tilde{u} = F(\varphi) + G(\psi)$, so GS for (3) is

$u = F(x+y) + G(xy)$, $F, G \in C^2(\mathbb{R})$.

If $u(x,y)$ is a solution to (3) satisfying the Cauchy conditions $u(x,1) = 0$ and $u_y(x,1) = g(x)$ for $x \geq 1$, then

(*) $F(x+1) + G(x) = 0$ for $x \geq 1$, and

(**) $F'(x+1) + xG'(x) = g(x)$ for $x \geq 1$.

Diff. (*) to get: (+) $F'(x+1) + G'(x) = 0$ for $x \geq 1$, i.e. for $x = 1$: $F'(2) + G'(1) = 0$.

Taking $x = 1$ in (**) yields $F'(2) + G'(1) = g(1)$, so we must have that $g(1) = 0$. Conclusion: if $g(1) \neq 0$ there is <u>no</u> solution!

Suppose $g(x) = (x-1)h(x)$.

(**) − (+): $(x-1)G'(x) = (x-1)h(x)$, so

$G'(x) = h(x)$ and therefore

$$G(x) = G(x_0) + \int_{x_0}^{x} h(t)\,dt.$$

From (*): $F(x+1) = -G(x) = -G(x_0) - \displaystyle\int_{x_0}^{x} h(t)\,dt$,

so $F(x) = -G(x_0) - \displaystyle\int_{x_0}^{x-1} h(t)\,dt$.

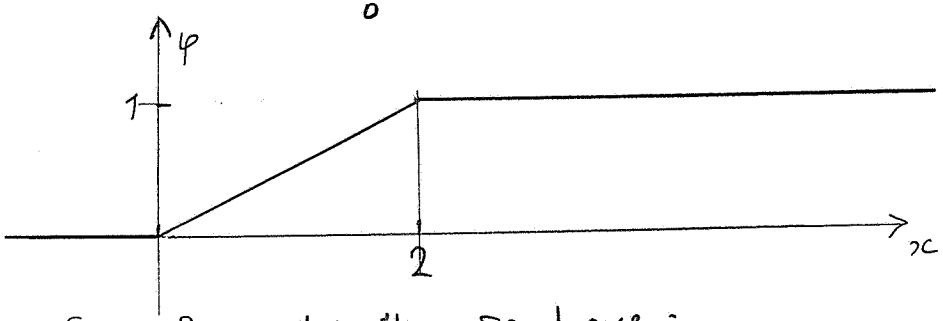
Hence PS is

$$u(x,y) = -G(x_0) - \int_{x_0}^{x+y-1} h(t)\,dt + G(x_0) + \int_{x_0}^{xy} h(t)\,dt$$
$$= \int_{x+y-1}^{xy} h(t)\,dt.$$

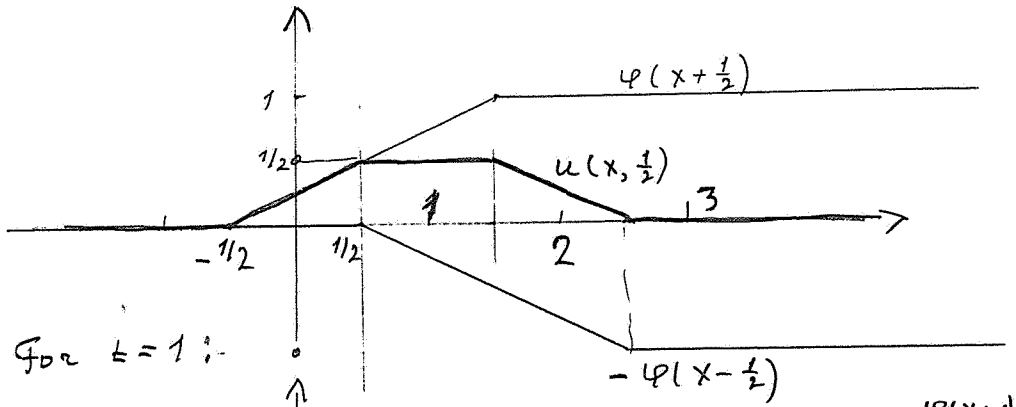
2.3. $u(x,t) \approx \frac{1}{2}\int_{x-t}^{x+t} \Pi(s)\,ds = \varphi(x+t) - \varphi(x-t)$,

where

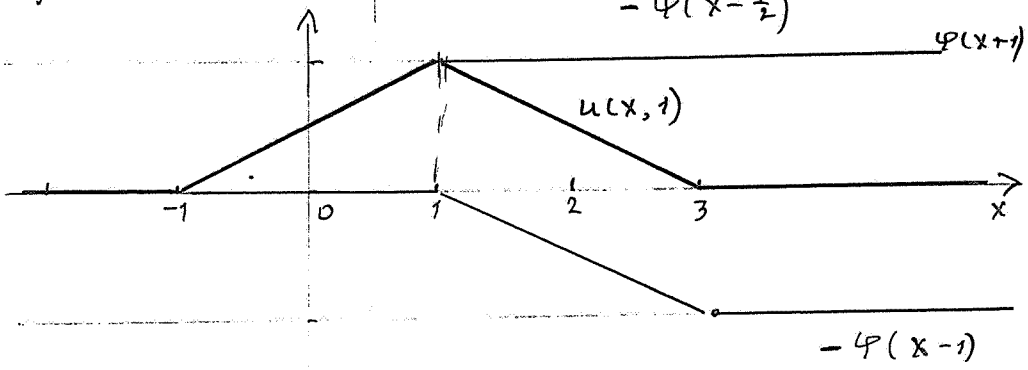
$$\varphi(x) = \frac{1}{2}\int_0^x \Pi(s)\,ds,$$

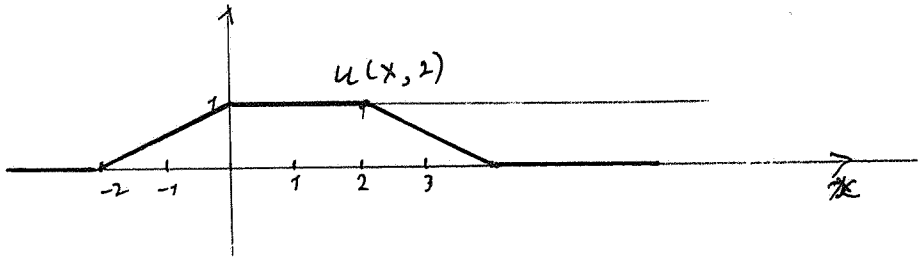


So, for $t = 1/2$ we have:

