Bestvina-Brady Groups and the Plus Construction

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1 Introduction

A recent result of Bestvina and Brady [1], Theorem 8.7, shows that one of two outstanding questions has a negative answer: either there exists a group of cohomological dimension 2 and geometric dimension 3 (a counterexample to the Eilenberg-Ganea Conjecture [4]), or there exists a nonaspherical subcomplex of an aspherical 2-complex (a counterexample to the Whitehead Conjecture [11]). More precisely, they consruct a family of groups which are potential counterexamples to the Eilenberg-Ganea Conjecture, each of which has cohomological dimension 2. These are also examples of groups of type FP_2 which are not finitely presented (see [1]). For one of these examples, they show that any 2-dimensional classifying space would give rise to a counterexample to the Whitehead conjecture.

We will refer to the examples cited above as *Bestvina-Brady groups.* These come equipped with natural, nonpositively curved cubical 3-dimensional classifying complexes, which we will call *Bestvina-Brady complexes.* In this short note, we show that these Bestvina-Brady complexes are (up to homotopy equivalence) formed by applying the Quillen plus construction to certain finite 2-complexes. From this, together with known facts about 2-complexes with aspherical plus constructions, we recover the result of Bestvina and Brady [1] that the Bestvina-Brady groups act freely on acyclic 2-complexes, and hence have cohomological dimension at most 2. It also follows that these groups have free relation modules of finite rank, and so are of type FF. Finally, we use our construction to give an alternative proof of the cited theorem of Bestvina and Brady: at least one of the Eilenberg-Ganea and Whitehead Conjectures is false.

2 Preliminaries on Graph Groups

In this section we prove a result on graph groups that is probably well-known. Recall that the *graph group* (or *right-angled Artin group*) $G = G(K)$ of a graph K is the group given by the presentation whose generators are the vertices of K, and whose defining relators are the commutarors $[u, v]$ for every edge $\{u, v\}$ of K.

Our result says that graph groups are *indicable throughout* in the sense of Higman [7]. In particular, they are *locally indicable* (every nontrivial, finitely generated subgroup admits an epimorphism onto the infinite cyclic group). Moreover, graph groups have no nontrivial perfect subgroups: this is the property that we will apply to the Bestvina-Brady groups in sections 4 and 5

Lemma 2.1 *Let G be a graph group, and H a nontrivial subgroup of G. Then there is an epimorphism $H \to \mathbb{Z}$.*

Proof. We first prove the result for finite graphs. Let $G = G(K)$ be the graph group on a finite graph K. We argue by induction on the number of vertices of K. If K is empty then $G = \{1\}$ and there is nothing to prove. Otherwise let v be a vertex of K and let L be the full subgraph of K on the vertex set $V(K)\backslash\{v\}$. Let N be the free group on a basis in one-to-one correspondence with the set of right cosets of U in $G(L)$, where U is the subgroup generated by the vertices adjacent to v in K. Now let $H = N \rtimes G(L)$, where $G(L)$ acts on N by permuting basis elements. Then H has a presentation

$$\langle V(L) \cup \{v\} \mid [a, b] = 1 \ (\{a, b\} \in E(L)),\ [u, v] = 1 \ (\{u, v\} \in E(K)) \rangle$$

which is identical to the graph group presentation of G. Hence $G \cong H$. By induction, we can assume the result for $G(L)$, and it follows for $G(K)$ by the isomorphism $G(K) \cong N \rtimes G(L)$.

To prove the general result, note that a graph group $G = G(K)$ is the union of the subgroups $G(L)$ for all finite subgraphs L of K, and that $G(L)$ is a retract of $G(K)$ for all L (see eg [10], Lemma 2). Thus if H is a nontrivial subgroup of $G(K)$, there is a finite subgraph L of K and an epimorphism $\phi : G(K) \to G(L)$ with $\phi(H) \neq \{1\}$. Composing $\phi|_H$ with an epimorphism $\phi(H) \to \mathbb{Z}$ gives the desired epimorphism $H \to \mathbb{Z}$.

Corollary 2.2 *Let G be a graph group. Then G is locally indicable, and has no nontrivial perfect subgroups.*

3 Preliminaries on the Plus Construction

Let K be a connected CW-complex and P a perfect normal subgroup of $\pi_1(K)$. Then a construction due to Quillen gives a complex K^{+P} such that

1. K^{+P} contains K as a subcomplex, and is obtained by attaching to K cells in dimensions less than or equal to 3.

2. $K \to K^{+P}$ is a homology equivalence.

3. $\pi_1(K) \to \pi_1(K^{+P})$ is an epimorphism with kernel P.

