Cockcroft Properties of Thompson's Group

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Abstract

In a study of the word problem for groups, R. J. Thompson considered a certain group F of self-homeomorphisms of the Cantor set and showed, among other things, that F is finitely presented. Using results of K. S. Brown and R. Geoghegan, M. N. Dyer showed that F is the fundamental group of a finite two-complex Z^2 having Euler characteristic one and which is *Cockcroft*, in the sense that each map of the two-sphere into Z^2 is homologically trivial. We show that no proper covering complex of Z^2 is Cockcroft. A general result on Cockcroft properties implies that no proper regular covering complex of any finite two-complex with fundamental group F is Cockcroft.

1 Introduction

Thompson's group F is defined by the presentation

$$\mathcal{P} = (x_0, x_1, x_2, \ldots : x_{j+1}^{-1} x_i^{-1} x_j x_i,\ 0 \leq i < j \in \mathbf{Z}).$$

For a discussion of this group and of its discovery by Thompson in 1965, see [Bri, Bro, BG] and the references cited there. The group F is torsion-free [BG, Corollary 1.5] and the derived subgroup $[F, F]$ is simple [T, Corollary 1.9], [Bro, Theorem 4.16]. Brown and Geoghegan [BG, Corollary 4.2] constructed an infinite-dimensional cubical aspherical CW complex Y with fundamental group F and showed that Y has the homotopy type of a CW complex Z with just two cells in each positive dimension. We refer to the complex Y as the *Brown-Geoghegan complex*. Using the complex Z, Brown and Geoghegan [BG, Theorem 7.1] computed the homology of F with trivial integer coefficients, finding that $H_n(F) = \mathbf{Z} \oplus \mathbf{Z}$ for each positive integer n.

This provided the first known example of a torsion-free group of type FP_∞ and with infinite cohomological dimension.

It turns out [BG] that the two-skeleton Z^2 of the complex Z is modeled on the following two-generator two-relator presentation for F.

$$\mathcal{Q} = (x_0, x_1 : [x_1^{x_0}, x_0 x_1^{-1}], [x_1^{x_0 x_1}, x_1 x_1^{-x_0}])$$

Here, $g^h = h^{-1}gh$, $g^{-h} = h^{-1}g^{-1}h = (g^h)^{-1}$, and $[g,h] = g^{-1}g^h = h^{-g}h$. A second finite presentation for Thompson's group [Bri, BG] is the following.

$$\mathcal{Q}' = (x_0, x_1 : [x_1^{x_0}, x_0 x_1^{-1}], [x_1^{x_0^2}, x_1 x_0^{-1}])$$

As we shall see, the two-complexes $K(\mathcal{Q})$ and $K(\mathcal{Q}')$ that model these presentations have the same simple homotopy type. In fact, these two-complexes are related by three-deformations. Our immediate objective is to study the second homotopy module of finite two-complexes with fundamental group F. Our main result on Thompson's group can be formulated as follows.

Theorem 1.1 *Let $K(\mathcal{Q})$ be the two-dimensional cellular model of the finite presentation $\mathcal{Q}$ for Thompson's group F.*

1. *Each map of the two-sphere into $K(\mathcal{Q})$ is homologically trivial.*
2. *Each proper covering complex of $K(\mathcal{Q})$ supports a map of the two-sphere that is homologically nontrivial.*

These results obviously apply to any two-complex having the same homotopy type as $K(\mathcal{Q})$. The first conclusion of the Theorem is due to M. N. Dyer [D]; see Lemma 2.1 below. Our proof of the second statement in the Theorem begins with an explicit description of the three-skeleton Y^3 of the Brown-Geoghegan complex. This description is given in Section 3. In Section 4, we follow the proof of Theorem 5.3 in [BG] to show how the finite two-complex $K(\mathcal{Q})$ is obtained from the model of a finite subpresentation of $\mathcal{P}$. Computations in second homotopy modules will be carried out in the cellular chain complex of the universal covering $\widetilde{Y}$ of Y, and in certain subcomplexes. We view these as chain complexes of right $\mathbf{Z}F$-modules.

In Section 2, we describe the *Cockcroft properties* of a given two-complex K and their relationship to the *Fox ideal*, which is a two-sided ideal in the integral group ring $\mathbf{Z}\pi_1(K)$ of the fundamental group. These topics are developed in detail in [D, GH]. In Section 5 the analysis of Section 4 is used to describe generators for the Fox ideal of a certain finite two-complex with fundamental group F. The proof of Theorem 1.1 is completed in Section 6. The paper concludes in Section 7 with a result on the invariance of Cockcroft properties for two-complexes with fixed fundamental group and Euler characteristic. Using Theorem 1.1, this invariance result yields the following Corollary.

Corollary 1.2 *Suppose that L is a finite connected two-complex with Euler characteristic one and fundamental group isomorphic to Thompson's group F.*

1. *Each map of the two-sphere into L is homologically trivial.*
2. *Each proper regular covering complex of L supports a map of the two-sphere that is homologically nontrivial.*

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2 Cockcroft properties and the Fox ideal

Consider the finite connected two-complex $K(\mathcal{Q})$ that is modeled on the finite presentation $\mathcal{Q}$ for Thompson's group F. Our study of $\pi_2(K(\mathcal{Q}))$ begins with the following result of M. N. Dyer, which relies on the homology calculations of [BG].

