On the abstract groups $(3, n, p; 2)$

Martin Edjvet
Department of Mathematics
University of Nottingham
Nottingham NG7 2RD
me@maths.nott.ac.uk

James Howie
Department of Mathematics
Heriot-Watt University
Edinburgh EH14 4AS
jim@cara.ma.hw.ac.uk

§0. Introduction.

The groups $(m, n, p; q) = \langle a, b \mid a^m = b^n = (ab)^p = [a, b]^q = 1\rangle$ were introduced and studied by Brahana [?, ?], Coxeter [?, ?, ?, ?] and Sinkov [?, ?, ?, ?, ?]. This initial work was followed by a later paper of Sinkov [?], and a paper by Leech and Mennicke [?]. More recently, interest in deciding which values of the parameters m, n, p, q yield finite groups and which infinite has been revived by Grove and McShane [?], Chaltin [?], Howie and Thomas [?], Holt and Plesken [?], and Edjvet [?, ?].

Since any permutation of m, n, p does not change the resulting group, we may assume that $2 \leq m \leq n \leq p$. The case $m = n = 2$ is straightforward, and the case $(m, n) = (2, 3)$ has been investigated by Howie and Thomas [?], Holt and Plesken [?], and Edjvet [?]. The only group in this category which remains unresolved is $(2, 3, 13; 4)$, which is known to have a homomorphic image $E \times PSL(2, 25)$ of order $2^{19}.3^4.5^2.13^2$, where E is an extension of the elementary abelian group of order 2^{12} by $PSL(3, 3)$, but it has not yet been shown to be infinite. (See [?] for further discussion of this group.)

With the above exception, Edjvet [?] has completely determined which groups $(m, n, p; q)$ are finite and which infinite, provided that $(m, q) \neq (3, 2)$. This leaves the groups $(3, n, p; 2)$, and these are the subject of the present paper. For any p, the group $(3,3,p;2)$ is a homomorphic image of $\langle a,b \mid a^3 = b^3 = [a,b]^2 = 1\rangle$, which is finite of order 288. Hence we may assume that $4 \leq n \leq p$. We prove the following.

Theorem A. *Let* $G = (3,n,p;2)$, *where either* (i) $n = 4$ *and* $p \geq 17$, *or* (ii) $n \geq 5$ *and* $p \geq 11$. *Then* G *is infinite.*

This reduces the number of open cases to 35: (2,3,13;4); (3,4,p;2) for $4 \leq p \leq 16$; and (3,n,p;2) for $5 \leq n \leq p \leq 10$. The groups $(3,4,4;2)$ and $(3,4,5;2)$ were shown to be finite in [?], and further inroads into these cases have been made in recent work by D. F. Holt, M. F. Newman and others. The current status of the problem, so far as we are aware, is that only six cases remain unsolved: namely $(2,3,13;4)$, $(3,4,9;2)$, $(3,4,11;2)$, $(3,4,13;2)$, $(3,5,6;2)$ and $(3,5,7;2)$. For a discussion of this and similar problems, we refer the reader to the survey article [?].

Our method of proof, as in [?, ?, ?], is that of pictures. In §1 below we recall the basic definitions of pictures in our context. In §2 we discuss a curvature function on the vertices of our pictures, and in §3 we apply this to show that our presentations, while not being (combinatorially) aspherical, have second homotopy modules generated by spherical pictures arising from the finite subgroups $\langle b\rangle \cong C_n$, $\langle ab\rangle \cong C_p$, and $\langle a, bab^{-1}\rangle \cong A_4$. We also show that the presentations *do not collapse*, in the sense of [?] (see also [?]), in other words, the above subgroups really are isomorphic to C_n, C_p and A_4 respectively, and not to proper homomorphic images of these groups. Finally, we collect these various lemmas together in §4 using a cohomological argument to show that the groups in Theorem A are infinite.

§1. Pictures.

In this section we discuss briefly the type of pictures to be used throughout the paper. For a more detailed discussion we refer the reader to [?, ?, ?, ?, ?].

Let $(3,n,p;2)$ be the group defined by the presentation

$$\langle a,b | a^3, b^n, (ab)^p, [a,b]^2 \rangle \quad (4 \leq n \leq p),$$

which will be denoted throughout by $\mathcal{P}$.

Let $A = \langle a | a^3 \rangle$ and $B = \langle b | b^n \rangle$. Then $(3,n,p;2)$ can be regarded as the quotient of the free product $A * B$ by the normal closure of the words $\alpha = (ab)^p$ and $\beta = [a,b]^2$.