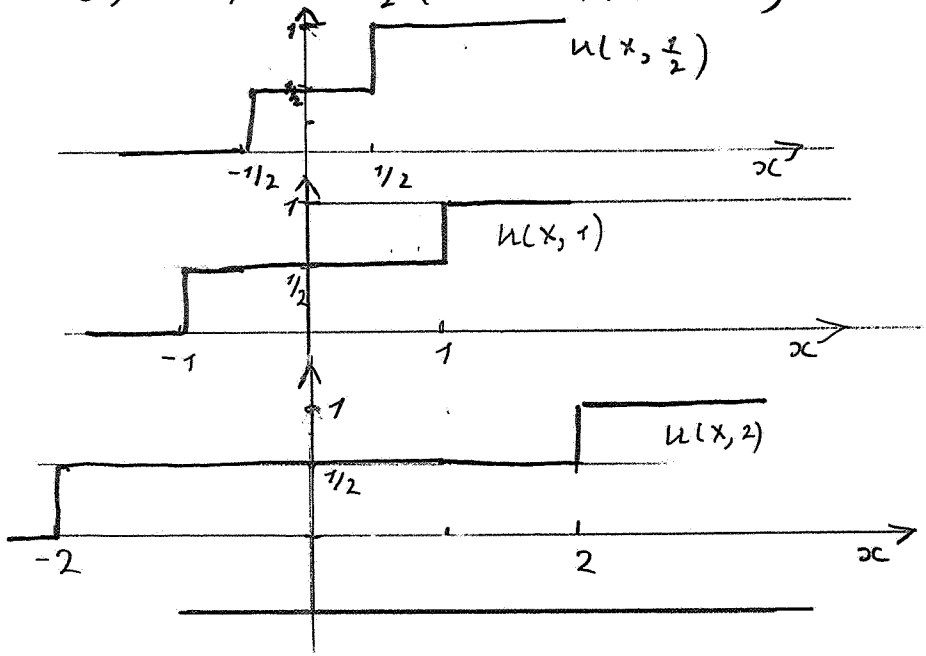


For $t = 1$:





b) $u(x,t) \approx \frac{1}{2}\left(H(x-t) + H(x+t)\right)$



PDE TUTORIAL 3 : SOLUTIONS

3.1 (a) Because $f(x)$ is bounded,

$$u_t(x,t) = \int_{-\infty}^{\infty} k_t(x-y,t)\, f(y)\, dy,$$

$$u_{xx}(x,t) = \int_{-\infty}^{\infty} k_{xx}(x-y,t)\, f(y)\, dy,$$

so $$u_t(x,t) - u_{xx}(x,t) = \int_{-\infty}^{\infty} \Big(k_t(x-y,t) - k_{xx}(x-y,t)\Big) f(y)\, dy.$$

$$k_t(x,t) = \dots = \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right) k(x,t)$$

$$k_x(x,t) = \dots = -\frac{x}{2t}\, k(x,t)$$

$$k_{xx}(x,t) = \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right) k(x,t)$$

$$\therefore\ k_t(x-y,t) - k_{xx}(x-y,t) =$$

$$\left(-\frac{1}{2t} + \frac{x^2}{4t^2} - \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right)\right) k(x-y,t) = 0,$$

so $u_t(x,t) - u_{xx}(x,t) = 0$ for $x \in \mathbb{R}, t>0$.

Suppose $f(x)$ is continuous at x_0, i.e. $\lim_{x\to x_0} f(x) = f(x_0)$. Then

$$u(x_0,t) = \int_{-\infty}^{\infty} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x_0-y)^2}{4t}} f(y)\, dy \overset{z=\frac{x_0-y}{\sqrt{4t}}}{=}$$

$$\int_{\infty}^{-\infty} (4\pi t)^{-\frac{1}{2}} e^{-z^2} f(x_0 - \sqrt{4t}\, z)\cdot(-\sqrt{4t})\, dz =$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x_0 - \sqrt{4t}\, z)\, dz,$$ so

$$\lim_{t\to 0^+} u(x_0,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x_0 - \sqrt{4t}\, z)\, dz$$

$$\underset{\text{because } f(x) \text{ is bounded}}{=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \lim_{t\to 0^+} \left(e^{-z^2} f(x_0 - \sqrt{4t}\, z)\right) dz \underset{\text{because } f(x) \text{ is continuous}}{=}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x_0)\, dz = f(x_0)$$

because $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$.
(see e.g. lecture notes)

(b) $$u(x,t) = \int_0^a (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \frac{1}{a}\, dy \overset{z=\frac{x-y}{\sqrt{4t}}}{=}$$

$$\int_{\frac{x}{\sqrt{4t}}}^{\frac{x-a}{\sqrt{4t}}} (4\pi t)^{-\frac{1}{2}} e^{-z^2} \frac{1}{a} \left(-\sqrt{4t}\right) dz =$$

$$\frac{1}{a\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4t}}}^{\frac{x}{\sqrt{4t}}} e^{-z^2} dz = \frac{1}{a\sqrt{\pi}} \left(\int_{\frac{x-a}{\sqrt{4t}}}^{0} e^{-z^2} dz + \int_0^{\frac{x}{\sqrt{4t}}} e^{-z^2} dz \right)$$

$$= \frac{1}{2a} \left(\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-z^2} dz - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-a}{\sqrt{4t}}} e^{-z^2} dz \right) =$$

$$\frac{1}{2a} \left(\Phi\left(\frac{x}{\sqrt{4t}}\right) - \Phi\left(\frac{x-a}{\sqrt{4t}}\right) \right).$$

(c) Observe that $\lim_{x\to 0} \Phi(x) = \Phi(0) = 0$,

$$\lim_{x\to -\infty} \Phi(x) = -\lim_{x\to\infty} \Phi(x) = -1.$$

Because $f(x)$ is continuous for $x \neq 0$ and $x \neq a$ it follows from (a) that

$$\lim_{t\to 0^+} u(x,t) = \begin{cases} \frac{1}{a} & \text{for } 0<x<a \\ 0 & \text{for } x<0 \text{ or } x>a. \end{cases}$$

$$\lim_{t\to 0^+} u(0,t) = \lim_{t\to 0^+} \frac{1}{2a} \left(\Phi(0) - \Phi\left(\frac{-a}{\sqrt{4t}}\right) \right) =$$

$$-\frac{1}{2a} \lim_{t\to 0^+} \Phi\left(\frac{-a}{\sqrt{4t}}\right) = \frac{1}{2a},$$

$$\lim_{t\to 0^+} u(a,t) = \lim_{t\to 0^+} \frac{1}{2a} \left(\Phi\left(\frac{a}{\sqrt{4t}}\right) - \Phi(0) \right) =$$

$$\frac{1}{2a}\lim_{t\to 0+}\Phi\left(\frac{a}{\sqrt{4t}}\right) = \frac{1}{2a}.$$

Thus:

$$\lim_{t\to 0+} u(x,t) = \begin{cases} \frac{1}{a} & \text{for} \quad 0<x<a \\ \frac{1}{2a} & \text{for} \quad x=0 \text{ or } x=a \\ 0 & \text{for} \quad x<0 \text{ or } x>a. \end{cases}$$

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\frac{1}{2a}\left(\Phi\left(\frac{x}{\sqrt{4t}}\right) - \Phi\left(\frac{x-a}{\sqrt{4t}}\right)\right) =$$

$$\frac{1}{2a}\left(\lim_{t\to\infty}\Phi\left(\frac{x}{\sqrt{4t}}\right) - \lim_{t\to\infty}\Phi\left(\frac{x-a}{\sqrt{4t}}\right)\right) = 0$$