Lemma 2.1 ([D]) *Each map of the two-sphere into $K(\mathcal{Q})$ is homologically trivial. That is, the Hurewicz homomorphism $h : \pi_2(K(\mathcal{Q})) \to H_2(K(\mathcal{Q}))$ is trivial.*

Proof: Since both of the relators in $\mathcal{Q}$ are commutators, the second homology of $K(\mathcal{Q})$ is free abelian of rank two: $H_2(K(\mathcal{Q})) \cong \mathbf{Z} \oplus \mathbf{Z}$. The Brown-Geoghegan homology calculations for F provide that $H_2(F)$ is also free abelian of rank two. Since $\pi_1(K(\mathcal{Q})) \cong F$, the exact Hopf sequence

$$\pi_2(K(\mathcal{Q})) \xrightarrow{h} H_2(K(\mathcal{Q})) \to H_2(F) \to 0$$

reveals that the Hurewicz map h is trivial. ◇

A connected two-dimensional CW complex K is *Cockcroft* if each map of the two-sphere into K is homologically trivial. For a subgroup $H \leq \pi_1(K)$, the two-complex K is *H-Cockcroft* if the covering complex of K that corresponds to H is Cockcroft. In this case, we also say that H is a *Cockcroft subgroup* of $\pi_1(K)$. The set of Cockcroft subgroups of $\pi_1(K)$ is denoted by $\mathbf{c}(K)$:

$$\mathbf{c}(K) = \{H \leq \pi_1(K) : K \text{ is } H\text{-Cockcroft}\}.$$

This set is partially ordered by inclusion and is closed under conjugation by elements of $\pi_1(K)$. A fundamental result due independently to Harlander [H] and to Gilbert and Howie [GH] provides that the partially ordered set $\mathbf{c}(K)$ has minimal elements. The two-complex K is *absolutely Cockcroft* if $\mathbf{c}(K)$ consists of $\pi_1(K)$ alone, that is, if K is Cockcroft and no proper covering complex of K is Cockcroft. Theorem 1.1 therefore states that the two-complex $K(\mathcal{Q})$ is absolutely Cockcroft. Other known examples of absolutely Cockcroft two-complexes [GH] include all two-complexes with finite fundamental group and Euler characteristic one and all two-spines of closed orientable aspherical three-manifolds.

For a connected two-complex K with universal covering $p : \widetilde{K} \to K$, a choice of basepoint $* \in \widetilde{K}$ determines an embedding of the second homotopy module $\pi_2(K)$ in the free right $\mathbf{Z}\pi_1(K)$-module $C_2(\widetilde{K})$ of two-dimensional cellular chains in $\widetilde{K}$.

$$\pi_2(K) \cong \pi_2(\widetilde{K}) \cong H_2(\widetilde{K}) = \ker(C_2(\widetilde{K}) \stackrel{\partial_2}{\to} C_1(\widetilde{K})) \leq C_2(\widetilde{K})$$

The chain module $C_2(\widetilde{K})$ is the free right $\mathbf{Z}\pi_1(K)$-module with basis determined by a set of basic lifts of the two-cells of K to the covering complex $\widetilde{K}$. The *Fox ideal* $\mathcal{F}(K)$ of K is the two-sided ideal in the group ring $\mathbf{Z}\pi_1(K)$ that is generated by the $\mathbf{Z}\pi_1(K)$-coefficients that appear when elements of $\pi_2(K)$ are expressed in terms of this free basis for $C_2(\widetilde{K})$. The Fox ideal is independent of the choice of basis for $C_2(\widetilde{K})$, and of the choice of basepoints used to identify $\pi_2(K)$ in $C_2(\widetilde{K})$. Clearly, the Fox ideal is generated by the $\mathbf{Z}\pi_1(K)$-coefficients that arise from any chosen set of $\mathbf{Z}\pi_1(K)$-module generators for $\pi_2(K)$. See [D, GH]. We will rely on the following relationship between Cockcroft properties and the Fox ideal.

Lemma 2.2 ([D, GH]) *Let K be a connected two-complex and let H be a subgroup of the fundamental group $\pi_1(K)$. The two-complex K is H-Cockcroft if and only the the Fox ideal $\mathcal{F}(K)$ is contained in the left ideal $\mathbf{Z}\pi_1(K) \cdot (H-1)$ of the integral group ring $\mathbf{Z}\pi_1(K)$.*

3 The Brown-Geoghegan complex

The Brown-Geoghegan complex Y has two-skeleton Y^2 modeled on the infinite presentation $\mathcal{P}$ for Thompson's group F. The one-skeleton Y^1 is thus a bouquet of circles with a single zero-cell and with an oriented one-cell for each generator from the presentation $\mathcal{P}$. For each pair of nonnegative integers $q_1, q_2 \in \mathbf{Z}$ with $q_2 - q_1 \geq 2$, the complex Y has an oriented two-cell whose attaching map is indicated in Figure 1.

Placement of the asterisk in Figure 1 refers to the choice of a fixed zero-cell $*$ in the universal covering complex $\widetilde{Y}^2$, and determines a set of basic lifts for the cells of Y^2. These basic lifts in turn determine a basis for the free cellular chain complex $C_*(\widetilde{Y}^2)$ of right $\mathbf{Z}F$-modules: $C_2(\widetilde{Y}) \stackrel{\partial_2}{\to} C_1(\widetilde{Y}) \stackrel{\partial_1}{\to} C_0(\widetilde{Y})$. Here, $C_0(\widetilde{Y}) = C_0(\widetilde{Y}^2) = \mathbf{Z}F$ is the free right $\mathbf{Z}F$-module of rank one with basis corresponding to the zero-cell $*$. The right $\mathbf{Z}F$-module $C_1(\widetilde{Y}) = C_1(\widetilde{Y}^2)$ has free basis determined by basic lifts of the oriented one-cells of Y. In keeping with the notation developed in [BG], we denote the basis element that corresponds to the generator x_q

by $(q) \in C_1(\widetilde{Y})$. The boundary homomorphism $\partial_1 : C_1(\widetilde{Y}) \to C_0(\widetilde{Y}) = \mathbf{Z}F$ is the right $\mathbf{Z}F$-homomorphism determined on this basis by $\partial_1(q) = 1 - x_q$. The chain module $C_2(\widetilde{Y}) = C_2(\widetilde{Y}^2)$ is the free right $\mathbf{Z}F$-module with basis consisting of all ordered pairs $(q_1\ q_2)$ of nonnegative integers with $q_2 - q_1 \geq 2$. The basis element $(q_1\ q_2)$ corresponds to the basic lift of the two-cell shown in Figure 1, and the boundary homomorphism $\partial_2 : C_2(\widetilde{Y}) \to C_1(\widetilde{Y})$ is the right $\mathbf{Z}F$-homomorphism determined on this basis by $\partial_2(q_1\ q_2) = -(q_2) + (q_2 - 1) \cdot x_{q_1} + (q_1) \cdot (1 - x_{q_2})$.