A *picture* Π over $\mathcal{P}$ consists of the following: a collection of disjoint closed discs in the interior of D^2 called *vertices*; a finite number of disjoint arcs called *edges*, each of which is either: (i) a simple closed curve in the interior of D^2 that meets no vertex, (ii) an arc joining two vertices (or one vertex to itself), (iii) an arc joining a vertex to the boundary ∂D^2 of D^2, or (iv) an arc joing ∂D^2 to ∂D^2; a collection of *labels*, one at each corner of each *region* of Π (i.e. connected component of the complement in D^2 of the edges and vertices) at a vertex, and one along each component of the intersection of the region with ∂D^2.

Each label of Π is one of $\{a^{\pm 1}, b^{\pm 1}\}$, with the exception of labels on (segments of) ∂D^2, which may be any element of $A \cup B$.

Reading the labels round a vertex in the *clockwise* direction yields either $\alpha^{\pm 1}$ or $\beta^{\pm 1}$ (up to cyclic permutation), as a cyclically reduced word in $A * B$. We use the term *α-vertex* or *β-vertex* to denote a vertex with label $\alpha^{\pm 1}$ or $\beta^{\pm 1}$ respectively. We also introduce the notion of *orientation* of a vertex: this is *positive* if the label is α or β, and *negative* if the label is α^{-1} or β^{-1}. A *boundary* vertex is one that is joined to ∂D^2 by at least one arc.

A region is a *boundary* region if it meets ∂D^2, and an *interior* region otherwise.

If no edges of Π meet ∂D^2 then Π is called *spherical.* In this case ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that Π contains at least one vertex or edge), which is called the *distinguished* region. All other regions are *interior.*

The labels of any region Δ of Π are required all to belong to either A or B. We talk of A-regions and B-regions accordingly. Each edge is required to separate an A-region from a B-region. Observe that this is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from A and B. A region bounded by edges that are closed curves will have no labels; nevertheless the above convention requires that it be designated an A- or B-region.

The *boundary label* of Π is the word obtained by reading the labels on ∂D^2 in an *anticlockwise* direction. This word represents the identity element of $(3,n,p;2)$. It may be asumed to be cyclically reduced as a word in $A * B$. If Π is spherical, then the boundary label is an element of $A \cup B$ determined by the other labels in the distinguished region.

Two distinct vertices of a picture are said to *cancel* along an edge e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $A * B$. An example of two cancelling β-vertices is given in Figure 1.1.

The point is that such vertices can be removed from a picture via a sequence of so-called *bridge moves*, followed by deletion of a *dipole* (see below) without changing the boundary label (see [?] for details). This gives an alternative picture with the same boundary label and two fewer vertices.

We say that a picture is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. Any cyclically reduced word in $A * B$ representing the identity element of $(3,n,p;2)$ occurs as the boundary label of some reduced picture. A picture is *connected* if the union of its vertices and edges is connected. In particular, no edge of a connected picture is a closed curve or joins two points of ∂D^2, unless the picture conists only of that edge. A *dipole* is a connected spherical picture containing precisely two vertices. It is easy to check that the vertices of a dipole are either both α-vertices or both β-vertices, and that they cancel. A connected component Π_0 of a picture Π is a *dipole* if it contains precisely two vertices, does not meet ∂D^2, and none of its interior regions contain other components of Π.

If (3,n, p;2) does not collapse, then every spherical picture has boundary label the identity element of A or of B. Any such spherical picture represents an element of the second homotopy group $\pi_2(\mathcal{P})$, where we regard the presentation $\mathcal{P}$ as a 2-dimensional CW-complex. Similarly, a spherical picture with nontrivial boundary label (in A or B) represents an element of the second relative homotopy group $\pi_2(\mathcal{P}, \mathcal{P}_A)$, or $\pi_2(\mathcal{P}, \mathcal{P}_B)$, where $\mathcal{P}_A$ is the subpresentation $\langle a|a^3\rangle$ and $\mathcal{P}_B$ is the subpresentation $\langle b|b^n\rangle$. Conversely every element of $\pi_2(\mathcal{P})$ (resp. $\pi_2(\mathcal{P}, \mathcal{P}_A)$, $\pi_2(\mathcal{P}, \mathcal{P}_B)$) can be so represented. The element of π_2 represented by a picture depends, strictly speaking, on some choices of base point, so is defined only up to the action of the fundamental group $\pi_1(\mathcal{P}) = (3, n, p; 2)$ and the addition of (elements representable by) dipoles. It makes sense therefore, to consider the quotient of π_2 by the submodule generated by dipoles. It turns out that this quotient is still non-zero. The connected spherical picture in Figure 1.2 represents a non-zero element. We shall denote this picture by $\mathcal{S}$ throughout the paper.

Two edges of a picture Π are said to be *parallel* if they are the only two edges in the boundary of some simply-connected region Δ of Π. In fact, by performing bridge moves as in Figure 1.3, we may assume that any B-region of Π having sublabel bb^{-1} is simply connected with precisely two edges (which are therefore parallel by definition). We shall make this assumption throughout the rest of the paper.