3.2 (a) $u_x(x,t) = \int_{-\infty}^{\infty} k_x(x-y,t)\, f(y)\, dy$,

$k_x(x,t) = -\frac{x}{2t}(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$, so

$$|u_x(x,t)| \leq \int_{-\infty}^{\infty}\left|\frac{x-y}{2t}(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} f(y)\right| dy =$$

$$\int_{-\infty}^{\infty}\frac{1}{2t}(4\pi t)^{-\frac{1}{2}}|x-y|\, e^{-\frac{(x-y)^2}{4t}}|f(y)|\, dy \leq$$

$$\int_{-\infty}^{\infty}\frac{1}{2t}(4\pi t)^{-\frac{1}{2}}|x-y|\, e^{-\frac{(x-y)^2}{4t}}\, dy \cdot \sup_y |f(y)| \overset{z=\frac{x-y}{\sqrt{4t}}}{=}$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi t}}|z|\, e^{-z^2}\, dz \cdot \sup_y |f(y)| =$$

$$\frac{2}{\sqrt{\pi t}}\int_0^{\infty} z\, e^{-z^2} dz \cdot \sup_y |f(y)| = \frac{1}{\sqrt{\pi t}}\sup_y |f(y)|$$

since $\int_0^{\infty} z\, e^{-z^2} dz \underset{y=z^2,\ dy=2z\,dz}{=} \frac{1}{2}\int_0^{\infty} e^{-y} dy =$

$$\frac{1}{2}\left[-e^{-y}\right]_{y=0}^{y\to\infty} = \frac{1}{2}.$$

(b) $$\int_{-\infty}^{\infty} k(x,t)^2\, dx = \int_{-\infty}^{\infty}(4\pi t)^{-1} e^{-\frac{x^2}{2t}}\, dx \overset{y=\frac{x}{\sqrt{2t}}}{=}$$

$$\int_{-\infty}^{\infty}(4\pi t)^{-1} e^{-y^2}\sqrt{2t}\, dy = \frac{\sqrt{2}}{4\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-y^2} dy =$$

$$\frac{\sqrt{2}}{4\pi\sqrt{t}}\sqrt{\pi} = \frac{\sqrt{2\pi}}{2\cdot 2\pi\sqrt{t}} = \frac{1}{2\sqrt{2\pi t}}.$$

(c) $$|u(x,t)| = \left|\int_{-\infty}^{\infty} k(x-y,t)\, f(y)\, dy\right| \overset{\text{Cauchy-Schwartz}}{\leq}$$

$$\left(\int_{-\infty}^{\infty} k(x-y,t)^2\, dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} f(y)^2\, dy\right)^{\frac{1}{2}}.$$

Here $$\int_{-\infty}^{\infty} k(x-y,t)^2\, dy = \int_{-\infty}^{\infty} k(y,t)^2\, dy \overset{(b)}{=}$$

$\frac{1}{2\sqrt{2\pi t}}$, so $$|u(x,t)| \leq \left(\frac{1}{2\sqrt{2\pi t}}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} f(y)^2 dy\right)^{\frac{1}{2}}$$

$= c\cdot t^{-\frac{1}{4}}$, where

$$c = \left(\frac{1}{2\sqrt{2\pi}}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} f(y)^2\, dy\right)^{\frac{1}{2}}.$$

(4.1) Maximum principle: if

(1) $\Omega = (a,b) \subset \mathbb{R}^1$, $T>0$

$\Omega_T = \Omega \times (0,T)$, $\partial'\Omega_T = (\Omega\times\{0\}) \cup (\partial\Omega \times [0,T])$

(2) $u \in C^0(\overline{\Omega_T}) \cap C^2(\Omega_T)$, and

(3) $u_t - \Delta u \leq 0$ in Ω_T,

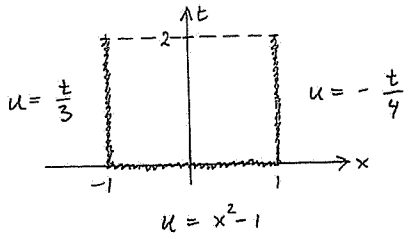
then $u(x,t) \leq \max\limits_{\partial'\Omega_T} u$ for $(x,t) \in \overline{\Omega_T}$.

The minimum principle follows from the maximum principle by substituting $-u$ for u. Thus:

Minimum principle: if (1), (2) and

(3') $u_t - \Delta u \geq 0$ in Ω_T, then

$u(x,t) \geq \min\limits_{\partial'\Omega_T} u$ for $(x,t) \in \overline{\Omega_T}$.



$n=1$, $\Omega = (-1,1)$, $T=2$

By minimum & maximum principle: $\min\limits_{\square} u \leq u(x,t) \leq \max\limits_{\square} u$

By inspection:

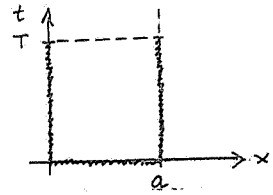
$\min\limits_{\square} u = u(0,0) = -1$,

$\max\limits_{\square} u = u(-1,2) = \frac{2}{3}$.

Thus: $\min\limits_{(x,t)\in[-1,1]\times[0,2]} u(x,t) = -1$,

$\max\limits_{(x,t)\in[-1,1]\times[0,2]} u(x,t) = \frac{2}{3}$.

(4.2) Let $T>0$.



By minimum & maximum principle

$\min\limits_{\square} u \leq u(x,t) \leq \max\limits_{\square} u$

for $(x,t) \in [0,a]\times[0,T]$.

By assumption $A \leq u \leq B$ on $\square$,

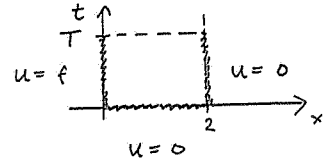
so $A \leq \min\limits_{\square} u$ and $\max\limits_{\square} u \leq B$, i.e.

$A \leq u(x,t) \leq B$ for $(x,t) \in [0,a]\times[0,T]$.

Because T was arbitrary it follows that $A \leq u(x,t) \leq B$ for $x \in [0,a]$, $t \geq 0$.

(4.3)

(i) Let $T>0$.



By minimum principle: $u(x,t) \geq \min\limits_{\square} u$ for $x \in [0,2]$, $t \in [0,T]$.

By inspection: $\min\limits_{\square} u = 0$, since $f(t) \geq 0$ by assumption.

To prove upper bound put $v(x,t) = u(x,t) - 1 + \frac{x}{2}$.

Observe: $v \in C^0([0,2]\times[0,T]) \cap C^2((0,2)\times(0,T))$

$v_t - v_{xx} = u_t - u_{xx} = 0$

$v(x,0) = u(x,0) - 1 + \frac{x}{2} = -1 + \frac{x}{2}$

$v(0,t) = u(0,t) - 1 + 0 = f(t) - 1$

$v(2,t) = u(2,t) - 1 + 1 = 0$.

By maximum principle: $v(x,t) \leq \max v$

By inspection: $\max v = 0$, since $f(t) \leq 1$ by assumption. Therefore

$u(x,t) \leq 1 - \frac{x}{2}$ for $x \in [0,2]$, $t \in (0,T]$.

Because $T > 0$ was arbitrary it follows that $0 \leq u(x,t) \leq 1 - \frac{x}{2}$ for $x \in [0,2]$, $t \geq 0$.