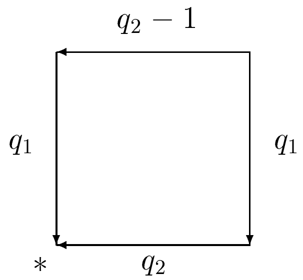

Figure 1. The two-cell $(q_1\ q_2)$

The three-skeleton Y^3 of the Brown-Geoghegan complex is obtained by attaching a cubical three-cell for each ordered triple (q_1, q_2, q_3) of nonnegative integers satisfying $q_{i+1} - q_i \geq 2$, $i = 1, 2$. The attaching map for this three-cell is indicated in Figure 2.

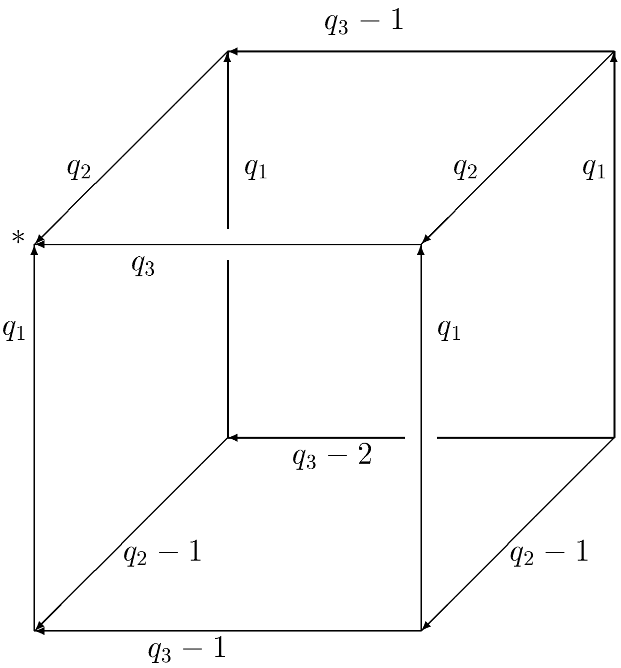

Figure 2. The three-cell $(q_1\ q_2\ q_3)$

Placement of the asterisk in Figure 2 refers to the selection of a basic lift of the pictured three-cell to the universal cover $\widetilde{Y}^3$, and determines a $\mathbf{Z}F$-basis element $(q_1\ q_2\ q_3)$ for the free right chain module $C_3(\widetilde{Y}) = C_3(\widetilde{Y}^3)$. The cellular boundary map $\partial_3 : C_3(\widetilde{Y}) \to C_2(\widetilde{Y})$ is the $\mathbf{Z}F$-homomorphism determined on this basis by the following formula.

$$\begin{aligned}\partial_3(q_1\ q_2\ q_3) \quad = \quad & (q_2\ q_3) - (q_2 - 1\ q_3 - 1) \cdot x_{q_1} \\ & -(q_1\ q_3) + (q_1\ q_3 - 1) \cdot x_{q_2} \\ & +(q_1\ q_2) \cdot (1 - x_{q_3})\end{aligned}$$

These boundaries determine elements $\partial_2(q_1\ q_2\ q_3) \in \pi_2(Y^2)$ under the identifications $\pi_2(Y^2) \cong \pi_2(\widetilde{Y}^2) \cong H_2(\widetilde{Y}^2) \leq C_2(\widetilde{Y})$. Since Y is aspherical, these three-cell boundaries actually generate $\pi_2(Y^2)$ as a right $\mathbf{Z}F$-module.

4 Finite presentations for F

For the purposes of computing within second homotopy modules, it is convenient to start with the following finite subpresentation $\mathcal{R}$ of the infinite presentation $\mathcal{P}$ for Thompson's group.

$$\mathcal{R} = (x_0,\ x_1,\ x_2,\ x_3,\ x_4\ :\ x_1^{x_0} = x_2,\ x_2^{x_1} = x_3,\ x_3^{x_2} = x_4,\ x_2^{x_0} = x_3,\ x_3^{x_1} = x_4),$$

The cellular model $K(\mathcal{R})$ is a finite subcomplex of the two-skeleton $Y^2 = K(\mathcal{P})$ of the Brown-Geoghegan complex Y.

Lemma 4.1 *1. The inclusion of the finite two-complex $K(\mathcal{R})$ in $K(\mathcal{P})$ induces an isomorphism of fundamental groups.*

2. The two-complex $K(\mathcal{R})$ three-deforms to the model $K(\mathcal{Q})$ of the presentation $\mathcal{Q}$ and also to the model $K(\mathcal{Q}')$ of the presentation $\mathcal{Q}'$.

In particular, the presentations $\mathcal{Q}$, $\mathcal{Q}'$, and $\mathcal{R}$ are all finite balanced presentations for Thompson's group F.