We now consider parallel edges. Suppose that Π is reduced and connected. A routine check shows that the maximum number of parallel edges between two α-vertices is 0, between an α- and a β-vertex is 3, (see Figure 1.4), and between two β-vertices is 2. For any reduced, connected spherical picture, it is now easy to show that each β-vertex has degree at least 4.

§2. Curvature.

Let Π be a reduced, connected, spherical picture over $\mathcal{P}$. Form a graph , from Π by contracting the boundary ∂D^2 to a point and removing it, and then identifying edges whenever they are parallel. We will sometimes refer to a *single, double, or triple* edge of , , meaning one formed by the identification of 1, 2 or 3 edges of Π. The vertices of , are simply the vertices of Π. Thus a region of , is interior if it was interior in Π; otherwise it is a boundary region. Thus , forms a tessellation of the sphere S^2 in which each region is a topological disc. The *distinguished region* of , is that formed from the distinguished region of Π, and all other regions of , are called *interior.*

Our curvature arguments will be based on those of [**?**, **?**].

Give each corner at a vertex of degree d the *angle* $2\pi/d$. The *curvature*, denoted $c(\Delta)$, of a region Δ of , of degree k, whose vertices have degree d_i $(1 \leq i \leq k)$ is then

$$(2-k)\pi + 2\pi \sum_{i=1}^{k} \frac{1}{d_i}$$

which will sometimes be denoted by $c(d_1, \ldots, d_k)$.

It follows from Euler's formula that the sum of the curvatures of regions of , is 4π. We shall usually seek to show that this value cannot be attained. Our method will be to locate those interior regions Δ of , of positive curvature ($c(\Delta) > 0$), and show that this can be compensated by negatively curved neighbours. To this end we introduce the notation c^* for an amended curvature function obtained by redistributing all positive $c(\Delta)$ to neighbouring regions, where the method of redistribution will be described in Lemma 3.1 below.

§3. Lemmas.

Choose a CW- complex X with $\pi_1(X) \cong A \cong C_3$ and a CW-complex Y with $\pi_1(Y) \cong B \cong C_n$. Form a space Z by taking the disjoint union of X and Y connected at a base point, and adjoining 2-cells along paths representing the elements α and β of $\pi_1(X \cup Y) \cong A * B$. Then $\pi_1(Z) \cong G$. It is pointed out in [**?**] that the reduced spherical pictures being considered here represent (free) homotopy classes of maps $S^2 \to Z$, and hence elements of $\pi_2(Z)$. Moreover, bridge moves do not change the homotopy class. In [?] it was shown that $\pi_2(Z)$ is generated by dipoles for many values of $(m, n, p; q)$. Here we consider the case $m = 3, q = 2$ and show that, for sufficiently high values of n, p, $\pi_2(Z)$ is generated by dipoles together with the element $\mathcal{S}$ of Figure 1.2. Let $\mathcal{D}$ denote the set of elements of $\pi_2(Z)$ represented by dipoles.

Lemma 3.1 *If either* (i) $n = 4$ *and* $p \geq 17$*, or* (ii) $n \geq 5$ *and* $p \geq 11$*, then* $\pi_2(Z) = \pi_2(Z, X) = \pi_2(Z, Y) = \langle \mathcal{D}, \mathcal{S} \rangle$.

Proof. Suppose by way of contradiction that $\pi_2(Z, X) \neq \langle \mathcal{D}, \mathcal{S} \rangle$ or $\pi_2(Z, Y) \neq \langle \mathcal{D}, \mathcal{S} \rangle$. Then there exist pictures over $\mathcal{P}$ representing elements of $\pi_2(Z, X) \backslash \langle \mathcal{D}, \mathcal{S} \rangle$ or of

$\pi_2(Z, Y) \backslash \langle \mathcal{D}, \mathcal{S} \rangle$. Among all such pictures, choose one, Π say, with fewest possible vertices. This ensures, for example, that Π is reduced and connected.

Our first observation is that Π cannot contain a subpicture of the type shown in Figure 3.1, for then we may obtain a picture with fewer vertices by introducing a copy of $\mathcal{S}$, and then performing bridge moves and removing four dipoles.

Form the graph , from Π as described in §2. Clearly , is also connected. We will show that the positive $c(\Delta)$ can be redistributed in such a way as to ensure that $c^*(\Delta_1) \leq 0$ for each interior region Δ_1 of , . We will consider the distinguished region at the end of the proof. First we describe all possibilities for regions Δ of , of positive curvature, and indicate how $c(\Delta)$ is redistributed.

Since $p \geq 11$ by assumption, it follows from the observations made at the end of §1 that deg(α-vertex)≥ 8 and deg(β-vertex)≥ 4 in , . Moreover, since $c(4, 4, 4, 4) = c(4, 8, 8) = 0$, it follows that if $c(\Delta) > 0$ then $\deg(\Delta) = 3$ and Δ contains at most one α-vertex. Note also that, since $\deg(\Delta) = 3$, Δ must be an A-region with boundary label $a^{\pm 3}$.

