Free subgroups in groups of small deficiency

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1 Introduction

The *deficiency* of a finite group presentation $\mathcal{P} : \langle\ X \mid R\ \rangle$ is $\mathrm{def}(\mathcal{P}) = |X| - |R|$, the difference between the numbers of generators and defining relators. For a given finitely presented group G, the deficiencies of finite presentations for G are bounded above (for example by the torsion free rank of the abelianisation G/G'). The *deficiency* of G, $\mathrm{def}(G)$, is the maximum of the deficiencies of all finite presentations of G.

The starting point for this paper is the following result of B Baumslag and Pride.

Theorem 1.1 [1] *If G is a finitely presented group with $\mathrm{def}(G) \geq 2$, then there is a subgroup H of finite index in G such that that H has a nonabelian free homomorphic image.*

In the language of [1], G is *as large as* F_2. Following Gromov [7], we will abbreviate this to *large*. Thus a group is defined to be *large* if some subgroup of finite index has a nonabelian free homomorphic image. Any large group is SQ-universal, and hence contains a nonabelian free subgroup, but the converse is not true: for example the Higman group

$$\langle\ a,\ b,\ c,\ d \mid a^b = a^2,\ b^c = b^2,\ c^d = c^2,\ d^a = d^2\ \rangle$$

has no proper subgroup of finite index [9], and hence also no free homomorphic image.

Theorem 1.1 was used by G Baumslag, Morgan and Shalen [2] to prove a result about *generalised triangle groups*. A *generalised triangle group* (of *type* (p, q, r)) is one with a presentation of the form

$$\langle\ x,\ y \mid x^p = y^q = W(x, y)^r = 1\ \rangle,$$

where p, q, r are integers greater than 1 and W is a word in x, y that is not equivalent (modulo the relations $x^p = y^q = 1$) to a conjugate of a power of x or of y, or to a proper power. Without loss of generality, we can write W in the form

$$x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_k} y^{\beta_k},$$

where $k \geq 1$ is the *length* of W and $1 \leq \alpha_i < p$, $1 \leq \beta_i < q$ for each i. We say that two words W, W' (and the corresponding presentations $\mathcal{P}, \mathcal{P}'$) are *equivalent* if they are related by the equivalence relation generated by cyclic permutation of words, inversion of words, and automorphisms of the finite cyclic groups $\langle x | x^p \rangle$, $\langle y | y^q \rangle$. Clearly equivalent generalised triangle group presentations present isomorphic groups. Let us define the *curvature* of a generalised triangle group presentation

$$\mathcal{P} : \langle\ x,\ y \mid x^p = y^q = W(x,y)^r = 1\ \rangle$$

to be $\kappa = \kappa(\mathcal{P}) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$.

Theorem 1.2 [2] *A generalised triangle group with $\kappa < 0$ is large.*

Now generalised triangle group presentations have deficiency -1 (and indeed it can easily be shown that these are efficient, so that $\text{def}(G) = -1$ for any generalised triangle group G). Thus in one sense this result can be regarded as an improvement on Theorem 1.1, whereby largeness of groups given by a class of presentations of deficiency less than 2 is proved, subject to a condition on relators being proper powers. There are other generalisations of Theorem 1.1 that fall into this category.

Theorem 1.3 [14, 7] *If G has a deficiency 1 presentation in which one of the relators is a proper power, then G is large.*

Theorem 1.4 [4] *If G has a deficiency 0 presentation in which two relators are p-th powers for some prime p and G/G' is infinite, then G is large.*

Theorem 1.5 [4] *If G/G' is finite, and G has a deficiency 0 presentation in which either three relators are proper powers, or two relators are proper powers with at least one being a third or higher power, then G is large.*

In [7], Gromov also gives another proof of the result of Baumslag and Pride, Theorem 1.1.

In a slightly different direction there is the result of Wilson and Zelmanov [19].

Theorem 1.6 [19] *Let G be a finitely generated group and let d denote the minimal number of generators of G/G'. If $d \geq 2$ and*

$$\text{def}(G) + \frac{d^2}{4} - d > 0 \tag{1}$$

then, for some prime p, the pro-p-completion of G contains a nonabelian free (abstract) subgroup.

In particular, no soluble group satisfies the inequality (1). Recall [8] that the Golod-Shafarevich theorem says that

$$\mathrm{def}(G) + \frac{d^2}{4} - d < 0$$

for any finite p-group G, so Theorem 1.6 can be regarded as a kind of 'characteristic zero Golod-Shafarevich Theorem'. Also of interest in this context is the following result of Wilson.

Theorem 1.7 [17] *If G is a soluble group of deficiency 1, then G is isomorphic to the metabelian group $\langle\ a,\ t \mid a^t = a^k\ \rangle$ for some $k \in \mathbb{Z}$.*

The present paper aims to obtain further generalisations of Theorem 1.1, motivated by two conjectures.

Conjecture 1.8 (Rosenberger [13]) *A generalised triangle group is either soluble-by-finite or contains a nonabelian free subgroup.*

The property described in this conjecture is known as a Tits alternative, since Tits [16] proved the analogous property for all linear groups. By Theorem 1.2 the conjecture is true when $\kappa < 0$. Rosenberger [13] and Levin and Rosenberger [11] have shown that the conjecture is also true if $r \geq 3$, and if the length of W is small. In this paper we will show that the conjecture is true when $\kappa = 0$.

Conjecture 1.9 (Wilson [18]) *Let G be a finitely presented group satisfying the hypotheses of Theorem 1.6. Then G contains a nonabelian free subgroup.*

Our principal result is the following.

Theorem A *Suppose that $\bar{K}$ is a connected regular covering complex of a finite 2-complex K, with nontrivial free abelian covering transformation group A. Suppose also that $H_2(\bar{K}, F)$ has a free FA-submodule of rank at least $1+\chi(K)$ for some field F, where χ denotes euler characteristic. Then $G = \pi_1(K)$ is large.*

Theorem 1.1 can be recovered from this as the special case $\chi(K) \leq -1$, where the existence of $\bar{K}$ follows from the negativity of the euler characteristic, and the existence of a rank 0 free submodule of H_2 is trivial. Theorems 1.3 and 1.4 can also be recovered from this by taking F to be a suitable finite prime field. Theorem 1.5 cannot be recovered from Theorem A, since there is no infinite abelian covering $\bar{K}$ of K when $H_1(K) \cong G/G'$ is finite.