Proof: (1) The two-complex $K(\mathcal{R})$ has a single zero-cell, one one-cell for each of the five generators $x_0, \ldots x_4$, and the five two-cells corresponding to the pairs $(0\ 2)$, $(1\ 3)$, $(2\ 4)$, $(0\ 3)$, and $(1\ 4)$. Following the proof of [BG, Theorem 5.3], let Y^2_- denote the subcomplex of Y^2 consisting of the entire one-skeleton Y^1 together with the two-cells $(q\ q+2)$, $q \geq 0$ (called *collapsible two-cells* in [BG]), and the two-cells $(0\ 3)$ and $(1\ 4)$ (the *essential two-cells*). The collapsible two-cells $(q\ q+2)$, $q \geq 3$, support group theoretic relations $x_{q+1}^{x_q} = x_{q+2}$, which show how to inductively express the generators $x_5,\ x_6, \ldots$ in terms of the generators of $\mathcal{R}$. Topologically, the inclusion $K(\mathcal{R}) \subseteq Y^2_-$ is a homotopy equivalence. Thus it suffices to show that the inclusion $Y^2_- \subseteq Y^2$ induces an isomorphism of fundamental groups.

We argue by induction on the lexicographically ordered pairs $(q_1\ q_2)$ to show that the relation supported by each two-cell of Y^2 is a consequence of the relations supported by the two-cells of Y^2_-. The base of the induction is trivial since the two-cell $(0\ 2)$ lies in Y^2_-. Now consider a fixed two-cell $(q_1\ q_2)$ that does not lie in Y^2_-, so that $q_2 - q_1 \geq 3$. Suppose first that $q_2 = q_1 + 3$, in which case we also have that $q_1 \geq 2$. The attaching map for the three-cell $(q_1 - 2\ q_1\ q_1 + 3)$ shows that the relation supported by the two-cell $(q_1\ q_1 + 3)$ is a consequence of the relations supported by the two-cells $(q_1 - 1\ q_1 + 2)$, $(q_1 - 2\ q_1 + 3)$, $(q_1 - 2\ q_1 + 2)$, and $(q_1 - 2\ q_1)$. Algebraically, this is seen as follows.

$$
\begin{aligned}
x_{q_1+2}^{x_{q_1}} &= x_{q_1+1}^{x_{q_1-2}x_{q_1}} && \text{using } (q_1 - 2\ q_1 + 2) \\
&= x_{q_1+1}^{x_{q_1-1}x_{q_1-2}} && \text{using } (q_1 - 2\ q_1) \text{ twice} \\
&= x_{q_1+2}^{x_{q_1-2}} && \text{using } (q_1 - 1\ q_1 + 2) \\
&= x_{q_1+3} && \text{using } (q_1 - 2\ q_1 + 3)
\end{aligned}
$$

Thus the relation $(q_1\ q_1{+}3)$ is a consequence of preceding relations in the lexicographic ordering. If $q_2 - q_1 \geq 4$ and $q_1 \geq 0$, then a similar argument using the attaching map for the three-cell $(q_1\ q_2 - 2\ q_2)$ can be used to show that the relation supported by $(q_1\ q_2)$ is a consequence of those supported by the two-cells $(q_2 - 2\ q_2)$, $(q_2 - 3\ q_2 - 1)$, $(q_1\ q_2 - 1)$, and $(q_1\ q_2 - 2)$, each of which either lies in Y^2_- or else precedes $(q_1\ q_2)$ in the lexicographic ordering on two-cells.

(2) To see that the two-complex $K(\mathcal{R})$ three-deforms to $K(\mathcal{Q})$, use the relations corresponding to the collapsible two-cells $(0\ 2)$, $(1\ 3)$, and $(2\ 4)$ to rewrite the generators x_2, x_3, and x_4 in terms of x_0 and x_1 (giving rise to a three-deformation of $K(\mathcal{R})$), and then eliminate these collapsible two-cells by collapsing across the free one-cells corresponding to the eliminated generators. We leave it to the reader to check that the resulting presentation is precisely $\mathcal{Q}$, up to free equivalence of the relators. To see that $K(\mathcal{R})$ three-deforms to $K(\mathcal{Q}')$, first carry out an elementary expansion in dimension three by attaching the two-cell $(0\ 4)$ and the three-cell $(0\ 2\ 4)$ to $K(\mathcal{R})$. The two-cell $(1\ 4)$ is then a free face of the attached three-cell and so $K(\mathcal{R})$ three-deforms to the model of the presentation

$$\mathcal{R}' = (x_0,\ x_1,\ x_2,\ x_3,\ x_4\ :\ x_1^{x_0} = x_2,\ x_2^{x_1} = x_3,\ x_3^{x_2} = x_4,\ x_2^{x_0} = x_3,\ x_3^{x_0} = x_4).$$

Now use the two-cells $(0\ 2)$, $(0\ 3)$, and $(0\ 4)$ to rewrite x_2, x_3, and x_4 in terms of x_1 and x_0 and then collapse these three two-cells across free one-cells. The result is the model $K(\mathcal{Q}')$ of the presentation $\mathcal{Q}'$. ◇

5 The Fox ideal

Let $K = K(\mathcal{R})$ be the cellular model of the balanced presentation $\mathcal{R}$ for Thompson's group F. We now wish to identify certain elements in the Fox ideal $\mathcal{F}(K)$. In the proof of Lemma 4.1, the two-cells (0 3) and (1 4), identified as essential by Brown and Geoghegan, are seen to determine the defining relations in the two-relator presentation $\mathcal{Q}$ for F. Along the same lines, Brown and Geoghegan identified the three-cells (0 3 6) and (1 4 7) as essential. With this is mind, we now begin with the complex K and perform a series of elementary expansions to form a three-complex W that is contained in Y^3 and is large enough to support the attaching maps for these essential three-cells. The added cells in dimension two, together with the accompanying free one-dimensional faces, are as follows.

$$(5) + (3\ 5); \quad (6) + (4\ 6); \quad (7) + (5\ 7)$$

The elementary expansions in dimension three are these.

$$\begin{array}{lll} (0\ 4) + (0\ 2\ 4); & (0\ 5) + (0\ 3\ 5); & (0\ 6) + (0\ 4\ 6); \\ (1\ 5) + (1\ 3\ 5); & (1\ 6) + (1\ 4\ 6); & (1\ 7) + (1\ 5\ 7); \\ (2\ 5) + (0\ 2\ 5); & (2\ 6) + (2\ 4\ 6); & (2\ 7) + (2\ 5\ 7); \\ (3\ 6) + (1\ 3\ 6); & (4\ 7) + (2\ 4\ 7) & \end{array}$$