(ii) Let $T > 0$ and $v(x,t) = u(x,t) - 1 + \frac{x}{2} + e^{-\frac{\pi^2 t}{16}} \cos(\frac{\pi x}{4})$

Observe: $v \in C^0([0,2]\times[0,T]) \cap C^2((0,2)\times(0,T))$,

$$v_t - v_{xx} = u_t - \frac{\pi^2}{16} e^{-\frac{\pi^2 t}{16}} \cos(\tfrac{\pi x}{4}) - u_{xx} + e^{-\frac{\pi^2}{16}t} \cos(\tfrac{\pi x}{4}) \cdot (\tfrac{\pi}{4})^2 = 0,$$

$$v(x,0) = u(x,0) - 1 + \frac{x}{2} + \cos(\tfrac{\pi x}{4}) = -1 + \frac{x}{2} + \cos(\tfrac{\pi x}{4}),$$

$$v(0,t) = u(0,t) - 1 + 0 + e^{-\frac{\pi^2}{16}t} = f(t) - 1 + e^{-\frac{\pi^2}{16}t},$$

$$v(2,t) = u(2,t) - 1 + 1 + 0 = 0.$$

By minimum principle: $v(x,t) \geq \min v$.

$$\min v = \min\{\min_{0\leq x\leq 2} v(x,0), \min_{0\leq t\leq T} v(0,t), 0\},$$

$\min_{0\leq x\leq 2} v(x,0) = ?$

$v_x(x,0) = \frac{1}{2} - \frac{\pi}{4}\sin(\frac{\pi x}{4}) = 0$, i.e. if

$\sin(\frac{\pi x}{4}) = \frac{2}{\pi}$ .. Impossible to solve explicitly.

$v_{xx}(x,0) = -\frac{\pi^2}{16}\cos(\frac{\pi x}{4})$

For $x \in [0,2]$, $\frac{\pi x}{4} \in [0, \frac{\pi}{2}]$, so

$\cos(\frac{\pi x}{4}) > 0$ for $x \in (0,2)$, and

therefore $v_{xx}(x,0) < 0$ for $x \in (0, \frac{\pi}{2})$.

Consequently there is no minimum in $(0,2)$.

$v(0,0) = 0$, $v(2,0) = 0$, so

$\min_{0\leq x\leq 2} v(x,0) = 0$.

$\min_{0\leq t\leq T} v(0,t) = \min_{0\leq t\leq T} (f(t) - 1 + e^{-\frac{\pi^2}{16}t}) \geq 0$

by assumption. Thus: $\min v = 0$,

i.e. $u(x,t) \geq 1 - \frac{x}{2} - e^{-\frac{\pi^2}{16}t}\cos(\frac{\pi x}{4})$

for $x \in [0,2]$, $t \in [0,T]$. Because $T > 0$ was arbitrary the inequality holds for all $x \in [0,2]$, $t \geq 0$.

(5.1) (a): Maximum principle:

Assume: (1) $\Omega \subset \mathbb{R}^n$ open, bounded

(2) $f \in C^0(\overline{\Omega}) \cap C^2(\Omega)$

(3) $\Delta f \geqq 0$ in Ω

Then: $f(x) \leqq M$ for all $x \in \overline{\Omega}$, where $M = \max_{x \in \partial\Omega} f(x)$.

Proof. Step 1. Suppose (1), (2) and

($3^>$) $\Delta f > 0$ in Ω.

Take $x_0 \in \overline{\Omega}$ so $f(x_0) = \max_{x \in \overline{\Omega}} f(x)$ (it is possible because $f \in C^0(\overline{\Omega})$ and Ω is bounded) Assume that $x_0 \in \Omega$. Then it follows that $f_{x_i x_i}(x_0) \leqq 0$, $i = 1, \dots, n$, and hence that $\Delta f(x_0) \leqq 0$. But this is impossible by the assumption ($3^>$). Therefore $x_0 \in \overline{\Omega} \setminus \Omega = \partial\Omega$, and it follows that $f(x) \leqq \max_{\partial\Omega} f$ for $x \in \overline{\Omega}$ whenever (1), (2) and ($3^>$) hold.

Step 2. Assume (1), (2), (3). Let $\varepsilon > 0$ and put $v(x) = f(x) + \varepsilon x_1^2$, $x \in \overline{\Omega}$. Then $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $\Delta v = \Delta f + 2\varepsilon \geqq 2\varepsilon > 0$ in Ω. From Step 1 we therefore have: $v(x) \leqq \max_{\partial\Omega} v$ in $\overline{\Omega}$. Hence $f(x) \leqq v(x) \leqq \max_{\partial\Omega} v = \max_{\partial\Omega}(f(x) + \varepsilon x_1^2)$

and since Ω is bounded there exists an $R > 0$, so $|x| < R$ for all $x \in \overline{\Omega}$. But $x_1^2 \leqq |x|^2 < R^2$ for $x \in \overline{\Omega}$, so

$$f(x) \leqq \max_{x \in \partial\Omega}(f(x) + \varepsilon x_1^2) \leqq \max_{x \in \partial\Omega}(f(x) + \varepsilon R^2)$$

$= M + \varepsilon R^2$ for $x \in \overline{\Omega}$. Because $\varepsilon > 0$ was arbitrary we deduce that $f(x) \leqq M$ for $x \in \overline{\Omega}$. $\square$

(b): Minimum principle:

Assume: (1) $\Omega \subset \mathbb{R}^n$ open, bounded

(2) $f \in C^0(\overline{\Omega}) \cap C^2(\Omega)$

(-3) $\Delta f \leqq 0$ in Ω

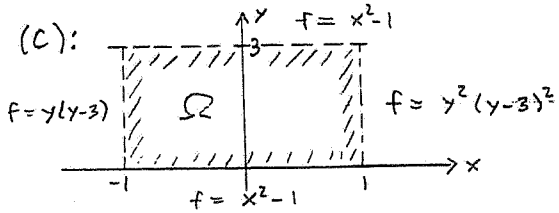
Then: $f(x) \geqq m$ for all $x \in \overline{\Omega}$, where $m = \min_{x \in \partial\Omega} f(x)$.

Proof. Put $\tilde{f}(x) = -f(x)$, $x \in \overline{\Omega}$. Then $\tilde{f}$ satisfies (1), (2), (3) and therefore $\tilde{f}(x) \leqq \max_{\partial\Omega} \tilde{f}$ in $\overline{\Omega}$ by the maximum principle. But

$$-f(x) \leqq \max_{\partial\Omega} -f(x),$$

is the same as $f(x) \geqq \min_{\partial\Omega} f$. $\square$

(c):



By minimum and maximum principle: $\min_{\overline{\Omega}} f = \min_{\partial\Omega} f$, $\max_{\overline{\Omega}} f = \max_{\partial\Omega} f$

By inspection: $\min_{\partial\Omega} f = f(-1, \frac{3}{2}) = -\frac{9}{4}$,

$\max_{\partial\Omega} f = f(1, \frac{3}{2}) = \frac{81}{16}$, so

$\max_{\overline{\Omega}} f = \underline{\underline{\frac{81}{16}}}$, $\min_{\overline{\Omega}} f = \underline{\underline{-\frac{9}{4}}}$.