Indeed, the complex K^{+P} is determined up to homotopy equivalence by these properties. There is a choice of K^{+P} that is obtained from K by adding cells only in dimensions 2 and 3, but we will not assume that this holds below. Any complex K^{+P} satisfying the above conditions will be called a *plus construction* on K.

In the special case where P is the (unique) *maximal* perfect subgroup of $\pi_1(K)$, we will write K^+ for K^{+P}.

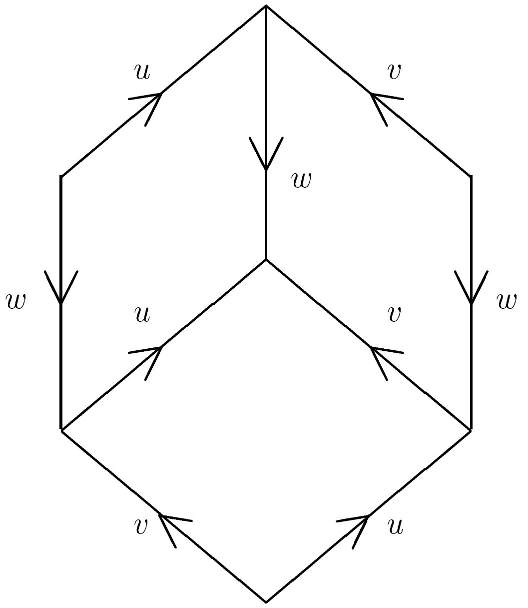
Plus constructions have been applied to 2-complexes in the context of the Whitehead conjecture and related asphericity questions by Haussman [6] and Gilbert [5].

4 Bestvina-Brady Groups

Let X be a finite, acyclic 2-dimensional simplicial complex that is a *flag complex* (every set of mutually adjacent vertices in the graph $X^{(1)}$ is the vertex set of a simplex of X). Define a new 2-complex $Y = Y(X)$ as follows. Y has two 0-cells, 0 and 1. For each 0-simplex v of X, there is a 1-cell y_v of Y, with initial vertex 0 and terminal vertex 1. For each 2-simplex $\{u, v, w\}$ of X, there is a 2-cell attached according to the word $y_u y_v^{-1} y_w y_u^{-1} y_v y_w^{-1}$.

Define a cubical 3-complex $Z = Z(X)$ as follows. Z has a single 0-cell, a 1-cell z_v for each 0-simplex of X, a 2-cell attached according to the commutator word $[z_u, z_v]$ for each 1-simplex $\{u, v\}$ of X, and a 3-cube for each 3-simplex $\{u, v, w\}$ of X, with faces the commutator 2-cells $[z_u, z_v]$, $[z_u, z_w]$ and $[z_v, z_w]$, put together in the obvious way. Then $\pi_1(Z)$ is the graph group $G = G(X^{(1)})$ on the 1-skeleton of X, and indeed (see eg [1]) Z is nonpositively curved, and so a $K(G, 1)$-complex.

There is a combinatorial map $\phi : Y \to Z$ that sends each y_v to z_v (with positive orientation), where the 2-cells of Y are mapped according to the Dehn diagram



On the π_1 level, the image , of ϕ_* is the kernel of the homomorphism $G \to \mathbb{Z}$ that sends each generator z_v of G to the same generator $+1$ of $\mathbb{Z}$. Let P denote the kernel of ϕ_*, and let Z_Γ denote the regular covering of Z corresponding to the normal subgroup , . We call , the *Bestvina-Brady group* determined by X, and Z_Γ the *Bestvina-Brady complex* determined by X. Dicks and Leary [3] construct a presentation for , . In the case where X is simply connected, they show that a certain subpresentation presents , : this subpresentation in general presents $\pi_1 Y$. Thus this theorem can be regarded as saying that $P = 1$ when X is simply connected. Our principal result concerns P for a general acyclic X.

Theorem 4.1 *The subgroup P is the maximal perfect subgroup of $\pi_1(Y)$, and Z_Γ is homotopy equivalent to Y^+, the Quillen plus construction on Y.*

Proof. We begin by noting that the covering complex Z_Γ has 0-skeleton $\mathbb{Z} = G/,$, and that the map $\phi : Y \to Z$ lifts to a map $\phi' : Y \to Z_\Gamma$ sending the zero-cells $0, 1$ to $0, 1$ respectively.

For integers $a < b$, let $Z_{[a,b]}$ denote the full subcomplex of Z_Γ with vertex set $\{a, a{+}1, \ldots, b\}$. Then $Z_{[0,1]}$ is 1-dimensional, and indeed ϕ restricts to an isomorphism $Y^{(1)} \to Z_{[0,1]}$.