There is a strong deformation retraction $r : W \to K$ and we use the induced map $r_* : C_2(\widetilde{W}) \to C_2(\widetilde{K})$ to compute the coefficients of $r_*(\partial_3(0\ 3\ 6))$ and $r_*(\partial_3(1\ 4\ 7))$ in terms of the $\mathbf{Z}F$-basis for $C_2(\widetilde{K})$ consisting of the two-cells (0 2), (0 3), (1 3), (1 4), and (2 4). For example, since K is two-dimensional we have $r_*(\partial_3(0\ 2\ 4)) = 0$. Using the fact that r fixes K, we can thus render the added two-cell (0 4) in terms of the two-cells of K as follows.

$$r_*(0\ 4) = (2\ 4) - (1\ 3) \cdot x_0 + (0\ 3) \cdot x_2 + (0\ 2) \cdot (1 - x_4)$$

In this way, all of the free faces of the added three-cells of W can be expressed in terms of the two-cells of K.

Lemma 5.1 *The Fox ideal $\mathcal{F}(K) = \mathcal{F}(K(\mathcal{R}))$ contains the following two elements in the integral group ring $\mathbf{Z}F$.*

$$\begin{aligned} \xi_0 &= x_0^2 + x_3^2 + x_6 - x_0x_1 - x_3x_4 - 1 \\ \xi_1 &= x_0x_1^2 + x_0x_4^2 + x_0x_7 + x_1x_4x_5 + x_3x_4x_5 + x_2x_4^2 \\ &\quad + x_3 + x_1 + x_5 + x_2x_7 + x_4x_7 + x_7 \\ &\quad - x_0x_1x_2 - x_0x_4x_5 - x_0 - x_2x_4x_5 - x_3x_4^2 - x_1x_4^2 \\ &\quad - x_3x_7 - x_2 - x_4 - x_1x_7 - x_5x_7 - 1 \end{aligned}$$

Proof: The elements ξ_0 and ξ_1 are the negatives of the (1 4)-coefficients of $r_*(\partial(0\ 3\ 6))$ and $r_*(\partial_3(1\ 4\ 7))$ respectively. Details are left to the industrious reader. Persons finding errors in these calculations are encouraged to keep mum about it. ◇

Remark. The proof of Theorem 5.3 in [BG] actually shows that the elements $r_*(\partial_3(0\ 3\ 6))$ and $r_*(\partial_3(1\ 4\ 7))$ generate $\pi_2(K) \cong H_2(\widetilde{K})$ as a $\mathbf{Z}F$-submodule of $C_2(\widetilde{K}) \leq C_2(\widetilde{Y})$. A set of generators for the Fox ideal is obtained by calculating the coefficients of these homotopy generators. Lemma 5.1 is enough for our purposes.

6 Proof of Theorem 1.1

Our goal in this section is to prove that the two-complex $K = K(\mathcal{R})$ is absolutely Cockcroft. Theorem 1.1 follows at once from Lemma 2.1 and Lemma 4.2 since Cockcroft properties are preserved by homotopy equivalences. The proof of Theorem 1.1 proceeds directly from Lemma 5.1.

Throughout this section, we let H be a subgroup of $\pi_1(K) = F$ such that K is H-Cockcroft. Our aim is to prove that $H = F$. By Lemma 2.2, the Fox ideal $\mathcal{F}(K)$ is contained in the left ideal $\mathbf{Z}F \cdot (H - 1)$, which is the kernel of the canonical projection $\mathbf{Z}F \to \mathbf{Z}[F/H]$ from the free module of rank one onto the transitive permutation module $\mathbf{Z}[F/H]$ with $\mathbf{Z}$-basis consisting of left cosets gH, $g \in F$.

Lemma 6.1 *If some F-conjugate of x_1 belongs to H, then $H = F$.*

Proof: The generators x_n, $n \geq 1$, are all conjugate in F. Replacing H by an F-conjugate if necessary, we can assume that $x_6 \in H$. Since ξ_0 lies in the two-sided ideal $\mathcal{F}(K)$, we have

$$x_0^{-1} \cdot \xi_0 \cdot x_0 = x_0^2 + x_4^2 + x_7 - x_0 x_2 - x_4 x_5 - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H - 1).$$

Since this element lies in the kernel of $\mathbf{Z}F \to \mathbf{Z}[F/H]$, one of the cosets $x_0^2 H,\ x_4^2 H,\ x_7 H$ must equal $1H = H$. If $x_4^2 \in H$, then $x_8 = x_6^{x_4^2} \in H$ and so $x_7 = x_8^{x_6^{-1}} \in H$. Similarly, if $x_0^2 \in H$, then $x_7 \in H$. In all cases we therefore have $x_6,\ x_7 \in H$, and hence $x_n \in H$, $n \geq 6$. Now

$$x_0^{-5} \cdot \xi_0 \cdot x_0^5 = x_0^2 + x_8^2 + x_{11} - x_0 x_6 - x_8 x_9 - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H - 1)$$

and $x_8^2,\ x_{11},\ x_8 x_9,\ 1 \in H$ so $x_0^2 H = x_0 x_6 H = x_0 H$, whence $x_0 \in H$. Finally, $x_1 = x_6^{x_0^{-5}} \in H$, which implies that $H = F$ since F is generated by x_0 and x_1. ◇

Lemma 6.2 $x_0^2 \in H$.