Assume for the moment that deg(α-vertex)$\geq k \geq 8$.

If $c(\Delta) > 0$ and Δ contains an α-vertex, then two of the edges of Δ connect the α-vertex to β-vertices. Let Δ_1 and Δ_2 be the neighbouring regions separated from Δ by these edges. Note that neither Δ_1 nor Δ_2 can be an A-region of degree 3, so they each have negative curvature. We add $\frac{1}{2}c(\Delta) \leq \frac{1}{2}c(4,4,k) = \pi/k$ to each of Δ_1 and Δ_2.

Suppose now that $c(\Delta) > 0$ and that Δ does not contain an α-vertex. There are two possibilities: either the three vertices of Δ have the same orientation, or one of them has the opposite orientation from the other two.

Case 1. All three vertices of Δ have the same orientation.

Then each of the three edges of Δ arises from a pair of parallel edges in Π, so the three regions Δ_1, Δ_2 and Δ_3 that share an edge with Δ are all A-regions. Let β_1 and β_2 denote the common vertices of Δ and Δ_1. Suppose first that Δ_1 has degree greater than 3, but that there is an α-vertex adjacent to β_1 in the boundary of Δ_1 as in Figure 3.2 (and hence a subword $a^{\pm 3}$ in the label of Δ_1). Then we add $c(\Delta) \leq c(4,4,4) = \pi/2$ to $c(\Delta_1)$. A similar move can be performed if Δ_2 or Δ_3 has this form.

From now on we assume that none of Δ_1, Δ_2 and Δ_3 have such a form. If all the vertices of Δ have degree 4, then from our various observations and assumptions, we deduce that each of Δ_1, Δ_2, Δ_3 is a 3-sided region with an α-vertex. Indeed, the configuration must be as in Figure 3.3 (or its mirror image).

Here we also have positive curvature in Δ_1, Δ_2 and Δ_3, so we must distribute the curvature further. Note that, for $i = 1, 2, 3$ we have already redistributed $\frac{1}{2}c(\Delta_i)$ to each of the two neighbours of Δ_i other than Δ. In addition to this, we now add $\frac{1}{6}c(\Delta) = \pi/12$ to each of these three regions Δ_4, Δ_5 and Δ_6. Note that the maximum contribution from Δ, Δ_1, Δ_2 and Δ_3 to each of Δ_4, Δ_5, Δ_6 is $\pi/k + \pi/12$.

Next, suppose that precisely two of the vertices of Δ have degree 4. Our previous assumptions force the edge joining these two vertices to separate Δ from a region Δ_1 of degree 3. There are essentially only two possible configurations, shown in Figure 3.4.

We redistribute the positive curvature of Δ, Δ_1 and Δ_3 as follows. In Figure 3.4 (i), we add $\frac{1}{2}(c(\Delta_1) + \pi/6) \leq \pi/k + \pi/12$ to Δ_6, $\frac{1}{2}(c(\Delta_1) + c(\Delta_3) + 3\pi/10) \leq 2\pi/k + 3\pi/20$ to Δ_4, and $\frac{1}{2}(c(\Delta_3) + 2\pi/15) \leq \pi/k + \pi/15$ to Δ_5. In Figure 3.4 (ii) we transfer π/k to Δ_4, $2\pi/k + \pi/6$ to Δ_5, and $\pi/k + \pi/6$ to Δ_6.

Now we consider the case where precisely one of the vertices of Δ has degree 4. If all four of the edges of , incident at this vertex correspond to double edges of Π, then two of the adjacent vertices must be α-vertices, as in Figure 3.5 (i). In this case we redistribute the positive curvature of Δ and Δ_3 by adding $7\pi/60$ to Δ_1, π/k to Δ_4, and $\pi/k + \pi/12$ to Δ_5.

Otherwise we have a triple edge, as in Figure 3.5 (ii). If the vertex marked β in Figure 3.5 (ii) has degree 6, then we redistribute the positive curvature of Δ and Δ_3 by adding $\pi/k + \pi/30$ to each of $c(\Delta_4)$ and $c(\Delta_5)$. Otherwise, this vertex has degree 5, and there are three possible configurations, as shown in Figure 3.6.

We redistribute the positive curvature of Δ, Δ_3 and (where appropriate) Δ_2 as follows. Add $\pi/k + \pi/12$ to each of Δ_4 and Δ_5. In Figure 3.6 (i) add π/k to Δ_6 and $\pi/6 - \pi/k$ to Δ_7. In Figure 3.6 (ii) add $\pi/30$ to Δ_2. In Figure 3.6 (iii) add $2\pi/15$ to Δ_6 and $\pi/10$ to Δ_7.