By tensoring the cellular chain complex of $\bar{K}$ over FA with the field of fractions of FA, it can readily be checked that $H_2(\bar{K}, F)$ always contains a free FA-submodule of rank $\geq \chi(K)$ (see Proposition 2.1 below), so the hypothesis on H_2 in Theorem A cannot be significantly weakened. Our proof of Theorem A is based on a calculation of the growth of H_1 of finite subcomplexes of $\bar{K}$, inspired by Gromov's proofs of Theorems 1.1 and 1.3 in [7].

We apply Theorem A to obtain evidence in favour of Conjectures 1.8 and 1.9 as follows.

Theorem B *A generalised triangle group*

$$G = \langle\ x,\ y \mid x^p = y^q = W(x,y)^r = 1\ \rangle$$

with $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 = 0$ *contains a free subgroup of rank 2, except in the case where* W *is equivalent to* xy*, in which case* G *is isomorphic to the (soluble) euclidean triangle group* $T(p,q,r)$*.*

Theorem C *If* G *is a finitely presented group satisfying the hypotheses of Theorem 1.6, and if in addition* $\mathrm{def}(G) \geq 1$*, then* G *is large.*

A corollary is that Wilson's Conjecture holds when $d \leq 4$.

When $\mathrm{def}(G) \leq 0$, then $H_2(\bar{K}, F)$ will always contain a nonzero free FA-module. If there is a free FA-module of rank greater than $1 - \mathrm{def}(G)$, then we can apply Theorem A to find free subgroups, indeed to show that G is large. This trick is not available to us when $1 - \mathrm{def}(G)$ is the maximum rank of a free FA-submodule for all F. On the other hand, if $H_2(\bar{K}, F)$ itself is free as an FA-module, then there are other tricks we can use.

Theorem D *Let* $\bar{K} \to K$ *be a regular covering of a finite 2-complex* K*, with free abelian covering transformation group* A *of rank at least 2. Suppose* F *is a field such that* $H_1(\bar{K}, F) \neq 0$ *and* $H_2(\bar{K}, F)$ *is a free* FA*-module. Then there is a subgroup* $B \subset A$ *of such that* $A/B \cong \mathbb{Z}^2$ *and* $H_1(\bar{K}, F)$ *contains a nonzero free* FB*-submodule.*

An elementary lemma about Laurent polynomial rings shows that $H_2(\bar{K}, F)$ is indeed a free FA-module of rank 1, in the case where $\mathrm{def}(\pi_1(K)) = 0$ and $H_2(\bar{K}, F)$ contains no free FA-submodule of rank 2. Combining this with results of Bieri and Strebel [3], we obtain the following.

Corollary 3.2 *Let* G *be a group having a finite presentation of deficiency 0. Suppose* $N \lhd G$ *is such that* G/N *is free abelian of rank 3, and* $N/N' \neq 0$*. Suppose in addition that* G *admits an automorphism* ϕ *such that* $\phi(N) = N$ *and* ϕ *induces the antipodal automorphism* $\alpha \mapsto -\alpha$ *on the free abelian group* G/N*. Then* N *contains a free subgroup of rank 2.*

This in turn can be applied to generalised triangle groups as follows. If G is a generalised triangle group with presentation $\langle\ x,\ y \mid x^p = y^q = W(x,y)^r = 1\ \rangle$, then a homomorphism $\phi : G \to H$ is said to be *essential* if $\phi(x)$, $\phi(y)$ and $\phi(W)$ have orders p, q, r respectively in H. (Note that this definition depends on the equivalence class of the given presentation for G.)

Theorem E *Let* G *be a generalised triangle group of type* $(2, 2q, 2)$ *with* $q \geq 4$*. If* G *admits an essential epimorphism onto the cyclic group of order* $2q$*, then* G *contains a nonabelian free subgroup.*

The rest of the paper is organised as follows. In §2 we prove Theorem A and give some applications, including Theorem C above. In §3 we prove the results on free modules over Laurent polynomial rings, and apply these to the study of groups of deficiency zero with abelianisation of rank at least 4. Finally, in §4 we apply our results about groups of small deficiency to the question of the existence of free subgroups in generalised triangle groups.

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2 Growth and homology of free abelian covers

Consider the following situation. We have a finite connected 2-complex K and a normal subgroup N of $\pi_1(K)$ such that $A = \pi_1(K)/N$ is free abelian of rank $n > 0$. Up to homotopy equivalence, we may assume that K has a single 0-cell, and 1-cells $x_1, \dots, x_n, y_1, \dots, y_m$, where the x_i form a basis for A and the y_i represent elements of N. Let $\bar{K}$ denote the regular covering of K corresponding to N. Then the 0-cells of $\bar{K}$ can be identified with the elements of A (or, equivalently, by n-tuples of integers), the 1-cells lying over the y_i are loops, while those lying over the x_i connect adjacent integer lattice points in $\mathbb{Z}^n \cong A$.

For each positive integer k, we define the *cube of side* k in $\bar{K}$ to be the full subcomplex C_k on the set of 0-cells $\{(a_1, \dots, a_n) \ ; \ |a_i| \leq k \ \forall i\}$. By this we mean that this set is the set of 0-cells of C_k, and a 0- or 1-cell of $\bar{K}$ belongs to C_k if and only if all the 0-cells in its boundary belong to C_k.

Note that the number of cells of C_k in each dimension is given by a polynomial of degree n in k. We are interested in the asymptotic behaviour of the first Betti number $\beta_1(C_k)$ for large k. Clearly this is bounded above by a polynomial of degree n.

Proposition 2.1 *Let F be a field and K and $\bar{K}$ be as above. Then $H_2(\bar{K}, F)$ contains a free FA-submodule of rank at least* $\max\{0, \chi(K)\}$. *If $H_2(\bar{K}, F)$ contains a free FA-submodule of rank greater than $\chi(K)$, then* $\dim_F(H_1(C_k, F))$ *grows (for large k) like a polynomial of degree n in k.*

Proof. Let $\mathcal{F}$ denote the field of fractions of the integral domain FA. Let $\mathcal{C}$ denote the cellular chain complex of $\bar{K}$ with F-coefficients, and let $\mathcal{D}$ denote $\mathcal{C} \otimes_{FA} \mathcal{F}$. Then $\mathcal{D}_i$, for $i = 0, 1, 2$, is an $\mathcal{F}$-vector space whose dimension is equal to the number of i-cells in K. Moreover, $H_0(\mathcal{D}) = 0$, so $H_2(\mathcal{D})$ must have dimension at least $\max\{0, \chi(K)\}$.