Proof: Just suppose that $x_0^2 \notin H$, and so H contains no conjugate of x_1 by Lemma 6.1. For any $n \geq 1$,

$$x_0^{3-n} \cdot \xi_0 \cdot x_0^{n-3} = x_0^2 + x_n^2 + x_{n+3} - x_0 x_1^{x_0^{n-3}} - x_n x_{n+1} - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$$

so $H \in \{x_0^2 H,\ x_n^2 H,\ x_{n+3} H\}$. Since $x_0^2,\ x_{n+3} \notin H$, we obtain $x_n^2 \in H$ for all $n \geq 1$. Similarly, for any $m \geq 1$, we have

$$x_0^{1-m} \cdot \xi_0 \cdot x_0^{m-1} = x_0^2 + x_{m+2}^2 + x_{m+5} - x_0 x_m - x_{m+2} x_{m+3} - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$$

and so $x_0^2 H \in \{x_0 x_m H,\ x_{m+2} x_{m+3} H\}$. Suppose now that $m \geq 1$ is such that $x_0^2 H = x_0 x_m H$. Then for any $n > m+1$, it follows that $x_0^2 H = x_{n+2} x_{n+3} H$; for otherwise $x_0^2 H = x_0 x_m H = x_0 x_n H$ so that $x_0 H = x_m H = x_n H$ where $1 \leq m < n-1$, whence

$$x_0^2 H = x_0 x_n H = x_{n-1} x_0 H = x_{n-1} x_m H = x_m x_n H = x_m^2 H = H,$$

which contradicts the fact that $x_0^2 \notin H$. We conclude that there is a positive integer N such that $x_0^2 H = x_n x_{n+1} H$ for all $n \geq N$. For any $n \geq N$, we have

$$x_0^{-n} \cdot \xi_0 \cdot x_0^n = x_0^2 + x_{n+3}^2 + x_{n+6} - x_0 x_{n+1} - x_{n+3} x_{n+4} - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$$

and so $x_{n+6} H = x_0 x_{n+1} H = x_n x_0 H$. Thus, for each $n \geq N$, we have $x_n^{-1} x_{n+6} H = x_0 H$. Now, $x_N^{-1} x_{N+6} H = x_{N+1}^{-1} x_{N+7} H$ so that $x_{N+6}^{-1} x_N x_{N+1}^{-1} x_{N+7} = x_{N+6}^{-1} x_{N+7} x_N x_{N+1}^{-1} \in H$. Hence, $x_{N+6} x_{N+7} H = x_N x_{N+1} H = x_N x_{N+1}^{-1} H = x_{N+7}^{-1} x_{N+6} H = x_{N+6} x_{N+8}^{-1} H = x_{N+6} x_{N+8} H$. This implies that $x_{N+7} H = x_{N+8} H$, which leads to the contradiction

$$x_0^2 H = x_{N+7} x_{N+8} H = x_{N+7}^2 H = H.$$

This final contradiction completes the proof, showing that $x_0^2 \in H$. ◇

Our goal is to prove that $H = F$. Consider the element $x_0^{-1} \cdot \xi_0 \cdot x_0 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$, the fact that $x_0^2 \in H$ implies that

$$x_4^2 + x_7 - x_0 x_2 - x_4 x_5 \in \mathbf{Z}F \cdot (H-1).$$

There are these two cases to consider.

A1 $x_4^2 H = x_0 x_2 H$ and $x_7 H = x_4 x_5 H$

B1 $x_4 H = x_5 H$ and $x_7 H = x_0 x_2 H$

Case 1. Suppose first that the conditions **B1** hold. Since $x_0^2 \in H$, we have $x_{2k}H = x_{2k+1}H$ for all $k \geq 1$. Consider the image of the element $\xi_1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$ under the projection $\mathbf{Z}F \to \mathbf{Z}[F/H]$. Using **B1**, we have the following identifications of terms with opposite signs.

$$\begin{aligned} x_0x_4^2H &= x_0x_4x_5H \\ x_1x_4x_5H &= x_1x_4^2H \\ x_3x_4x_5H &= x_3x_4^2H \\ x_2x_4^2H &= x_2x_4x_5H \\ x_3H &= x_2H \\ x_5H &= x_4H \\ x_2x_7H = x_6x_2H &= x_6x_3H = x_3x_7H \\ x_4x_7H = x_6x_4H &= x_6x_5H = x_5x_7H \end{aligned}$$

Working modulo $\mathbf{Z}F \cdot (H-1)$, we cancel these terms and find that

$$x_0x_1^2 + x_0x_7 + x_1 + x_7 - x_0x_1x_2 - x_0 - x_1x_7 - 1 \in \mathbf{Z}F \cdot (H-1).$$

It follows that $H = 1H \in \{x_0x_1^2H,\ x_0x_7H,\ x_1H,\ x_7H\}$. If H is equal to x_1H or x_7H, then $H = F$ by Lemma 6.1. Similarly, if $H = x_0x_7H = x_0^2x_2H$, then $x_2 \in H$ (using **B1** and Lemma 6.2), and so $H = F$ by Lemma 6.1. We may therefore assume that $x_0x_1^2 \in H$. Next, we have $x_1H \in \{x_0x_1x_2H,\ x_0H,\ x_1x_7H\}$. If $x_1H = x_1x_7H$, then $x_7 \in H$, and so $H = F$ by Lemma 6.1. If $x_1H = x_0H$, then $H = x_0x_1^2H = x_0x_1x_0H = x_0^2x_2H$, and so $x_2 \in H$, whence $H = F$ as before. We may therefore assume that $x_1H = x_0x_1x_2H$, which implies that $x_7H \in \{x_0H,\ x_1x_7H\}$. If $x_7H = x_1x_7H$, then $x_1^{x_7} \in H$ and so $H = F$ by Lemma 6.1. We can therefore assume that $x_7H = x_0H$ and $x_0x_7H = x_1x_7H$. With all these assumptions in place, we have

$$H = x_0^2H = x_0x_7H = x_1x_7H = x_1x_0H = x_0x_2H = x_7H$$

with the last equality coming thanks to **B1**. We conclude finally that $x_7 \in H$ and so $H = F$ by Lemma 6.1.