Suppose then that all three vertices of Δ have degree greater than 4. Clearly at least one vertex has degree 5. If Δ has an A-neighbour Δ_1 of degree greater than 3, then we merely add $c(\Delta) \leq \pi/5$ to $c(\Delta_1)$, so suppose all three neighbours of Δ have degree 3. If Δ has a vertex with the configuration of β_1 in Figure 3.7 then we redistribute the positive curvature of Δ and Δ_3 by adding π/k to each of Δ_4 and Δ_5, so suppose there is no such vertex. Then Δ has a configuration like that of Figure 3.8. Redistribute the positive curvature of Δ and Δ_i, $(i = 1, 2, 3)$ by adding $2\pi/15$ to each of $c(\Delta_j)$, $(j = 4, \ldots, 9)$.

Case 2. One vertex of Δ has the opposite orientation from the other two.

Then this vertex is connected to each of the other two by a single edge in Π, so two of the neighbours of Δ are B-regions. It is easy to see that at most one of the vertices of Δ can have degree 4. If all three have degree greater than 4, then we redistribute $c(\Delta) \leq \pi/5$ by adding $\pi/10$ to the curvature of each of the B-region neighbours of Δ.

Suppose then that Δ has a vertex of degree 4. This vertex has two incident double edges, one single edge and one triple edge. If the triple edge is opposite to the single edge, then the configuration is as in Figure 3.9. If the region Δ_2 in Figure 3.9 has degree greater than 3, we redistribute $c(\Delta)$ by adding $\pi/5$ to $c(\Delta_2)$ and $\pi/10$ to $c(\Delta_3)$. If Δ_2 has degree 3 then we redistribute $c(\Delta) + c(\Delta_2) \leq 2\pi/k + \pi/5$ by adding $\pi/10$ to $c(\Delta_3)$ and $\pi/k + \pi/20$ to each of $c(\Delta_4)$ and $c(\Delta_5)$.

If the triple edge is adjacent to the single edge, then we have the configuration of Figure 3.10. If the edge joining β_1 to γ_1 in Figure 3.10 is a triple edge, then we redistribute $c(\Delta) \leq 3\pi/10$ by adding $\pi/10$ to $c(\Delta_3)$ and $\pi/5$ to $c(\Delta_1)$. Suppose then that this edge is not a triple edge. We next assume that β_1 has degree 5. Then the edge must be a double edge. If Δ_2 has degree greater than 3, then we redistribute $c(\Delta)$ by adding $\pi/10$ to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_3)$. If Δ_2 has degree 3 and γ_2 is an α-vertex, then we redistribute $c(\Delta) + c(\Delta_2) \leq 2(\pi/k + \pi/10)$ by adding $\pi/10$ to each of $c(\Delta_1)$ and $c(\Delta_3)$, and π/k to each of $c(\Delta_4)$ and $c(\Delta_5)$.

Finally, if Δ_2 has degree 3 and γ_2 is a β-vertex, we see that Δ_2 is a positively curved region with three β-vertices of the same orientation, and either one or two of them have degree 4. We note that our method of redistributing curvature for such regions has not resulted in an increase to $c(\Delta)$. Similarly, in redistributing $c(\Delta)$, we must be careful not to increase $c(\Delta_2)$. We achieve this as follows. If β_3 has degree greater than 5, then we add $\pi/10$ to $c(\Delta_1)$ and $2\pi/15$ to $c(\Delta_3)$. Otherwise we add $\pi/10$ to $c(\Delta_1)$ and $\pi/5$ to $c(\Delta_3)$.

Hence we may assume that β_1 in Figure 3.10 has degree at least 6. If β_3 also has degree at least 6, then we redistribute $c(\Delta)$ by adding $\pi/12$ to each of $c(\Delta_1)$ and $c(\Delta_3)$. We may now assume that β_3 has degree 5. If Δ_2 has degree greater than 3, then we redistribute $c(\Delta)$ by adding $\pi/12$ to $c(\Delta_1)$, $\pi/20$ to $c(\Delta_2)$, and $\pi/10$ to $c(\Delta_3)$. If Δ_2 has degree 3 and γ is an α-vertex, then we redistribute $c(\Delta)$ by adding $\pi/12$ to $c(\Delta_1)$, $\pi/10$ to $c(\Delta_3)$, π/k to $c(\Delta_5)$, and $\pi/k-\pi/20$ to $c(\Delta_6)$. If Δ_2 has degree 3 and γ is a β-vertex, then we redistribute $c(\Delta)$ by adding $\pi/12$ to $c(\Delta_1)$ and $3\pi/20$ to $c(\Delta_3)$.

This completes the classification of positively curved regions, and the instructions on how to redistribute curvature. If c^* denotes the resulting amended curvature function, then we show that $c^*(\Delta)\leq 0$ for all interior regions Δ. Suppose that Δ is an interior region with $c(\Delta)<c^*(\Delta)$. We first note that, with five exceptions illustrated in Figure 3.11, The contribution to $c^*(\Delta)$ across any given edge is at most $\pi/10$.