(d): Put $w = v - u$. Then $w \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $\Delta w = \Delta v - \Delta u \geqslant 0$ in Ω. By the maximum principle, $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$. But $w = v - u \leqq 0$ on $\partial\Omega$, so $\max_{\partial\Omega} w \leqq 0$, i.e. $w \leqq 0$ on $\overline{\Omega}$, or $v \leqq u$ in $\overline{\Omega} \supset \Omega$. $\square$

(5.2) $u(x,y)$ is composed by C^∞ functions and is therefore a C^∞ function in its domain of definition, which is $\mathbb{R}^2 \setminus \{(0,0)\}$. By the chain rule:

$$u_x = \frac{1}{4\pi} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{1}{2\pi} \cdot \frac{x}{x^2+y^2},$$

$$u_{xx} = \frac{1}{2\pi} \cdot \frac{x^2+y^2 - x \cdot 2x}{(x^2+y^2)^2} = \frac{1}{2\pi} \cdot \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

Because $u(x,y) = u(y,x)$ we get immediately that $u_{yy} = \frac{1}{2\pi} \cdot \frac{x^2 - y^2}{(x^2+y^2)^2}$, and hence $\Delta u = u_{xx} + u_{yy} = 0$ for all $(x,y) \neq (0,0)$.

(5.3)

$$w = |\nabla u|^2 = \sum_{j=1}^n (u_{x_j})^2,$$

$$w_{x_k} = \sum_{j=1}^n 2 u_{x_j} u_{x_j x_k},$$

$$w_{x_k x_k} = \sum_{j=1}^n 2\left((u_{x_j x_k})^2 + u_{x_j} u_{x_j x_k x_k} \right),$$

so $\Delta w = \sum_{k=1}^n w_{x_k x_k} = 2 \sum_{j,k=1}^n \left((u_{x_j x_k})^2 + u_{x_j} u_{x_j x_k x_k} \right)$

$$= 2 \sum_{j,k=1}^n (u_{x_j x_k})^2 + 2 \sum_{j=1}^n \sum_{k=1}^n u_{x_j} u_{x_j x_k x_k} =$$

$$2 \sum_{j,k=1}^n (u_{x_j x_k})^2 + 2 \sum_{j=1}^n u_{x_j} \left(\sum_{k=1}^n u_{x_k x_k} \right)_{x_j} =$$

$$2 \sum_{j,k=1}^n (u_{x_j x_k})^2 + 2 \sum_{j=1}^n u_{x_j} (\Delta u)_{x_j} =$$

$$2 \sum_{j,k=1}^n (u_{x_j x_k})^2.$$

$(\Delta u)_{x_j} = (\alpha)_{x_j} = 0$

$\therefore$ $w \in C^0(\overline{\Omega}) \cap C^2(\Omega)$,

$$\Delta w = 2 \sum_{j,k=1}^n (u_{x_j x_k})^2 \geqslant 0 \text{ in } \Omega.$$

By maximum principle: $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$, and since

$$\max_{\overline{\Omega}} |\nabla u|^2 = \left(\max_{\overline{\Omega}} |\nabla u|\right)^2,$$

$$\max_{\partial\Omega} |\nabla u|^2 = \left(\max_{\partial\Omega} |\nabla u|\right)^2,$$

it follows that $\max_{\overline{\Omega}} |\nabla u| = \max_{\partial\Omega} |\nabla u|$. $\square$

(5.4) $\Delta(fg) = \sum_{j=1}^n (fg)_{x_j x_j} = \sum_{j=1}^n (f_{x_j} g + f g_{x_j})_{x_j} = \sum_{j=1}^n (f_{x_j x_j} g + 2 f_{x_j} g_{x_j} + f g_{x_j x_j}) = \Delta f \cdot g + 2 \nabla f \cdot \nabla g + f \Delta g$. If $\Delta f = \Delta(f^2) = 0$, take $g = f$ above to get $|\nabla f|^2 = 0$, i.e. $\nabla f = 0$ in Ω. Because Ω is connected f must be constant in Ω.

(5.5)

(a) Take $v=1$ in Green's first identity to get

$$\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} \, dS .$$

Since $\Delta u = f$ in Ω, $\frac{\partial u}{\partial N} = g$ on $\partial\Omega$ it follows that $\int_\Omega f \, dx = \int_{\partial\Omega} g \, dS$. □

(b) By (a) we must require that

$$\int_\Omega \alpha \, dx = \int_{\partial\Omega} 2 \, dS ,$$ i.e. that

$$\alpha \text{ area}(\Omega) = 2 \text{ length}(\partial\Omega).$$

But $\text{area}(\Omega) = \pi$, $\text{length}(\partial\Omega) = 2\pi$,

so $\alpha\pi = 4\pi$, or $\alpha = 4$.

(c) Suppose $u(x) = \lambda(x_1^2 + x_2^2)$.

$\nabla u = \lambda(2x_1, 2x_2)$, so

$$\frac{\partial u}{\partial N}(x) = \nabla u(x) \cdot x = 2\lambda(x_1^2 + x_2^2) = 2\lambda$$

for $|x| = 1$.

$\Delta u = 4\lambda$ for $|x| < 1$.

Hence if we take $\lambda = 1$, then

$$\begin{cases} \Delta u = 4 & \text{in } \Omega \\ \frac{\partial u}{\partial N} = 2 & \text{on } \partial\Omega . \end{cases}$$

$\therefore \quad u(x) = x_1^2 + x_2^2, \quad (x_1, x_2) \in \overline{\Omega}$.

PDE Solutions 6: Fourier Series

Qu.1 Calculate the Fourier coefficients for the square wave

$$f(x) = 0 \quad if \quad -2 < x < -1$$
$$f(x) = k \quad if \quad -1 < x < 1$$
$$f(x) = 0 \quad if \quad 1 < x < 2$$

with $f(x+4) = f(x)$. (Note that the period in this case is 4, so $L = 2$)

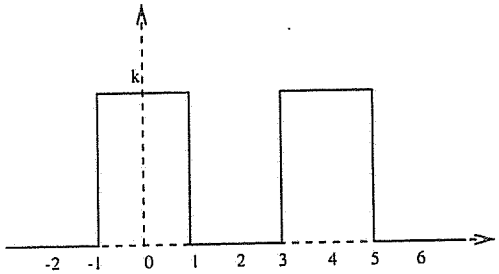


Figure 1: The square wave

According to the formulae on the sheet, we have (with $L = 2$)

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx.$$

As $f(x) = 0$ when $x < -1$ or $x > 1$ and $f(x) = k$ when $-1 < x < 1$ this gives

$$a_0 = \frac{1}{4}\int_{-1}^{1} kdx = \frac{1}{2}[kx]_{-1}^{1} = k/2 \,.$$

Similarly the integrals for a_n can be carried out:

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx.$$

which gives explicitly

$$a_n = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\frac{2k}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-1}^{1} = \frac{k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right)\right]$$

We could leave this as it stands, or notice that $\sin(-x) = -\sin(x)$ which allows us to tidy this up too

$$a_n == \frac{2k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)\right].$$

Even this isn't the end of the story, actually putting some n values in shows that $a_n = 0$ if n is even, and

$$a_{2m+1} = \frac{2k}{(2m+1)\pi}(-1)^m, \quad n = 2m+1 \;\; odd$$

~~but the formula with sin above is fine~~.