The first part of the statement follows immediately. If $H_2(\bar{K}, F)$ contains a free FA-submodule of rank greater than $\chi(K)$, then $\dim_{\mathcal{F}}(H_2(\mathcal{D})) > \chi(K)$, so $\dim_{\mathcal{F}}(H_1(\mathcal{D})) > 0$, and so $H_1(\bar{K}, F)$ contains a free FA-module of rank at least 1. Let α be a 1-chain representing a generator of a rank 1 free FA-submodule of $H_1(\bar{K}, F)$, and let δ be the least positive integer such that α is supported in C_δ. Then, for $k > \delta$ there are at least $(2k - 2\delta + 1)^n$ translates of α supported in C_k. Since these are F-linearly independent in $H_1(\bar{K}, F)$, they must also be F-linearly independent in $H_1(C_k, F)$, so $\dim_F(H_1(C_k, F)) \geq (2k - 2\delta + 1)^n$ for $k > \delta$. □

Theorem 2.2 *With the above notation, if* $\dim_F(H_1(C_k, F))$ *grows faster than any polynomial of degree* $n - 1$*, then for any sufficiently large natural number* p *the subgroup* $N \cdot \pi_1(K)^p$ *admits a homomorphism onto the free group of rank 2.*

Proof. Clearly we may assume that K contains at least one 2-cell, for otherwise $\pi_1(K)$ is free of rank greater than 1. Let δ denote the maximum diameter of any 2-cell of $\bar{K}$, in other words the maximum norm of $\alpha \in A$ such that some 2-cell of $\bar{K}$ meets 0-cells x and $\alpha(x)$. Here the norm of $\alpha = x_1^{a_1} \ldots x_n^{a_n}$ is the L_∞ norm $\max\{|a_1|, \ldots, |a_n|\}$, and δ is finite by the hypotheses on K.

For each $k > \delta$, consider the subcomplex D_k of $C_k^{(1)}$ formed by removing all the 0-cells of $C_{k-\delta-1}$, together with any 1-cell incident with such a 0-cell. The number of cells in each dimension of D_k is given by a polynomial in k of degree $n - 1$. Since the first Betti number $\beta_1(C_k) = \dim_F(H_1(C_k, F))$ grows faster than any such polynomial, it follows that $\beta_1(C_k)$ is greater than the number of 1-cells in D_k for $k \gg 0$, and hence that $\beta_1(C_k/D_k) > 0$ for $k \gg 0$. Note also that, if a 2-cell of $\bar{K}$ is not in C_k, then it meets no 0-cell of $C_{k-\delta-1}$.

Now choose k large enough so that $\beta_1(C_k/D_k) > 0$, and choose $p > 6k + 6$, so that the p^n-fold cover $K_p = \bar{K}/A^p$ of K contains three disjoint copies $C(1)$, $C(2)$ and $C(3)$ of C_k. Let $D(1)$, $D(2)$, $D(3)$ denote the copies of D_k contained in $C(1)$, $C(2)$ and $C(3)$ respectively, and let Z be the subcomplex of $K_p^{(1)}$ consisting of $D(1)$, $D(2)$, $D(3)$ and the complement of $C(1) \cup C(2) \cup C(3)$ in $K_p^{(1)}$. Then Z is connected if $n > 1$, and consists of 3 connected components if $n = 1$. We form a quotient space Y of K_p by shrinking each component of Z to a point.

Note that every 2-cell of K_p that is not contained in one of $C(1)$, $C(2)$ or $C(3)$ is bounded by a path that is entirely contained in Z. For $n > 1$ it follows that Y is homeomorphic to the wedge of $C(1)/D(1)$, $C(2)/D(2)$ and $C(3)/D(3)$. Since each of these has positive first Betti number, it follows that there are epimorphisms

$$\pi_1(K_p) \to \pi_1(Y) \to (\mathbb{Z}/q\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}),$$

where q is the the characteristic of the field F. In all cases, a subgroup of finite index in $\pi_1(K_p)$ admits a nonabelian free homomorphic image.

In the case $n = 1$, Y is homotopy equivalent to

$$C(1)/D(1) \vee C(2)/D(2) \vee C(3)/D(3) \vee S^1$$

and the result follows in a similar way. □

Theorem A follows immediately from Proposition 2.1 and Theorem 2.2. As explained in §1, Theorems 1.1, 1.3 and 1.4 follow from Theorem A. We give some further consequences below.

Corollary 2.3 *If G is a finitely presented group that admits a homomorphism onto the free metabelian group of rank 2, then G is large, and hence also SQ-universal.*

Proof. Let M denote the free metabelian group of rank 2, and let $\phi : G \to M$ be an epimorphism. Then let $N = \phi^{-1}([M, M])$ and choose a 2-complex K with $\pi_1(K) = G$. Since $A = G/N \cong M/[M, M]$ is free abelian of rank 2, we have $n = 2$ in Theorem 2.2. On the other hand $H_1(\bar{K}) = [M, M]$ is the free $\mathbb{Z}A$-module on the single generator $[x_1, x_2]$, and it follows that $\beta_1(C_k) = 4k^2$ for all k. Now apply Theorem 2.2 with $F = \mathbb{Q}$. □

Remark The above result may be compared with a theorem of Bieri and Strebel [3], Theorem 4.1: if $N \lhd G$ with G/N abelian, such that $G/[N, N]$ is not finitely presented, then G contains a nonabelian free subgroup. This applies in the situation of the corollary, since $M = G/[N, N]$ is not finitely presented. In the more general situation considered in [3], it is not clear whether the group G is necessarily SQ-universal.

Corollary 2.4 *If G has a finite, deficiency 1 presentation, and $N \lhd G$ is such that G/N is free abelian of rank $n > 0$, and $N/[N, N] \otimes_{\mathbb{Z}} F$ has F-dimension at least 2 for some field F, then G contains a nonabelian free subgroup.*

Proof. Choose a finite 2-complex K with $\pi_1(K) = G$ and $\chi(K) = 0$, and let $\bar{K}$ be the covering corresponding to N. If $H_2(\bar{K}, F) \neq 0$ for some field F, then $H_2(\bar{K}, F)$ contains a nonzero free submodule, since FA is a domain. By Theorem A it follows that G is large.