Case 2. Now suppose that the conditions **A1** hold. In particular, $x_7H = x_4x_5H$. Considering the image of ξ_1 under $\mathbf{Z}F \to \mathbf{Z}[F/H]$, we have the following identifications of terms having opposite signs.

$$\begin{aligned} x_0x_7H &= x_0x_4x_5H \\ x_1x_4x_5H &= x_1x_7H \\ x_3x_4x_5H &= x_3x_7H \\ x_2x_7H &= x_2x_4x_5H \end{aligned}$$

We conclude that the following element lies in $\mathbf{Z}F \cdot (H-1)$.

$$\lambda = x_0x_1^2 + x_0x_4^2 + x_2x_4^2 + x_3 + x_1 + x_5 + x_4x_7 + x_7 - x_0x_1x_2 - x_0 - x_3x_4^2 - x_1x_4^2 - x_2 - x_4 - x_5x_7 - 1$$

Now consider the element $\xi_0 = x_0^2 + x_3^2 + x_6 - x_0x_1 - x_3x_4 - 1 \in \mathcal{F}(K) \subseteq \mathbf{Z}F \cdot (H-1)$. Since $x_0^2 \in H$ we have that

$$x_3^2 + x_6 - x_0x_1 - x_3x_4 \in \mathbf{Z}F \cdot (H-1).$$

It follows that one of the following two conditions is satisfied.

A0 $x_3^2H = x_0x_1H$ and $x_6H = x_3x_4H$

B0 $x_3H = x_4H$ and $x_6H = x_0x_1H$

We treat these two possibilities as separate subcases.

Subcase 2(i) Suppose that the condition **B0** is satisfied. In particular, $x_{2k}H = x_{2k-1}H$ for all $k \geq 1$ since $x_0^2 \in H$. In this event, there are the following additional cancellations of terms when the element λ is projected into $\mathbf{Z}[F/H]$. We have

$$\begin{aligned} x_0x_1^2H &= x_0x_1x_2H \\ x_2x_4^2H = x_3^2x_2H &= x_3^2x_1H = x_1x_4^2H \\ x_3H &= x_4H \\ x_1H &= x_2H \end{aligned}$$

and also

$$\begin{aligned} x_4x_7H &= x_6x_4H = x_6x_3H = x_3x_7H = x_3x_4x_5H = x_3x_4x_6H = x_3x_5x_4H \\ &= x_3x_5x_3H = x_3^2x_6H = x_3^2x_5H = x_3x_4x_3H = x_3x_4^2H. \end{aligned}$$

We are left to conclude that $H \in \{x_0x_4^2H,\ x_5H,\ x_7H\}$. Since the hypothesis **A1** provides that $x_0x_4^2H = x_0^2x_2H$, we find that H contains a conjugate of x_1 and so $H = F$ by Lemma 6.1.

Subcase 2(ii) Here we assume that the conditions **A0** are satisfied. Consider again the image of λ in $\mathbf{Z}[F/H]$. One of the positive terms must cancel with the negative term $-1H$. If any of $x_3,\ x_1,\ x_5$, or x_7 lies in H, then $H = F$ by Lemma 6.1. If $H = x_0x_4^2H = x_0^2x_2H$, then $x_2 \in H$ and so $H = F$ (using **A1** and Lemma 6.1). We may therefore assume that

$$H \in \{x_0x_1^2H,\ x_2x_4^2H,\ x_4x_7H\}.$$

If $x_0x_1^2 \in H$, then conjugating by $x_0^2 \in H$ we have $x_0x_3^2 \in H$. Using **A0** we find

$$x_0x_1H = x_3^2H = x_0^{-1}H = x_0H$$

so that $x_1 \in H$, whence $H = F$.

Now suppose that $x_2x_4^2 = x_3^2x_2 \in H$. Then $x_3x_2H = x_3^{-1}H$ and using **A1** we find

$$H = x_2x_4^2H = x_2x_0x_2H = x_0x_3x_2H = x_0x_3^{-1}H = x_2^{-1}x_0H$$

and so $x_0H = x_2H$. Using **A0** together with conjugation by $x_0^{-2} \in H$, we obtain

$$x_4H = x_1x_2H = x_1x_0H = x_0x_2H = x_0^2H = H$$

whence $H = F$ by Lemma 6.1.

For the final case, suppose that $x_4x_7 = x_6x_4 \in H$. Conjugating by $x_0^{-2} \in H$, we have $x_4x_2 \in H$, so that $x_2H = x_4^{-1}H = x_7H$. Using **A0** and the fact that $x_6^{-1}H = x_4H$, it follows that $x_6H = x_3x_4H = x_3x_6^{-1}H = x_5^{-1}x_3H$ and so $x_3H = x_5x_6H = x_8H$. Conjugating by powers of x_0^2, it follows that $x_kH = x_{k+5}H$ for all $k \geq 1$. With **A0** and **A1**, conjugation by powers of x_0^2 reveals that $x_0x_kH = x_{k+2}^2H$ and $x_kx_{k+1}H = x_{k+3}H$ for all $k \geq 1$. Now,

$$x_1x_0H = x_0x_2H = x_4^2H = x_4x_9H = x_8x_4H = x_8x_9H = x_{11}H = x_1H$$

which implies that $x_0 \in H$. With this we have $x_4H = x_4x_0H = x_0x_5H = x_7^2H = x_7x_2H = x_2x_8H = x_2x_3H = x_5H$. Conjugating by powers of x_0, this leads to the conclusion that $x_kH = x_{k+1}H$ for all $k \geq 1$. Finally, $x_1^2H = x_1x_2H = x_4H = x_1H$ and so $x_1 \in H$. This implies that $H = F$ and completes the proof of Theorem 1.1.