Next, we construct a 3-complex W with 2-skeleton $Y \cup_{Z_{[0,1]}} Z_{[0,2]}$ as follows. For each 2-simplex $\{u, v, w\}$ of X we have a corresponding 2-cell α in Y: attach a 3-cell with boundary $\alpha \cup \phi(\alpha)$. (Note that α and $\phi(\alpha)$ have the same boundary in $Z_{[0,1]}$, so this makes sense.) Now we have a one-to-one correspondence between the 2-cells of Y and the 3-cells of W, with each 2-cell of Y appearing precisely once in the boundary of the corresponding 3-cell of W. Hence W collapses to $Z_{[0,2]}$, and so in particular $Z_{[0,2]}$ is homotopy equivalent to W. On the other hand, W is formed from Y by attaching cells in dimensions up to 3, $Y \to W$ is a homology equivalence, and $\pi_1(Y) \to \pi_1(W)$ is surjective with perfect kernel K say, so $W = Y^{+K}$. (Indeed, K is generated as a normal subgroup by paths of the form $(u_2 u_1^{-1}) \ldots (u_n u_{n-1}^{-1})(u_1 u_n^{-1})$, where $(u_1, \ldots, u_n, u_1)$ is a closed path in X. Those closed paths bounding 2-simplices of X correspond to relations in $\pi_1(Y)$, and since X is acyclic it follows that K is perfect.)

Given $Z_{[a,b]}$, we can construct $Z_{[a,b+1]}$ by attaching cells in dimensions up to 3. As with the map $Y \to W$ above, the map $Z_{[a,b]} \to Z_{[a,b+1]}$ is a homology equivalence, and at the fundamental group level we have an epimorphism with perfect kernel, so $Z_{[a,b+1]}$ is obtained by applying a plus construction to $Z_{[a,b]}$. Similar remarks apply to the map $Z_{[a,b]} \to Z_{[a-1,b]}$. Taking direct limits, we see that the map $Z_{[0,2]} \to Z_\Gamma$ is a plus construction map. Since $Z_{[0,2]}$ is homotopy equivalent to a plus construction on Y, so is Z_Γ.

Hence $P = \mathrm{Ker}(\pi_1(Y) \to \pi_1(Z_\Gamma))$ is a perfect normal subgroup of $\pi_1(Y)$, and Z_Γ is indeed homotopy equivalent to the plus construction Y^{+P} of Y associated to P. It remains to check that P is the maximal perfect subgroup of $\pi_1(Y)$, or equivalently that , has no nontrivial perfect subgroups. But this follows immediately from Corollary 2.2, since $G = G(X^{(1)})$ has no nontrivial perfect subgroups.

Corollary 4.2 (Bestvina and Brady [1]) , *acts freely on an acyclic 2-complex, and so has cohomological dimension at most 2.*

Proof. By [1], Z is nonpositively curved, and so aspherical. Hence so is Z_Γ, and hence Y^+. By [5] it follows that the covering complex Y_P of Y corresponding to the maximal perfect subgroup P of $\pi_1(Y)$ is acyclic, and clearly , $\cong \pi_1(Y)/P$ acts freely on Y_P.

Corollary 4.3 , *has a presentation on a finite generating set, in which the relation module is free of finite rank.*

Proof. Let $\Phi = \pi_1(Y^{(1)})$ and $N = \rho^{-1}(P)$, where $\rho : \Phi \to \pi_1(Y)$ is the natural epimorphism. Then Φ is a free group of finite rank, $1 \to N \to \Phi \to\ ,\ \to 1$ is a presentation of , , and the abelianisation N^{ab} of N is a relation module of , , so projective as a $\mathbb{Z},$ -module, since , has cohomological dimension at most 2. This relation module is also generated by the images of the 2-cells of Y. Let F be the free $\mathbb{Z},$ -module with the 2-cells of Y as basis, $\psi : F \to N^{ab}$ the corresponding epimorphism of $\mathbb{Z},$ -modules, and Q the kernel of ψ (so that $F \cong N^{ab} \oplus Q$). Note also that

$$N^{ab} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \cong H_2(,\ ) \cong F \otimes_{\mathbb{Z}\Gamma} \mathbb{Z},$$

since $N \subseteq [\pi_1 Y, \pi_1 Y]$ and $H_2(,\ ) \cong H_2(Y^+) \cong H_2(Y)$ is free abelian on the set of 2-cells of Y. Hence $Q \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} = 0$, in other words Q is a finitely generated, perfect projective $\mathbb{Z},$ -module. Since , has no nontrivial perfect subgroups, it follows that $Q = 0$ (see eg [9]), so that N^{ab} is free.