Assume then that $H_2(\bar{K}, F) = 0$ for all F. Then $H_1(\bar{K}, \mathbb{Z}) = N/[N, N]$ is torsion-free. Since $H_1(\bar{K}, F) \geq 2$ for some field F, this is true for all F, in particular for $F = \mathbb{Q}$. We can therefore find a finite connected subcomplex L of $\bar{K}^{(1)}$ such that $\beta_1(L) = 2$ and $H_1(L, \mathbb{Q}) \to H_1(\bar{K}, \mathbb{Q})$ is injective. Then $\pi_1(L)$ is free of rank 2, and $H_2(\bar{K}, L) = 0$, so $\pi_1(L) \to \pi_1(K)$ is injective (see [12] or [10]). □

Corollary 2.5 *If G has a finite, deficiency 1 presentation, and $M \lhd N \lhd G$ are such that G/N and N/M are free abelian of ranks r_1, r_2 respectively, with $r_1 + r_2 \geq 3$, then G contains a nonabelian free subgroup.*

Proof. If $r_1 = 1$ the result follows immediately from Corollary 2.4. If $r_1 \geq 3$ then we may replace M by N and N by a subgroup N_0 containing N with G/N_0 free abelian of rank 1, which reduces us to the previous case. Similarly if $r_1 = 0$ then replacing $N = G$ by N_0 containing N with G/N_0 free abelian of rank 1 also reduces us to a previous case.

Hence we may assume that $r_1 = 2$. Arguing as in the proof of the Corollary 2.4, if either $H_2(\bar{K}, F) \neq 0$ or $H_1(\bar{K}, F)$ has F-dimension greater than 1 for some field F, then we can find a free subgroup in G, so we may assume that $H_2(\bar{K}, F) = 0$ and $H_1(\bar{K}, F)$ has dimension 1 for all F. In particular $H_1(\bar{K}, \mathbb{Z})$ is a torsion-free abelian group of rank 1. It follows that $r_2 = 1$, and since the infinite cyclic group N/M is a homomorphic image of $H_1(\bar{K}, \mathbb{Z})$ we have $H_1(\bar{K}, \mathbb{Z}) \cong \mathbb{Z}$. Moreover $\alpha = [x_1, x_2]$ is a generator of the infinite cyclic group $N/M = H_1(\bar{K}, \mathbb{Z})$. The free abelian group $A = G/N$ acts on N/M, either trivially or non-trivially. In either case $(1 - x_1^2)\alpha = (1 - x_2^2)\alpha = 0$ in $H_1(\bar{K})$. Thus there are 2-chains γ_1, γ_2 of $\bar{K}$ such that $\partial(\gamma_1) = (1 - x_1^2)\alpha$ and $\partial(\gamma_2) = (1 - x_2^2)\alpha$. Since $H_2(\bar{K}) = 0$ it follows that $(1 - x_2^2)\gamma_1 = (1 - x_1^2)\gamma_2$, and hence that $\gamma_1 = (1 - x_1^2)\gamma_0$, $\gamma_2 = (1 - x_2^2)\gamma_0$ for some 2-chain γ_0. But then necessarily $\partial(\gamma_0) = \alpha$, a contradiction.

This completes the proof. □

Theorem C now follows immediately from Corollary 2.5 in the case where $d = 3, 4$, and from Theorem 1.1 in the case where $d = 2$.

3 Submodules of free modules over Laurent polynomial rings

The aim of this section is to prove Theorem D and the following consequences.

Theorem 3.1 *Let G be a group having a finite presentation of deficiency 0. Suppose $N \lhd G$ is such that G/N is free abelian of rank 3, and $N/N' \neq 0$. Then $(N/N') \otimes_{\mathbb{Z}} F$ contains a nonzero free FH-module for some nontrivial subgroup H of G/N and some field F.*

Corollary 3.2 *Let G and N be as in the Theorem. Suppose in addition that G admits an automorphism ϕ such that $\phi(N) = N$ and ϕ induces the antipodal automorphism $\alpha \mapsto -\alpha$ on the free abelian group G/N. Then N contains a free subgroup of rank 2.*

We first note some elementary properties of Laurent polynomial rings. The first holds for arbitrary noetherian unique factorisation domains.

Proposition 3.3 *Let Λ be a noetherian unique factorisation domain. Then the kernel of a homomorphism between two free Λ-modules of finite rank is either free of rank at most 1, or contains a free submodule of rank 2.*

Proof. Any cyclic submodule of a free Λ-module is free, since Λ is a domain. Suppose that $\phi : M_1 \to M_2$ is a homomorphism between two free Λ-modules of finite rank, and L is the kernel of ϕ. Then L is finitely generated, since Λ is noetherian. If L contains no free submodule of rank 2, we will show that any 2-generator submodule of L is contained in a cyclic submodule of L. It will follow by induction on the number of generators that L is cyclic, and hence free. Suppose then that $a, b \in L$. If a, b do not generate a free submodule of rank 2, then $\lambda a = \mu b$ for some $\lambda, \mu \in \Lambda$, which may be assumed to be nonzero, with no common irreducible factors. Then a is divisible by μ in the free module M_1. In other words, there exists $c \in M_1$ such that $\mu c = a$. Hence also $\mu(\lambda c - b) = 0$, so $\lambda c = b$. Finally, $\mu\phi(c) = \phi(\mu c) = 0$ in the free Λ-module M_2, so $\phi(c) = 0$ and $c \in L$. Hence a, b are contained in a cyclic submodule of L, as claimed. □

Proposition 3.4 *Let F be a field and Λ the ring of Laurent polynomials in three variables X_1, X_2, X_3. Let*

$$(\mathcal{C}, \partial) : C_2 \to C_1 \to C_0$$

be a chain complex of free Λ-modules with $H_1(\mathcal{C}) \neq 0$ and $H_2(\mathcal{C})$ free. Then $H_1(\mathcal{C})$ contains a nonzero free $F[X_i^{\pm 1}]$-module for some $i = 1, 2, 3$.