We can save ourselves some blood, sweat toil and tears by noticing that $f(x)$ is an even function, so we would expect all the b_n to be zero. For those who like to suffer carrying out the integrals confirms this

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx.$$

which gives explicitly

$$b_n = \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{1}{2}\left[\frac{2k}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-1}^{1} = -\frac{k}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{-n\pi}{2}\right)\right] = 0$$

as $\cos(-x) = \cos(x)$.

Qu.2 Calculate the Fourier coefficients for the "sawtooth wave"

$$f(x) = x, \quad -\pi < x < \pi$$

with $f(x + 2\pi) = f(x)$.

This is a bit more painful than the first question as we have to carry out some integration by parts (famous formula $\int udv = uv| - \int vdu$). I'll spare the trees of the world and only calculate

the b_n as I know that all the a's are zero since the diagram makes it clear that the sawtooth wave is *odd*. We have $L = \pi$ so

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx$$

$$b_n = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx$$

where we have integrated by parts in going to the second line. If we carry out the second integral we get

$$b_n = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$b_n = -\frac{2}{n}\cos(n\pi) = \begin{cases} -\frac{2}{n}, & n \text{ is even.} \\ \frac{2}{n}, & n \text{ is odd.} \end{cases}$$

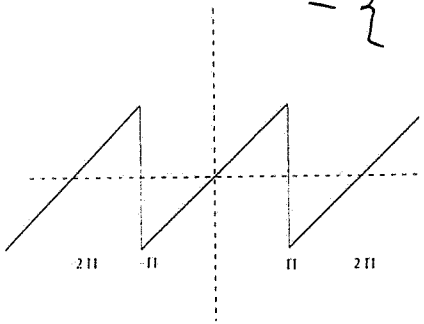


Figure 2: The sawtooth wave

Qu.3 Work out the odd Fourier half-range expansion for

$$f(x) = x, \quad 0 < x < L$$

We are told explicitly what to do here. All we need to do is calculate b_n in the manner of **Qu.2** using integration by parts

$$b_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\left[-\frac{xL\cos(nx)}{n\pi}\right]_0^L + \frac{L}{n\pi}\int_0^L \frac{\cos\left(\frac{n\pi x}{L}\right)}{n}dx$$

If we carry out the second integral we get

$$b_n = \frac{2}{L}\left[-\frac{xL\cos\left(\frac{n\pi x}{L}\right)}{n\pi}\right]_0^L + \frac{L^2}{n\pi^2}\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{n^2\pi^2}\right]_0^L$$

$$b_n = -\frac{2L}{n\pi}\cos(n\pi) = \begin{cases} -\frac{2L}{n\pi}, & n \text{ is even} \\ \frac{2L}{n\pi}, & n \text{ is odd.} \end{cases}$$

where the sin terms have vanished as $\sin(n\pi) = 0$.

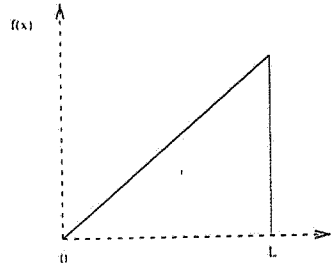


Figure 3: $f(x) = x$

Qu.4 Suppose we are interested in the general solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. We solved this in the lectures and found

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

As we can see the constants C_n, D_n are left undetermined. Suppose we are also told that $u(x,0) = f(x)$ where $f(x) = x, 0 < x < L$ as in question 3 and $u_t(x,0) = 0$. Show how this fixes C_n, D_n using the results of question 3.

The first thing to note is that $f(x)$ *is* the function in **Qu.3**, so we know its Fourier coefficients b_n. If we now set $t = 0$ in the expression for $u(x,t)$ above we get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right).$$

If this is to match up with what we calculated in **Qu.3** then

$$C_n = b_n = -\frac{2L}{n\pi}\cos(n\pi)$$

so we see that being told $u(x,0)$ fixes all the b_n.

What about the D_n's? Well, we have $u_t(x,0) = 0$, so work out $u_t(x,t)$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)\left(-C_n \sin\left(\frac{n\pi ct}{L}\right) + D_n \cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

and then set $t = 0$.

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right) D_n \sin\left(\frac{n\pi x}{L}\right) = 0$$

which means that all the D_n's must be zero.

PDE Tutorial 7, solutions.

This sheet is essentially about one thing, namely **SEPARATION OF VARIABLES**. We'll restrict ourselves to the 1D wave and heat equations.

$$u_{tt} = c^2 u_{xx} \qquad \text{1D Wave Equation} \tag{1}$$

$$u_t = \alpha^2 u_{xx} \qquad \text{1D Heat Equation} \tag{2}$$

Qu. 1 Use separation of variables to solve the 1D Heat Equation for $0 < x < 1$ with boundary conditions

$$\text{(a)} \quad u(0,t) = 0 \quad u_x(1,t) = 0$$

$$\text{(b)} \quad u_x(0,t) = 0 \quad u(1,t) = 0$$

Give the general form of the solution in both cases.

Let's do the whole process in detail for (a): First *separate the variables* by letting $u(x,t) = X(x)T(t)$, then substitute into the heat equation using

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t) \\ u_{xx}(x,t) &= X_{xx}(x)T(t) \end{aligned}$$

which gives $X(x)T_t(t) = \alpha^2 X_{xx}(x)T(t)$ or

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{\alpha^2 T(t)}.$$

Now the cunning bit - both sides of this equation must be a constant as the left depends only on x and the right only on t. We thus have

$$\begin{aligned} \frac{X_{xx}(x)}{X(x)} &= -\mu^2 \\ \frac{T_t(t)}{\alpha^2 T(t)} &= -\mu^2 \end{aligned} \tag{3}$$

where we have chosen a *negative* constant for the reasons explained in the lectures.
The boundary conditions are

$$\begin{aligned} u(0,t) &= 0 \ , \quad X(0)T(t) = 0 \\ u_x(1,t) &= 0 \ , \quad X_x(1)T(t) = 0 \end{aligned}$$

so to get a non-zero answer we must have $X(0) = X_x(1) = 0$ (taking the other possibility $T(t) = 0$ gives u(x,t)=0 for all time, which is a rather uninteresting solution...)
Let's solve equation(3) with these boundary conditions. Equation(3) may be rewritten as

$$X_{xx}(x) + \mu^2 X(x) = 0 \quad , \quad X(0) = X_x(1) = 0. \tag{4}$$

The solution of this equation is

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

and the boundary conditions imply that

$$\begin{aligned} X(0) &= A = 0 \\ X_x(1) &= -\mu A\sin(\mu) + \mu B\cos(\mu) = 0. \end{aligned}$$

As the first tells us $A = 0$, the second implies that $\cos(\mu) = 0$, or

$$\mu = (n + \frac{1}{2})\pi$$

(choosing the other possibility $B = 0$ would mean that $X(x)$ and hence $u(x,t)$ would be zero for all t).