Suppose first of all that Δ is an A-region of degree $n\geq 6$. It is clear from Figure 3.11 that $c^*(\Delta)\leq c(\Delta)+n\pi/4$, so if Δ contains two or more α-vertices, then $c^*(\Delta)\leq c(4,4,4,4,8,8)+6\pi/4=0$. If Δ has no α-vertices then none of its edges are of one of the exceptional forms shown in Figure 3.11, so $c^*(\Delta)\leq c(4,4,4,4,4,4)+6\pi/10<0$. Finally, if Δ has precisely one α-vertex, then it can readily be verified that $c^*(\Delta)\leq c(4,4,4,4,4,8)+\pi<0$.

Next suppose that Δ is an A-region of degree 5. If none of the vertices of Δ are α-vertices, then $c^*(\Delta)\leq c(4,4,4,4,4)+5(\pi/2)=0$, while if three or more are α-vertices then arguing as above we have $c^*(\Delta)\leq c(4,4,8,8,8)+5(\pi/4)=0$. We can assume therefore that Δ has either one or two α-vertices. Note that four of the corners of Δ have the same label (a, say), while the fifth has the opposite label (a^{-1}). If Δ has two α-vertices, then there are six ways (up to symmetry) in which these can be distributed around Δ (in relation to the a^{-1}-corner). For each of these in turn, it can readily be verified, using Figure 3.11, that $c^*(\Delta) - c(\Delta) \leq \pi = -c(4,4,4,8,8) \leq -c(\Delta)$. If Δ has precisely one α-vertex, and this is where the a^{-1} label is, then Figure 3.11 again yields $c^*(\Delta) \leq c(4,4,4,4,8) + 2(\pi/5) + \pi/10 = 0$, while if Δ has precisely one α-vertex elsewhere, then at least one of the β-vertices of Δ has degree greater than 4, so $c(\Delta) \leq c(4,4,4,5,8) = -17\pi/20$, while Figure 3.11 shows that $c^*(\Delta) - c(\Delta) \leq 2(\pi/5) + 3(\pi/10) = 16\pi/20$.

Now suppose Δ is an A-region of degree 4. The various ways in which positive curvature can be transferred to Δ is illustrated in Figure 3.12, with the maximum amount of curvature that can be transferred across a given edge indicated in each case. Suppose that Δ has two or more α-vertices. Then $c^*(\Delta) \leq c(5,5,8,8) + \pi/5 + 2(\pi/8) < 0$ (if the configuration of Figure 3.12 (iii) occurs in Δ), or $c^*(\Delta) \leq c(4,4,8,8) + 4(\pi/8) = 0$ (otherwise). We may therefore suppose that Δ has at most one α-vertex, in which case it has one of the three configurations shown in Figure 3.13 (up to symmetry). We have used information from Figure 3.12 to indicate in Figure 3.13 edges across which no positive curvature is transferred to Δ.

In Figure 3.13 (i) we have $c^*(\Delta) \leq c(4,4,4,8) + 2(\pi/8) = 0$. In Figure 3.13 (ii) we have $c^*(\Delta) \leq c(4,4,4,8) + \pi/8 < 0$, or $c^*(\Delta) \leq c(4,4,5,8) + 7\pi/60 < 0$, or $c^*(\Delta) \leq c(4,5,5,8) + \pi/5 + \pi/8 = 0$. In Figure 3.13 (iii) we have $c^*(\Delta) \leq c(4,4,5,5) + \pi/5 = 0$. Hence $c^*(\Delta) \leq 0$ whenever Δ is an interior A-region.

We now turn our attention to the case where Δ is a B-region. At this stage of the proof it becomes necessary to treat the cases $n = 4$ and $n > 4$ separately.

Case 1 $n = 4$.

In this case $k \geq 12$, since $p \geq 17$. Hence $\pi/k + \pi/12 \leq \pi/6$. Thus if Δ contains two or more α-vertices, then $c^*(\Delta) \leq c(4,4,12,12) + 4(\pi/6) = 0$. Next suppose that Δ has precisely one α-vertex. If two of the β-vertices of Δ have degree 5 or greater, then $c^*(\Delta) \leq c(4,5,5,12) + 2(\pi/6) + 2(\pi/10) = 0$. If one of the β-vertices has degree 7 or greater, then $c^*(\Delta) \leq c(4,4,7,12) + 2(\pi/6) + 2(\pi/10) < 0$. In the remaining three subcases, it is a routine verification that $c^*(\Delta) \leq c(4,4,6,12) + 2(\pi/6) +$

$2(\pi/12) = 0$, $c^*(\Delta) \leq c(4,4,5,12) + 2(\pi/6) + \pi/10 = 0$ and $c^*(\Delta) \leq c(4,4,4,12) + 2(\pi/6) = 0$ respectively.