Corollary 4.4 *There is a finite free* $\mathbb{Z},$ *-resolution of* $\mathbb{Z}$ *of length 2.*

Proof. Indeed, the cellular chain complex of Y_P is such a resolution.

Corollary 4.5 *Any* $K(,\ ,1)$*-complex with finite* 1*-skeleton has dimension at least 3.*

Proof. Suppose W is a $K(,\ ,1)$-complex of dimension 2 with $W^{(1)}$ finite. Then the number of 2-cells in W is given by $1 - \beta_1(,\ ) + \beta_2(,\ ) - \chi(W^{(1)})$. But this is finite, since $H_*(,\ ) = H_*(Y^+) = H_*(Y)$ and Y is finite. Hence , $= \pi_1(W)$ is finitely presented, a contradiction to the Main Theorem of [1].

Remark The group , has *infinite relation gap*, in the sense that it has a presentation $N \to \Phi \to\ ,$ (with Φ free of finite rank) such that N^{ab} is finitely generated as a $\mathbb{Z},$ -module, while N is not finitely generated as a normal subgroup. If one considers the sequence of finitely presented groups $,_n = \pi_1(Z_{[-n,n]})$ $(n \geq 1)$, then we have epimorphisms $,_n \to\ ,_{n+1}$ and , is the direct limit of the $,_n$. Each $,_n$ has a finite presentation with the same underlying free group Φ: $N_n \to \Phi \to\ ,_n$, where $N_1 \subset N_2 \subset \ldots$. Moreover, each $Z_{[-n,n]}$ is homotopy equivalent to a plus construction on Y, so N_n^{ab} is a free module over $\mathbb{Z},_n$ for all n, of fixed finite rank r, say. A natural question arises as to whether the minimal numbers of generators of the N_n as normal subgroups of Φ increase unboundedly as $n \to \infty$. If so, then the $,_n$ provide examples of finitely presented groups with arbitrarily high *finite* relation gaps. However, it is not *a priori* clear that N_n cannot be normally generated by r elements for all n.

5 Proof of Bestvina-Brady theorem

Throughout this section, we assume that the Eilenberg-Ganea and Whitehead conjectures are both true, and derive a contradiction. Recall [8] that a *tower map* between CW-complexes is a composite of a finite number of maps, each of which is either a covering or an embedding of a subcomplex. Since coverings of aspherical complexes are aspherical, the Whitehead conjecture implies that, whenever $K' \to K$ is a tower and K is an aspherical 2-complex, then K' is aspherical.

Now let X be a 2-dimensional simplicial (flag) complex which is finite, connected and acyclic, such that $\pi_1 X$ is isomorphic to the binary icosahedral group $SL(2,5)$, and let Γ and Z_Γ be the corresponding Bestvina-Brady group and complex, and let Y be the 2-complex constructed in section 4. Moreover, let g be any element of $\pi_1 X$ of order $k \geq 3$, so that $\pi_1 X$ is the normal closure of g, and let $(u_0, u_1, \ldots, u_m)$. Then the kernel P of $\pi_1 Y \to \Gamma$ is generated by $g_n = (u_0 u_1^{-1})^n \ldots (u_n u_0^{-1})^n$ for $n \in \mathbb{Z}$, and each g_n also has finite order in $\pi_1 Y$ (see, for example [3], Proposition 2).

Since Γ has cohomological dimension 2, the Eilenberg-Ganea conjecture implies the existence of a 2-dimensional $K(\Gamma, 1)$-complex W, say. There is a homotopy equivalence $\psi : Z_\Gamma \to W$, which we may assume without loss is reasonably nice: in particular we assume that W is a TCW-complex and that ψ is a *transverse map*, in the sense of [2], Chapter VII. It then follows from [8] that $\psi' = \psi \circ \phi' : Y \to W$ factors through a *maximal tower lift*

$$\begin{tikzcd}
 & W' \arrow[d, "p"] \\
Y \arrow[ur, "\psi''"] \arrow[r, "\psi'"'] & W
\end{tikzcd}$$

This is just a commutative diagram with p a tower map, that is maximal (ie universal) among all such commutative diagrams. Now from maximality of the lift ψ'', it follows that W' is finite and $\psi''_* : \pi_1 Y \to \pi_1 W'$ is surjective. Moreover the kernel of ψ''_* is contained in that of $\psi'_* : \pi_1 Y \to \pi_1 W$, which is P. If this kernel is equal to P, then $\pi_1 W' \cong \Gamma$, contradicting the fact that Γ is not finitely presented. Hence some g_n does not belong to the kernel of ψ''_*. But then $\psi''_*(g_n) \neq 1$ is an element of finite order in $\pi_1 W'$, and W' cannot be aspherical, contrary to Whitehead's conjecture.

This contradiction completes the proof.

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