We still have to solve the equation for $T(t)$, which can be rewritten

$$T_t(t) + \mu^2\alpha^2 T(t) = 0$$

which is a bit easier to solve, being first order. The answer is $T(t) = C\exp(-\mu^2\alpha^2 t)$. We can now stick the solutions for $X(x)$ and $T(t)$ together to get $u(x,t)$, namely $u(x,t) = X(x)T(t) = B\sin(\mu x)\cdot C\exp(-\mu^2\alpha^2 t)$. Since B and C are both unknown constants we might as well combine them and call the product $\bar{C}$

$$u(x,t) = X(x)T(t) = \bar{C}\sin(\mu x)\exp(-\mu^2\alpha^2 t).$$

There is one last tweak before we are finished. Since $\mu = (n + \frac{1}{2})\pi$ the above solution will work for any n, so we can get a general solution by summing all these up with arbitrary constants

$$u(x,t) = \sum_{n=0}^{\infty} \bar{C}_n \sin(\mu_n x)\exp(-\mu_n^2\alpha^2 t) \quad , \quad \mu_n = (n + \frac{1}{2})\pi$$

<u>I'll be a lot briefer for (b) as the idea is identical to (a):</u>

- Separate the variables by letting $u(x,t) = X(x)T(t)$
- Get the two ordinary differential equations, one for $X(x)$ one for $T(t)$.
- Solve these subject to the boundary conditions

The equation for $X(x)$ is now

$$X_{xx}(x) + \mu^2 X(x) = 0 \quad , \quad X_x(0) = X(1) = 0. \tag{5}$$

(note the slightly different boundary conditions to (a)) The solution of this equation is still

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

and the new boundary conditions imply that

$$\begin{aligned} X_x(0) &= \mu B = 0 \\ X(1) &= A\cos(\mu) + B\sin(\mu) = 0. \end{aligned}$$

The first tells us $B = 0$, the second implies that $\cos(\mu) = 0$, or

$$\mu = (n + \frac{1}{2})\pi$$

The solution for $T(t)$ is unchanged so the final answer for $u(x,t)$ is given by

$$u(x,t) = \sum_{n=0}^{\infty} \bar{C}_n \cos(\mu_n x) \exp(-\mu_n^2 \alpha^2 t) \quad , \quad \mu_n = (n + \frac{1}{2})\pi$$

which differs from the answer for (a) by having a cos instead of a sin.

Qu. 2. Use separation of variables to solve the 1D Wave Equation for $0 < x < 1$ with boundary conditions

$$\text{(a)} \quad u(0,t) = 0 \quad u_x(1,t) = 0$$
$$\text{(b)} \quad u_x(0,t) = 0 \quad u(1,t) = 0$$

Give the general form of the solution in both cases.

The method is the same as **Qu.1**: Now however the resulting ordinary differential equation for $T(t)$ is econd order. First *separate the variables* by letting $u(x,t) = X(x)T(t)$, then substitute into the wave equation using

$$\begin{aligned} u_{tt}(x,t) &= X(x)T_{tt}(t) \\ u_{xx}(x,t) &= X_{xx}(x)T(t) \end{aligned}$$

which gives $X(x)T_{tt}(t) = c^2 X_{xx}(x)T(t)$ or

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_{tt}(t)}{c^2 T(t)}.$$

Same stunt as before - both sides of this equation must be a constant as the left depends only on x and the right only on t. We thus have

$$\begin{aligned} \frac{X_{xx}(x)}{X(x)} &= -\mu^2 \\ \frac{T_{tt}(t)}{c^2 T(t)} &= -\mu^2 \end{aligned} \tag{6}$$

where we have chosen a *negative* constant again.
The boundary conditions are

$$\begin{aligned} u(0,t) &= 0 \ , \quad X(0)T(t) = 0 \\ u_x(1,t) &= 0 \ , \quad X_x(1)T(t) = 0 \end{aligned}$$

so to get a non-zero answer we must have $X(0) = X_x(1) = 0$.
Let's solve equation(6) with these boundary conditions. The first part of equation(6) may be rewritten as

$$X_{xx}(x) + \mu^2 X(x) = 0 \quad , \quad X(0) = X_x(1) = 0. \tag{7}$$

whose solution is, as before, $X(x) = A\cos(\mu x) + B\sin(\mu x)$. The boundary conditions imply that

$$\begin{aligned} X(0) &= A = 0 \\ X_x(1) &= -\mu A\sin(\mu) + \mu B\cos(\mu) = 0. \end{aligned}$$

As the first tells us $A = 0$, the second implies that $\cos(\mu) = 0$, or $\mu = (n + \frac{1}{2})\pi$.

The equation for $T(t)$ is very similar

$$T_{tt}(t) + \mu^2 c^2 T(t)$$

with the solution

$$T(t) = C\cos(\mu ct) + E\sin(\mu ct)$$

so sticking the two solutions for $X(x)$ and $T(t)$ together gives

$$u(x,t) = (\tilde{C}\cos(\mu ct) + \tilde{E}\sin(\mu ct))\sin(\mu x)$$

and different values of n in μ may be combined to give

$$u(x,t) = \sum_{n=0}^{\infty}(\tilde{C}_n\cos(\mu_n ct) + \tilde{E}_n\sin(\mu_n ct))\sin(\mu_n x) \quad , \quad \mu_n = (n + \frac{1}{2})\pi$$

.

I'll spare you the details again for (b): All that changes are the boundary conditions in the o.d.e. for X which means that the final solution becomes

$$u(x,t) = \sum_{n=0}^{\infty}(\tilde{C}_n\cos(\mu_n ct) + \tilde{E}_n\sin(\mu_n ct))\cos(\mu_n x) \quad , \quad \mu_n = (n + \frac{1}{2})\pi$$

(note the $\cos(\mu_n x)$ rather than the $\sin(\mu_n x)$ that appears in (a)).

Qu. 3 The problem

$$u_t = \alpha^2 u_{xx} \quad 0 < x < 1\ , \quad t > 0\ , \qquad u(0,t) = u(1,t) = 0 \quad t > 0$$

has general solution

$$u(x,t) = \sum_{k=1}^{n} A_k e^{-k^2\pi^2\alpha^2 t}\sin(k\pi x)$$

where the constants A_k are determined from the initial conditions. Find the constants A_k when the initial condition is

$$u(x,0) = \sin(12\pi x)$$

This is a easy one to finish off: If $u(x,0) = \sin(12\pi x)$ then we must have, by setting $t = 0$ in the sum above for $u(x,t)$

$$u(x,0) = \sin(12\pi x) = \sum_{k=1}^{n} A_k\sin(k\pi x)$$

so by inspection (i.e. looking and guessing!) we see that $A_{12} = 1$ and all the other A's must be zero. The solution that satifies the *initial condition* $u(x,0) = \sin(12\pi x)$ is thus

$$u(x,t) = e^{-144\pi^2\alpha^2 t}\sin(12\pi x)$$

11·413 P.D.E. Solutions to Examples 8

1 If $x_r(t)$ is the char. through y_r then

$$x_r(t) = f'(h(y_r))t + y_r \quad \ldots (1) \qquad r=1,2$$

If char. coincide at $(\bar{x}, \bar{t})$, then putting $t=\bar{t}$ in (1) gives $f'(h(y_1))\bar{t} + y_1 = f'(h(y_2))\bar{t} + y_2$ so that