Suppose then that Δ has no α-vertices. The various ways in which contributions to $c^*(\Delta)$ can be received across edges are illustrated in Figure 3.14. If each vertex of Δ has degree 5 or more, then $c^*(\Delta) \leq c(5,5,5,5) + 4(\pi/10) = 0$. If Δ has precisely one vertex of degree 4 then either $c^*(\Delta) \leq c(4,5,5,5) + \pi/20 + \pi/5 < 0$, or $c^*(\Delta) \leq c(4,5,5,5) + \pi/20 + \pi/10 + 2\pi/15 < 0$, or $c^*(\Delta) \leq c(4,5,5,6) + 2(3\pi/20) < 0$. If Δ has precisely two vertices of degree 4 then either $c^*(\Delta) \leq c(4,5,4,5)+2(\pi/20) < 0$, or $c^*(\Delta) \leq c(4,4,5,5)+\pi/5 = 0$, or $c^*(\Delta) \leq c(4,4,5,5)+2\pi/15+\pi/20 < 0$, or $c^*(\Delta) \leq c(4,4,5,5)+2(\pi/20)+\pi/10 = 0$, or $c^*(\Delta) \leq c(4,4,5,6)+3\pi/20+\pi/10 < 0$. Otherwise $c^*(\Delta) \leq c(4,4,4,5) + 2(\pi/20) = 0$.

Case 2 $n > 4$.

Here $k \geq 8$ and $\pi/k + \pi/12 \leq 5\pi/24$. If Δ contains no α-vertices then $c^*(\Delta) \leq c(4,4,4,4,4) + 5(\pi/10) = 0$. If Δ has degree greater than 5 and has at least one α-vertex, then $c^*(\Delta) \leq c(4,4,4,4,4,8) + 6(5\pi/24) = 0$. Hence we can assume that Δ has degree $n = 5$ and has at least one α-vertex. If Δ has three or more α-vertices, then $c^*(\Delta) \leq c(4,4,8,8,8) + 5(5\pi/24) < 0$. If Δ has precisely two α-vertices, and a β-vertex of degree greater than 4, then $c^*(\Delta) \leq c(4,4,5,8,8) + 5(5\pi/24) < 0$. If Δ has precisely two α-vertices, and all its β-vertices have degree 4, then $c^*(\Delta) \leq c(4,4,4,8,8) + 4(5\pi/24) < 0$ (using the fact that no positive curvature is distributed across an edge joining two β-vertices of degree 4). Finally, if Δ contains precisely one α-vertex, then either $c^*(\Delta) \leq c(4,4,4,5,8) + 2(5\pi/24) + 3(\pi/10) < 0$ or $c^*(\Delta) \leq c(4,4,4,4,8) + 2(5\pi/24) < 0$.

To complete the proof, we observe that $c^*(\Delta_0) < 4\pi$, where Δ_0 is the distinguished region. This follows immediately from the fact that each vertex of Δ_0 has degree at least 4, and the fact that $c^*(\Delta_0) - c(\Delta_0) \leq \ell(\pi/4)$, where ℓ is the degree of Δ_0.

Corollary *The elements a, b of (3,n,p;2) have orders 3 and n respectively.*

Proof. By the Lemma, $\pi_2(Z,X)$ is generated by $\mathcal{D} \cup \mathcal{S}$, every element of which has trivial boundary label. Hence no spherical picture has boundary a nontrivial element of $\pi_1(X) = \langle a | a^3 \rangle$, so a has order 3 as claimed. Similarly b has order n.

Lemma 3.2 *If either $n = 4$ and $p \geq 17$, or $n \geq 5$ and $p \geq 11$ then (3,n,p;2) does not collapse.*

Proof. By the Corollary to Lemma 3.1, it is enough to prove that ab has order p and $[a, b]$ has order 2 in (3,n, p;2).

Suppose first that ab has order $k < p$. Then $p = kr$ for some $r > 1$. Let $\mathcal{P}_1$ be a picture with boundary label $(ab)^k$. Then the disjoint union of r copies of $\mathcal{P}_1$ has boundary label $(ab)^p$. We can form a spherical picture $\mathcal{P}$ from this by adding a single α-vertex labelled $(ab)^{-p}$. The algebraic sum of the α-vertices in $\mathcal{P}$ (that is, the number with label $(ab)^p$ minus the number with label $(ab)^{-p}$) is congruent to -1 modulo r, and is therefore nonzero, since $r > 1$. But this contradicts Lemma 3.1, since $\mathcal{S}$ does not contain an α-vertex, and any dipole containing α-vertices contains precisely one of each sign. Hence ab has order p, as claimed.