7 Cockcroft properties and Euler characteristic

The Euler characteristic of a finite connected Cockcroft two-complex is determined by its fundamental group. For if L is a finite connected two-complex with fundamental group isomorphic to G, then $H_1(L) \cong H_1(G)$ and L is Cockcroft if and only if the canonical surjection $H_2(L) \to H_2(G)$ is an isomorphism, whence

$$\chi(L) = 1 - \mathrm{rk}_{\mathbf{Z}}(H_1(G)) + \mathrm{rk}_{\mathbf{Z}}(H_2(L)) \geq 1 - \mathrm{rk}_{\mathbf{Z}}(H_1(G)) + \mathrm{rk}_{\mathbf{Z}}(H_2(G)),$$

with equality holding if and only if L is Cockcroft. Applying this to Thompson's group F, let L be a finite connected two-complex with fundamental group isomorphic to F. Using the fact that $H_1(F)$ and $H_2(F)$ are both free abelian of rank two, the Euler characteristic formula above shows that L is Cockcroft if and only if $\chi(L) = 1$.

Lustig has asked [DGH, Problem 19] whether the Cockcroft properties of a finite connected two-complex are determined by the Euler characteristic and fundamental group. More precisely, suppose that K and L are finite connected two-complexes with isomorphic fundamental groups and equal Euler characteristics. Is there an isomorphism $\mathbf{c}(K) \to \mathbf{c}(L)$ of partially ordered sets? Is such an isomorphism of partially ordered sets induced by an isomorphism $\pi_1(K) \to \pi_1(L)$ of fundamental groups? Does each isomorphism $\pi_1(K) \to \pi_1(L)$ induce an isomorphism $\mathbf{c}(K) \to \mathbf{c}(L)$? Each of these questions has a positive answer if we restrict attention to normal subgroups.

Theorem 7.1 *Let K and L be two finite CW-complexes with equal Euler characteristics. If $\theta : \pi_1(K) \to \pi_1(L)$ is a group isomorphism and N is a normal subgroup of $\pi_1(K)$ such that L is $\theta(N)$-Cockcroft, then K is N-Cockcroft.*

Proof: Since the Cockcroft properties of a 2-complex depend only on the (simple) homotopy type, we may apply elementary expansions to K and L without changing the problem. Hence we may assume, without loss of generality, that K and L have isomorphic 1-skeleta, and that moreover this isomorphism of 1-skeleta induces the given isomorphism θ between the fundamental groups.

Let G denote the fundamental group $\pi_1(K)$. Then G acts on the universal covering $\tilde{L}$ via θ, and so the cellular chain complexes $C_*(\tilde{K})$ and $C_*(\tilde{L})$ are both chain complexes of free right $\mathbf{Z}G$-modules.

The isomorphism of 1-skeleta also induces an isomorphism of $\mathbf{Z}G$-chain complexes between the cellular chain complexes of the 1-skeleta of the universal covers of K and L. This can be extended to a $\mathbf{Z}G$-chain homomorphism $\phi : C_*(\tilde{K}) \to C_*(\tilde{L})$. Now consider the effect of applying the functor $- \otimes_{\mathbf{Z}N} \mathbf{Z}$ to these chain complexes and chain maps. We obtain the cellular chain complexes of the covers K_N and $L_{\theta(N)}$ of K and L corresponding to N and $\theta(N)$ respectively, and a $\mathbf{Z}(G/N)$-chain map between them that we will also call ϕ. Note that there is an isomorphism between the 1-skeleta of K_N and $L_{\theta(N)}$ that induces an isomorphism of fundamental groups, and that the corresponding isomorphism between chain-complexes is given by ϕ in dimensions 0 and 1. It follows that ϕ restricts to an isomorphism between the 1-dimensional cycle and boundary groups Z_1 and B_1 of K_N and $L_{\theta(N)}$.

Now consider the short exact sequences

$$0 \to H_2(K_N) \to C_2(K_N) \to B_1(K_N) \to 0$$

and

$$0 \to H_2(L_{\theta(N)}) \to C_2(L_{\theta(N)}) \to B_1(L_{\theta(N)}) \to 0.$$

The chain-map ϕ induces an isomorphism on B_1, and an epimorphism on H_2 (since $H_2(L_{\theta(N)}) \to H_2(N)$ is an isomorphism), and hence also an epimorphism on C_2, by the 5-Lemma. On the other hand, the fact that K and L have equal Euler characteristics and isomorphic 1-skeleta implies that the free $\mathbf{Z}(G/N)$-modules $C_2(K_N)$ and $C_2(L_{\theta(N)})$ have equal ranks. The epimorphism $\phi_* : C_2(K_N) \to C_2(L_{\theta(N)})$ is therefore an isomorphism, by Kaplansky's theorem [K, M]. Hence so is $\phi_* : H_2(K_N) \to H_2(L_{\theta(N)}) \cong H_2(N)$, by the 5-Lemma again. Hence K is N-Cockcroft, as claimed. $\diamond$

Corollary 7.2 *For any finite connected two-complex K, the automorphism group* $\mathrm{Aut}(\pi_1(K))$ *acts on the partially ordered set* $\mathbf{nc}(K)$ *of all normal Cockcroft subgroups of* $\pi_1(K)$.

Proof: Taking $L = K$ in Theorem 7.1, we see that $\mathbf{nc}(K) = \{N \lhd \pi_1(K) : K \text{ is } N\text{-Cockcroft}\}$ is invariant under the natural action of $\mathrm{Aut}(\pi_1(K))$ on the set of subsets of $\pi_1(K)$. $\diamond$

Theorem 1.1 shows that there is a finite connected absolutely Cockcroft two-complex K with fundamental group F and Euler characteristic one. The following restatement of Corollary 1.2 is an immediate consequence of Theorem 1.1 and Theorem 7.1.

Corollary 7.3 *Let L be a finite connected two-complex with fundamental group isomorphic to Thompson's group F. For a normal subgroup $N \lhd \pi_1(L)$, L is N-Cockcroft if and only if $\chi(L) = 1$ and $N = \pi_1(L)$.*

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