$$\bar{t} = \frac{y_2 - y_1}{f'(h(y_1)) - f'(h(y_2))} > 0$$

Using the above expression for $\bar{t}$ in (1) with $r=1$ gives,

$$\bar{x} = f'(h(y_1))\bar{t} + y_1$$

$$= \frac{f'(h(y_1))y_2 - y_1 f'(h(y_2))}{f'(h(y_1)) - f'(h(y_2))}$$

2 The answer is $u(x,t) = \begin{cases} 1, & x<t \\ (1-x)(1-t)^{-1}, & t \le x \le 1 \\ 0, & x>1 \end{cases}$

3 Clearly $u^- = 0$ and $f(u) = \frac{u^2}{2}$. also

$$u^+ = \lim_{x \to -\frac{t^2}{4}} -\frac{2}{3}\left[t + (3x+t^2)^{\frac{1}{2}}\right] = -t.$$

Now $\frac{d\gamma}{dt} = -\frac{d}{dt}\left(\frac{t^2}{4}\right) = -\frac{t}{2}$.

Jump condition is $\frac{d\gamma}{dt} = \frac{u^+ + u^-}{2}$, which is correct.

11.413 Solutions to Examples 8

4 $u_t = v_t g'(v)$, $u_x = v_x g'(v)$, $u_t + f'(u)u_x = g'(v)[v_t + v v_x]$

Thus $v_t + v v_x = 0$. The change of variables need not map discontinuous solutions of (*) into (**) ie for the 1st equ^n. $\frac{d\gamma}{dt} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$ while for **1

$$\frac{d\gamma}{dt} = \frac{v_+ + v_-}{2} = \frac{f'(u_+) + f'(u_-)}{2}$$

(5) Using the same notation as Q1, let $y_1 = \pi$, $y_2 = \pi + \epsilon$, $\epsilon > 0$. Then $h(y_1) = \sin\pi = 0$ and $h(y_2) = -\sin\epsilon$. Thus $h(y_1) > h(y_2)$ for $\epsilon > 0$ small. From Q1,

$$\bar{t} = \frac{\epsilon}{\sin\epsilon} \to 1 \quad \text{as} \quad \epsilon \to 0^+$$

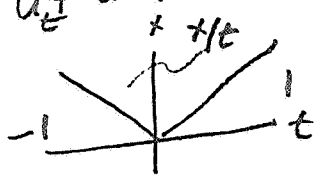
In particular, if $t > 1$ can find $\bar{t}$ with $\bar{t} < t$. Hence there cannot be a smooth solution for $t > 1$.

6 (a) $u^- = -1$, $u^+ = 1$, $f(u) = \frac{u^2}{2}$ and $\frac{d\gamma}{dt} = \frac{u^+ + u^-}{2} = 0$.

Hence $u(x,t) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ is a solution.

Since $u^- < u^+$ it is not admissible.

(b) $u = \frac{x}{t}$ satisfies $u_t + u u_x = 0$. Thus we look for a sol^n



Clearly $u = \begin{cases} -1, & x < -t \\ \frac{x}{t} & -t < x < t \\ 1 & x > t \end{cases}$

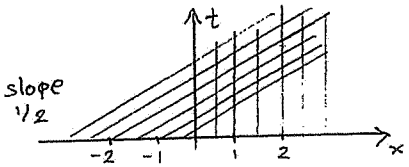
is a continuous solution.

9.1 (a): see lecture notes.

(b): Characteristics: $x = f'(h_i(x_0))t + x_0$, $t \geq 0$, where $x_0 \in \mathbb{R}$. Note $f(u) = \frac{1}{2}u^2$, so $f'(u) = u$.

(i=1): $x_0 < 0$: $x = 2t + x_0$, $t \geq 0$,

$x_0 \geq 0$: $x = x_0$, $t \geq 0$.

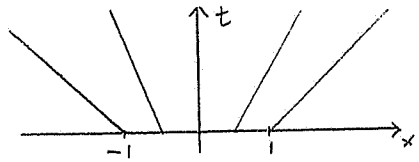


If $u(x,t)$ is a C^1 solution, then $u(x,t)$ is constant on the char.'s,

i.e. $u(x,t) = h_1(x_0) = \begin{cases} 2, & x = 2t + x_0 \quad ; x_0 < 0 \\ 0, & x = x_0 \quad ; x_0 \geq 0 \end{cases}$

Because $x = 2t + x_0$ intersects the t-axis $(x=0, t \geq 0)$ at $t = -\frac{1}{2}x_0$ we conclude that there can be <u>no</u> C^1 solution $u = u(x,t)$.

(i=2): $x = x_0 t + x_0$, $t \geq 0$; $x_0 \in \mathbb{R}$.



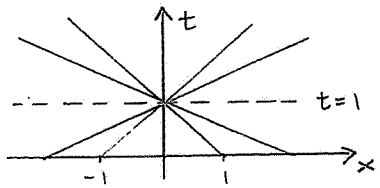
No intersecting char.'s. A C^1 solution is constant on char.'s, so

$u(x,t) = x_0$ when $x = x_0 t + x_0$, $t \geq 0$.

Hence, since $x = x_0 t + x_0$ yields $x_0 = \frac{x}{1+t}$,

$$\underline{u(x,t) = \frac{x}{1+t}, \quad x \in \mathbb{R}, \ t \geq 0.}$$

(i=3): $x = -x_0 t + x_0$, $t \geq 0$; $x_0 \in \mathbb{R}$

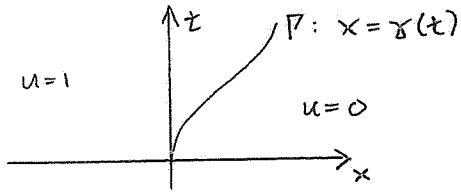


All char.'s intersect at $(x,t) = (0,1)$, and since a C^1 solution must be constant on char.'s and the constant is different for different char.'s it follows that there is <u>no</u> C^1 solution for $t \geq 1$. For $t < 1$ we get $u(x,t) = -x_0$ for $x = -x_0 t + x_0$, $t \geq 0$, i.e.

$$\underline{u(x,t) = \frac{x}{t-1}, \quad 0 \leq t < 1.}$$

9.2 (a), (b), (c): see lecture notes.

(d): We use Rankine-Hugoniot's theorem (= question (c)).



$\gamma(0) = 0$

$u^+ = 0$, $u^- = 1$

Try $u(x,t) = \begin{cases} 1 & \text{if } x < \gamma(t) \\ 0 & \text{if } x \geq \gamma(t). \end{cases}$

Note that (i) holds and that $u^+ = 0$, $u^- = 1$. By (ii) we infer

$$\gamma'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{4}, \quad \text{so}$$

$$\gamma(t) = \gamma(0) + \int_0^t \gamma'(\tau)\,d\tau = \frac{1}{4}t.$$

Consequently, $\underline{u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{4}t \\ 0 & \text{if } x \geq \frac{1}{4}t \end{cases}}$

is a weak solution.