Proof. Choose $\alpha \in C_1$ representing a nonzero element of $H_1(\mathcal{C})$. If α generates a free $F[X_i^{\pm 1}]$-submodule of $H_1(\mathcal{C})$ for some $i = 1, 2, 3$, then we are done. We may therefore assume that there are polynomials $f_i(X_i) \in F[X_i]$ $(i = 1, 2, 3)$ such that $f_i(X_i)\alpha = 0$ in $H_1(\mathcal{C})$. Hence also there are 2-chains $\gamma_i \in C_2$ such that $\partial(\gamma_i) = f_i(X_i)\alpha$ for $i = 1, 2, 3$. Now for $i, j = 1, 2, 3$ we have $\lambda_{ij} = f_i(X_i)\gamma_j - f_j(X_j)\gamma_i \in H_2(\mathcal{C})$. Note that $\lambda_{ji} = -\lambda_{ij}$. We also have an equation

$$f_1(X_1)\lambda_{23} + f_2(X_2)\lambda_{31} + f_3(X_3)\lambda_{12} = 0. \tag{2}$$

Now Λ is a free $F[X_1^{\pm 1}]$-module, with basis the set of monomials $X_2^m X_3^n$, $m, n \in \mathbb{Z}$. If Δ is a Λ-basis for the free module $H_2(\mathcal{C})$, then $\Phi = \{X_2^m X_3^n \delta,\ m, n \in \mathbb{Z},\ \delta \in \Delta\}$ is an $F[X_1^{\pm 1}]$-basis for $H_2(\mathcal{C})$. By the euclidean algorithm, we may write (uniquely)

$$\lambda_{12} = f_1(X_1)q_{12} + r_{12}$$

for $q_{12}, r_{12} \in H_2(\mathcal{C})$ such that each nonzero coordinate of r_{12} (in terms of the basis Φ) involves only non-negative powers of X_1, and has X_1-degree less than that of f_1. Similarly we have $\lambda_{13} = f_1(X_1)q_{13} + r_{13}$, with similar restrictions on r_{13}. From (2) we deduce that

$$f_2(X_2)r_{13} \equiv f_3(X_3)r_{12} \mod f_1(X_1),$$

and from the degree restrictions on the coordinates of r_{12} and r_{13} it follows that

$$f_2(X_2)r_{13} = f_3(X_3)r_{12}.$$

Since f_2 and f_3 clearly have no common irreducible factors, it follows that r_{12} is divisible by $f_2(X_2)$, say $r_{12} = f_2(X_2)s$. Now let $\gamma' = \gamma_1 + s \in C_2$. Then $\partial\gamma' = \partial\gamma = f_1(X_1)\alpha$, and moreover

$$f_2(X_2)\gamma' = f_1(X_1)\gamma_2 - (\lambda_{12} - f_2(X_2)s) = f_1(X_1)(\gamma_2 - q_{12}).$$

Since f_1 and f_2 have no common irreducible factors, it follows that γ' is divisible by $f_1(X_1)$ in C_2, that is, there is an element $\gamma'' \in C_2$ such that $f_1(X_1)\gamma'' = \gamma'$. Since $f_1(X_1)(\partial\gamma'' - \alpha) = 0$ in the free Λ-module C_1, it follows that $\partial\gamma'' = \alpha$, contradicting the assumption that $\alpha \neq 0$ in $H_1(\mathcal{C})$.

This contradiction completes the proof. □

Proof of Theorem D. Let B be a subgroup of A maximal with respect to the properties that A/B is free abelian and $H_1(\bar{K}, F)$ contains a nonzero free FB-submodule. Then it suffices to show that the rank of A/B is at most 2. We suppose that A/B has rank at least 3 and derive a contradiction. Let $\mathcal{C}$ the cellular chain complex of $\bar{K}$ over F. Then $\mathcal{C}$ consists of free FA-modules, which are therefore also free FB-modules. Let $\mathcal{F}$ denote the field of fractions of FB, and let $\mathcal{D} = \mathcal{C} \otimes_{FB} \mathcal{F}$. Since $H_1(\mathcal{C})$ contains a free FB-module, it follows that $H_1(\mathcal{D}) \neq 0$.

Also, since $H_2(\mathcal{C})$ is free over FA, it follows that $H_2(\mathcal{D})$ is free over $FA \otimes_{FB} \mathcal{F} = \mathcal{F}(A/B)$. Let X_1, X_2, X_3 be three elements of a basis for the free abelian group A/B, and consider $\mathcal{D}$ as a chain complex over the Laurent polynomial ring $\mathcal{F}[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. By Proposition 3.4 we see that, for some $i = 1, 2, 3$, $H_1(\mathcal{D})$ contains a nonzero free $\mathcal{F}[X_i^{\pm 1}]$-submodule. If $\hat{B}$ is the subgroup of A generated by B and X_i, then it follows that $H_1(\bar{K}, F) = H_1(\mathcal{C})$ contains a nonzero free $F\hat{B}$-submodule. Since also $A/\hat{B}$ is free abelian, we have a contradiction to the maximality of B, and the proof is complete. □

Proof of Theorem 3.1. Let K be a finite 2-complex of euler characteristic 1 with $\pi_1(K) \cong G$, and let $\bar{K}$ be the regular covering of K corresponding to the normal subgroup N. Since $H_1(\bar{K}) = N/N' \neq 0$, there is a field F of prime order such that $H_1(\bar{K}, F) = (N/N') \otimes_{\mathbb{Z}} F \neq 0$. If $H_2(\bar{K}, F)$ contains a free $F(G/N)$-module of rank 2, then (for example by tensoring the F-chain complex of $\bar{K}$ with the fraction field of $F(G/N)$) one can show that $H_1(\bar{K}, F)$ contains a free $F(G/N)$-module (see the proof of Proposition 2.1). Otherwise, by Proposition 3.3, $H_2(\bar{K}, F)$ is free (necessarily of rank 1). Now apply Theorem D.

Proof of Corollary 3.2. The abelian group N/N' is not finitely generated, since $(N/N') \otimes_{\mathbb{Z}} F$ contains a nonzero free FH-submodule for some infinite subgroup H of G/N and some field F (by Theorem 3.1). Hence by definition the Bieri-Strebel invariant $\Sigma \subseteq S^2$ of the $\mathbb{Z}(G/N)$-module N/N' is a proper subset of S^2. On the other hand, Σ is stable under the automorphism of S^2 induced by ϕ, which is the antipodal automorphism of S^2. Hence $\Sigma \cup -\Sigma = \Sigma \neq S^2$, and by Theorem 4.1 of [3] it follows that N has a free subgroup of rank 2. □

4 Applications to generalised triangle groups

In this section the result of the previous sections are applied to generalised triangle groups. The Rosenberger Conjecture 1.8 has been proved for $r \geq 3$ [13], so we will restrict attention here to the case $r = 2$. We begin by recalling a few well-known facts about generalised triangle groups.

Every generalised triangle group admits an essential representation into $\mathrm{PSL}(2, \mathbb{C})$ [2, 5]. Up to conjugacy, such a representation ρ is determined by the values of the traces of $\rho(x)$, $\rho(y)$ and $\rho(xy)$ (which are well-defined only up to multiplying an even number of them by -1, since we are working in $\mathrm{PSL}(2,\mathbb{C})=\mathrm{SL}(2,\mathbb{C})/\{\pm I\}$). In what follows, we will always assume that $\rho(x)$ and $\rho(y)$ have traces $2\cos(\pi/p)$ and $2\cos(\pi/q)$ respectively, which is enough to ensure that they have orders p and q respectively in $\mathrm{PSL}(2,\mathbb{C})$. It then follows that the trace of $\rho(W)$ is a polynomial of degree k (the length of W) in the unknown $\lambda=\mathrm{Tr}(\rho(xy))$ [2, 5]. We denote this polynomial by τ_W or $\tau_W(\lambda)$. In order to obtain an essential representation, we must ensure that $\mathrm{Tr}(\rho(W))=0$, in other words we must set λ equal to a root of τ_W.

Since finitely generated subgroups of linear groups are residually finite, a consequence of the existence of essential representations into $\mathrm{PSL}(2,\mathbb{C})$ is that G also admits an essential epimorphism onto a finite group H, say. The kernel N of such an epimorphism $G\to H$ has a presentation of deficiency $1-\kappa\cdot|H|$, where $\kappa=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1$ [15]. More precisely, if X is the 2-complex model of the generalised triangle presentation $\mathcal{P}$ for G, and $\bar{X}$ the regular covering corresponding to the normal subgroup N, then the 2-cells of $\bar{X}$ occur in groups of p, q or r with the same boundaries (since the relations of $\mathcal{P}$ are proper powers which do not lift to proper powers in $\pi_1(\bar{X}^{(1)})$). Removing all but one of each of these groups of 2-cells, we obtain a 2-complex K with fundamental group N and euler characteristic $\kappa\cdot|H|$.

We first consider generalised triangle groups

$$G=\langle\ x,\ y,\ |\ x^p\ =\ y^q\ =\ W(x,y)^2\ =\ 1\ \rangle$$

of type $(p,q,2)$, with $\kappa=0$. Without loss of generality $p\le q$, so we have $(p,q)=(4,4)$ or $(3,6)$. In each case the image of an essential representation into $\mathrm{PSL}(2,\mathbb{C})$ will contain a free subgroup of rank 2, unless it is cyclic, a euclidean triangle group, or (in the $(4,4)$ case) isomorphic to S_4. In each case the existence of a cyclic representation implies the existence of a euclidean triangle group representation also. We treat this situation first. Let $z=e^{\pi i/p}$, and consider representations $\rho:\langle x,y|x^p=y^q=1\rangle\to\mathrm{PSL}(2,\mathbb{C})$ with

$$\rho(x)=\begin{pmatrix} z & \lambda-2z\cos(\pi/q)\\ 0 & \bar{z}\end{pmatrix},\quad \rho(y)=\begin{pmatrix} 2\cos(\pi/q) & -1\\ 1 & 0\end{pmatrix},$$

so that $\lambda=\mathrm{Tr}(\rho(xy))$. We can obtain an essential representation of our generalised triangle group G by setting λ to be any root of τ_W. Now the image of ρ is a euclidean $(p,q,2)$-triangle group if and only if λ is either 0 or $4\cos(\pi/p)\cos(\pi/q)$. Note that the trace of $x^{-1}y$ is $4\cos(\pi/p)\cos(\pi/q)-\lambda$, so we can interchange the rôles of these two possible values for λ by replacing $W(x,y)$ by the equivalent word $W(x^{-1},y)$.

Lemma 4.1 *If both* 0 *and* $4\cos(\pi/p)\cos(\pi/q)$ *occur as roots of* τ_W*, then* G *contains a free subgroup of rank 2.*

Proof. Under the hypotheses of the lemma, G has two normal subgroups H_1 and H_2, with $G/H_i\cong\mathbb{Z}_q$, and two essential representations $\rho_1,\rho_2:G\to\mathrm{PSL}(2,\mathbb{C})$, such that $\rho_i(H_i)$ is free abelian of rank 2. We consider the image of $H=H_1\cap H_2$ under the map $\rho=\rho_1\times\rho_2:G\to\mathrm{PSL}(2,\mathbb{C})\times\mathrm{PSL}(2,\mathbb{C})$. It follows from the preliminary comments at the start of the section that H has a deficiency 1 presentation. We will show that $\rho(H)$

is free abelian of rank 4. It then follows from Corollary 2.5 that G contains a nonabelian free subgroup, as claimed. We treat the two values for (p,q) separately.

Case 1 $(p,q) = (4,4)$.

In this case, H_1 can be taken to be the normal closure in G of xy, and H_2 the normal closure of $x^{-1}y$. Then the representations ρ_1 and ρ_2 are obtained by setting λ equal to $4\cos(\pi/p)\cos(\pi/q) = 2$ and 0 respectively. The elements $a = xy$ and $b = yx$ of H_1 are mapped by ρ_1 to $\left(\begin{array}{cc} 2 & -z \\ \bar{z} & 0 \end{array}\right)$ and $\left(\begin{array}{cc} 1+i & \bar{z} \\ z & 1-i \end{array}\right)$ respectively, where $z = e^{i\pi/4}$, and by ρ_2 to elements of order 2 in $\mathrm{PSL}(2,\mathbb{C})$. Note that $A\rho_1(a)A^{-1} = \left(\begin{array}{cc} 1 & 0 \\ \bar{z} & 1 \end{array}\right)$ and $A\rho_1(b)A^{-1} = \left(\begin{array}{cc} 1 & 0 \\ z & 1 \end{array}\right)$, where $A = \left(\begin{array}{cc} 1 & -z \\ 0 & 1 \end{array}\right)$, so that $\rho_1(a)$ and $\rho_1(b)$ generate a free abelian subgroup of $\mathrm{PSL}(2,\mathbb{C})$ of rank 2. Note also that $\rho_2(x^{-1}) = \overline{\rho_1(x)}$ and $\rho_2(y) = \rho_1(y) = \overline{\rho_1(y)}$, so that if $c = x^{-1}y$ and $d = yx^{-1}$, then $\rho_2(c)$ and $\rho_2(d)$ also generate a free abelian subgroup of $\mathrm{PSL}(2,\mathbb{C})$ of rank 2, while $\rho_1(c)$ and $\rho_1(d)$ have order 2. Clearly $\rho(H)$ is a free abelian subgroup of $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$. To see that this subgroup has rank 4, it suffices to note that the elements $\rho(a^2)$, $\rho(b^2)$, $\rho(c^2)$ and $\rho(d^2)$ are linearly independent.

Case 2 $(p,q) = (3,6)$.

The argument in this case is similar. Here H_1 and H_2 are the normal closures of xy^2 and xy^{-2} respectively, while ρ_1 and ρ_2 are obtained by setting λ equal to $\sqrt{3}$ and 0 respectively.

The conjugates of xy^2 are mapped to elements of order 3 by ρ_2, while conjugates of xy^{-2} are mapped to elements of order 3 by ρ_1. A calculation shows that, if $a = xy^2$ and $b = y^2x$, then $\rho_1(a)$ and $\rho_1(b)$, conjugated by $\left(\begin{array}{cc} 1 & e^{-i\pi/6} \\ 0 & 1 \end{array}\right)$, are linearly independent lower triangular matrices, so $\rho_1(a)$ and $\rho_1(b)$ freely generate an abelian subgroup of $\mathrm{PSL}(2,\mathbb{C})$. The same holds for $\rho_2(c)$ and $\rho_2(d)$ where $c = xy^{-2}$ and $d = y^{-2}x$, since $\rho_2(x) = \overline{\rho_1(x^{-1})}$ and $\rho_2(y) = \overline{\rho_1(y)}$. Hence, as in the previous case, $a^3, b^3, c^3, d^3 \in H$ are mapped by ρ to a basis for a free abelian subgroup of rank 4 in $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$. $\square$

Lemma 4.2 *If either* 0 *or* $4\cos(\pi/p)\cos(\pi/q)$ *is a repeated root of* τ_W*, then* G *contains a nonabelian free subgroup.*

Proof. Replacing $W(x,y)$ by $W(x^{-1},y)$ if necessary, we may assume that 0 has a repeated root. Let $\Lambda = \mathbb{C}[\lambda]/((\lambda^2))$, and consider the representation $\rho : G \to \mathrm{PSL}(2,\Lambda)$ given by

$$\rho(x) = \left(\begin{array}{cc} z & \lambda - 2z\cos(\pi/q) \\ 0 & \bar{z} \end{array}\right), \quad \rho(y) = \left(\begin{array}{cc} 2\cos(\pi/q) & -1 \\ 1 & 0 \end{array}\right),$$

with $z = e^{i\pi/p}$. If H_2 is the subgroup of index q in G as in the proof of Lemma 4.1, then the image of $\rho(H_2)$ in $\mathrm{PSL}(2,\mathbb{C})$ under the natural map $\sigma : \mathrm{PSL}(2,\Lambda) \to \mathrm{PSL}(2,\mathbb{C})$

induced by $\Lambda \to \mathbb{C}$ is free abelian of rank 2. Since the kernel K of σ consists of matrices of the form $\pm(I + \lambda A)$, and $\lambda^2 = 0$ in Λ, it is clear that K is torsion-free abelian. By Corollary 2.4, it suffices to find two linearly independent elements of $\rho(H_2) \cap K$ (since H_2 has a presentation of deficiency 1).

In the case $(p, q) = (4, 4)$, we find that

$$\rho((xy)^2) = -I + \lambda \begin{pmatrix} 0 & -z \\ \bar{z} & 0 \end{pmatrix} \quad \text{and} \quad \rho((yx)^2) = -I + \lambda \begin{pmatrix} 1+i & -2z-\bar{z} \\ z & -1-i \end{pmatrix}$$

which satisfy our requirements.

In the case $(p, q) = (3, 6)$ we find that

$$\rho((xy)^2) = -I + \lambda \begin{pmatrix} 0 & -z \\ \bar{z} & 0 \end{pmatrix} \quad \text{and} \quad \rho((yx)^2) = -I + \lambda \begin{pmatrix} \sqrt{3}z & -3z-\bar{z} \\ z & -\sqrt{3}z \end{pmatrix}$$

which is again sufficient. □

Corollary 4.3 *If* $(p, q) = (3, 6)$ *then* G *has a free subgroup of rank 2, except when* $W = xy$*, up to equivalence.*

Proof. Since the only essential representations whose images do not contain free subgroups of rank 2 are those corresponding to the roots 0 and $\sqrt{3}$ of the trace polynomial τ_W, we may assume that τ_W has no roots other than these. By Lemma 4.1 we may assume that only one root occurs, and by Lemma 4.2 that this root has multiplicity 1. Hence τ_W is linear, so W has length 1, in other words, up to cyclic permutation and inversion we have $W(x, y) = xy^p$ for some $p \in \{1, \ldots, 5\}$. If p is not coprime to 6, then G is a nontrivial amalgamated free product, so has a nonabelian free subgroup. Hence we have $p = 1$ or $p = 5$ (corresponding to the roots 0 and $\sqrt{3}$ respectively for τ_W). But xy^5 and xy are equivalent, and the proof is complete. □

Remark We cannot yet apply our results to the case $(p, q) = (4, 4)$, since here there exists another possible essential representation, with image isomorphic to S_4, corresponding to the root 1 of the trace polynomial τ_W. In order to deal with this, we must also restrict the multiplicity of 1 as a root.

Lemma 4.4 *If* $(p, q) = (4, 4)$ *and* 1 *is a multiple root of* τ_W*, then* G *contains a free subgroup of rank 2.*

Proof. Put $\mu = \lambda - 1$ and $M = \mathbb{C}[\mu]/((\mu^2)) = \mathbb{C}[\lambda]/(((\lambda - 1)^2))$, and consider the representation $\rho : G \to \mathrm{PSL}(2, M)$ given by

$$\rho(x) = \begin{pmatrix} z & \mu + 1 - \sqrt{2}z \\ 0 & \bar{z} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix},$$

where $z = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$. As in Lemma 4.2, we have a natural map $\sigma : \mathrm{PSL}(2, M) \to \mathrm{PSL}(2, \mathbb{C})$. The kernel H of the composite map $\sigma \circ \rho$ has index 24 in G, and a admits a presentation of deficiency 1. The kernel K of σ is torsion free abelian, so it suffices to find three linearly independent elements in $K \cap \rho(H)$, by Corollary 2.5. Three such elements are:

$$\rho((xy)^3) = -I + \mu \begin{pmatrix} 1 & -2z \\ 2\bar{z} & -1 \end{pmatrix},$$

$$\rho((yx)^3) = -I + \mu \begin{pmatrix} 1+2i & -2z \\ 2z & -1-2i \end{pmatrix}$$

and

$$\rho((y^2xy^{-1})^3) = -I + \mu \begin{pmatrix} 1 & -2\bar{z} \\ 2z & -1 \end{pmatrix}.$$

□

Corollary 4.5 *If* $(p, q) = (4, 4)$ *then* G *contains a free subgroup of rank 2, unless* W *has the form* xy*, up to equivalence.*

Proof. By Lemmas 4.1, 4.2 and 4.4, we may assume that τ_W has at most two roots, and no root of τ_W has multiplicity greater than 1. Hence W has length at most 2. If W has length 1, then G is a free product with amalgamation, unless W is equivalent to xy. If W has length 2 then up to equivalence τ_W has roots 0 and 1, and $W = xyx^2y^2$. Then G has a subgroup of index 4 with a presentation

$$\langle z_1, z_2, z_3, z_4 \mid z_1z_2z_3z_4,\ z_1z_3z_4z_3z_1z_2,\ z_2z_4z_1z_4z_2z_3 \rangle.$$

The first relator allows us to eliminate z_4, and this transforms the second relator to $z_1z_2^{-1}z_1^{-1}z_3z_1z_2$. Since z_3 occurs only once in this, we may also eliminate z_3, which results in a 2-generator, 1-relator presentation. The standard representation of this group as an HNN extension of a 1-relator group with shorter relator clearly reveals the existence of nonabelian free subgroups (for example, as the associated subgroups in the HNN extension). □

Combining the results of this section so far, we have proved the following.

Theorem 4.6 *Let* G *be a generalised triangle group* $\langle x, y \mid x^p = y^q = W(x, y)^2 = 1\rangle$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. *Then* G *contains a free subgroup of rank 2, unless* W *has the form* xy*, up to equivalence.*

Now Fine, Levin and Rosenberger [6] proved the analogous result for generalised triangle groups of type (p, q, r) with $r \geq 3$ and $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 = 0$, so combining that with the above result completes the proof of Theorem B.

From now on, suppose that $\kappa > 0$, in other words that $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. In this case it is less easy to proceed, but the results of section 3 can be applied under suitable conditions.

Theorem 4.7 *Suppose* $H = \langle a, b \mid a^2 = b^{2q} = W(a,b)^2 = 1\rangle$ *has an essential cyclic representation, where* $q \geq 4$. *Then* H *contains a nonabelian free subgroup.*

Proof. The kernel G of an essential representation $H \to \mathbb{Z}_{2q}$ has a deficiency zero presentation with generators $x_1, \ldots, x_{2q}$ and relations $x_i x_{i+q} = 1$, $i = 1, \ldots, q$ and

$$W'(x_i, \ldots, x_{2q}, x_1, \ldots, x_{i-1})W'(x_{i+q}, \ldots, x_{2q}, x_1, \ldots, x_{i+q-1})\ (i = 1, \ldots, q),$$

where W' is a rewrite of W.

From this it is clear that G/G' is free abelian of rank $q \geq 4$, so there is a normal subgroup N of G with G/N free abelian of rank 3 and N/G' (and hence also N/N') nonzero. It follows from Theorem 3.1 that N/N' is not finitely generated as an abelian group. Since $G' \lhd N$ with N/G' finitely generated abelian, it follows that G'/N' is not finitely generated as an abelian group, and hence neither is G'/G''.

It is also clear that $x_i \mapsto x_{i+q} = x_i^{-1}$ is an automorphism of G that induces the antipodal automorphism on G/G', and hence by Corollary 3.2 G contains a nonabelian free subgroup. □

Example Let $H = \langle a, b \mid a^2 = b^6 = (abab^2)^2 = 1\rangle$. Then H admits an essential representation to $\mathbb{Z}_6$, and the kernel G of this representation has presentation

$$G = \langle x_1, x_2, x_3 \mid x_1 x_2^{-1} x_1^{-1} x_2 = x_2 x_3^{-1} x_2^{-1} x_3 = x_3 x_1 x_3^{-1} x_1^{-1} = 1\rangle.$$

Clearly G is free abelian of rank 3, so Theorem 4.7 cannot be directly extended to apply when $q < 4$. However, we can still prove the existence of free subgroups when $q = 3$ under the slightly weaker assumption that the roots of the trace polynomial corresponding to an essential cyclic representation are repeated.

Let $H = \langle\ a, b \mid a^2 = b^6 = W(a,b)^2 = 1\ \rangle$. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} z & \lambda \\ 0 & \bar{z} \end{pmatrix},$$

where $z = e^{i\pi/6}$ and $\lambda = \mathrm{Tr}(AB)$ is a variable. Then $\tau_W(\lambda) = \mathrm{Tr}(W(A,B))$ is an odd or even polynomial in λ, and H has a cyclic essential representation if and only if ± 1 are roots of τ_W.

Theorem 4.8 *With the above notation, if* 1 *is a repeated root of* τ_W, *then* H *has a nonabelian free subgroup.*

Proof. Setting $\lambda = 1$ yields an essential representation ρ of H into $\mathrm{PSL}(2, \mathbb{C})$ whose image is a euclidean triangle group of type $(2, 6, 3)$. The restriction of this to $G = H'$ has image the translation subgroup of the triangle group, which is free abelian of rank

2. Since we know that G/G' is free abelian of rank 3 (see the proof of Theorem 4.7), we can apply Theorem 3.1, provided we can show that $G'/G'' \neq 0$.

We do this by lifting ρ to a representation $\tilde{\rho} : H \to \mathrm{PSL}(2, \Lambda)$, where Λ is the ring $\mathbb{C}[\lambda]/((\lambda - 1)^2)$. We can do this with A, B as above, since 1 is a repeated root of τ_W. Consider the elements $(ab)^3$ and $(ba)^3$ of $\mathrm{Ker}(\rho)$. A calculation gives

$$\tilde{\rho}((ab)^3) = (AB)^3 = -I + (\lambda - 1)\begin{pmatrix} -1 & -2\bar{z} \\ 2z & 1 \end{pmatrix}$$

(using the fact that $(\lambda - 1)^2 = 0$ in Λ). Noting that BA is the transpose of $(AB)^{-1}$, it follows that

$$\tilde{\rho}((ba)^3) = (BA)^3 = -I + (\lambda - 1)\begin{pmatrix} 1 & -2z \\ 2\bar{z} & -1 \end{pmatrix}.$$

These two matrices generate a subgroup of rank 2 in the abelian group $\ker(\mathrm{PSL}(2, \Lambda) \to \mathrm{PSL}(2, \mathbb{C}))$, so G' has a free abelian homomorphic image of rank at least 1, and this is sufficient to prove the theorem. □

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