A similar argument shows that $[a, b]$ has order 2. If not, there is a picture $\mathcal{P}_2$ with boundary label $[a, b]$. Form a spherical picture $\mathcal{P}$ by adding a β-vertex with label $[a, b]^{-2}$ to the disjoint union of two copies of $\mathcal{P}_1$. Since $\mathcal{P}$ has an odd number of β-vertices, $\mathcal{S}$ has six β-vertices, and any dipole has either two β-vertices or none, we again obtain a contradiction to Lemma 3.1.

§4. Completion of proof.

Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array} \qquad (*)$$

be a pushout of groups. If K_A, K_B, K_C are Eilenberg-Maclane spaces of type $K(A,1)$, $K(B,1)$, $K(C,1)$ respectively, and $\tilde{f} : K_A \to K_B$, $\tilde{g} : K_A \to K_C$ are continuous maps realizing f, g at the fundamental group level, then we can form a space X with $\pi_1(X) = D$ by

$$X = M(\tilde{f}) \bigcup_{K_A} M(\tilde{g}) \ ,$$

where $M(\cdot)$ denotes the mapping cylinder. The homotopy type of X is independent of the various choices made. We say that (*) is *geometrically Mayer-Vietoris* if X is aspherical (that is, a $K(D,1)$-space).

The above can be mimicked algebraically using algebraic mapping cylinders as follows. Let R be a commutative ring with identity, and let $\mathbf{P}^A$, $\mathbf{P}^B$, $\mathbf{P}^C$ be projective resolutions over RA, RB, RC respectively of the trivial module R. Define a chain complex $\mathbf{P}$ of projective RD-modules as follows:

$$\mathbf{P}_n = (\mathbf{P}_n^B \otimes_{RB} RD) \oplus (\mathbf{P}_n^C \otimes_{RC} RD) \oplus (\mathbf{P}_{n-1}^A \otimes_{RA} RD) \ ,$$

and the boundary map of $\mathbf{P}$ is induced from those of $\mathbf{P}^B$ and $\mathbf{P}^C$ on the B- and C-factors; and on the A-factor by $d^A + \tilde{f} - \tilde{g}$, where d^A is the boundary in $\mathbf{P}^A$, and $\tilde{f} : (\mathbf{P}^A \otimes_{RA} RB) \to \mathbf{P}^B$ and $\tilde{g} : (\mathbf{P}^A \otimes_{RA} RC) \to \mathbf{P}^C$ are chain maps commuting with the identity map $R \to R$. In the above RB is made into a left RA-module via the homomorphism $A \to B$, etc. The chain homotopy equivalence class of $\mathbf{P}$ does not depend on the various choices made. We say that (*) is *algebraically Mayer-Vietoris over* R if $\mathbf{P}$ is a resolution of R over RD.

Remarks (1) If (*) is geometrically Mayer-Vietoris, then it is algebraically Mayer-Vietoris over all R. The space X can be assumed to be a CW-complex, with K_A, K_B and K_C subcomplexes, and then $\mathbf{P}$ is chain homotopy equivalent to the cellular R-chain complex of the universal cover of X.
(2) If (*) is algebraically Mayer-Vietoris over R, then the (co-)homology of A, B, C, D with coefficients in a given RD-module, is linked by a Mayer-Vietoris sequence. This explains our choice of terminology. We shall not require these sequences explicitly in what follows.

From now on, let us restrict attention to the ring $\mathbb{Q}$ of rational numbers. We first recall some definitions. Recall that a group G is *of type* $\mathrm{FP}_{\mathbb{Q}}$ if there is a projective $\mathbb{Q}G$-resolution $\mathbf{P}$ of $\mathbb{Q}$

of finite length, consisting of finitely generated modules. If E is an idempotent square matrix with entries from $\mathbb{Q}G$, and trace $\sum_{g\in G}\tau_g g$, then the image of E is a finitely generated projective $\mathbb{Q}G$-module P, say, and $r(P)=\tau_1$ is its *(Hattori-Stallings) rank*, an invariant of P. If $\mathbf{P}$ is a finite projective resolution as above, then

$$\chi_{\mathbb{Q}}(G)=\sum_{n=0}^{\infty}(-1)^n r(P_n)$$

is the *rational Euler characteristic* of G, which again is an invariant. (See for example [**?**] for details.)

Proposition 4.1 *If the pushout (*) of groups is algebraically Mayer-Vietoris over* $\mathbb{Q}$*, and each of* A,B,C *is of type* $\mathrm{FP}_{\mathbb{Q}}$*, then so is* D*, and moreover*

$$\chi_{\mathbb{Q}}(D)=\chi_{\mathbb{Q}}(B)+\chi_{\mathbb{Q}}(C)-\chi_{\mathbb{Q}}(A).$$

Now let us return to the situation of our main theorem.

Lemma 4.2 *If* $\mathcal{P}$ *as in* **§1** *does not collapse and* $\pi_2(\mathcal{P})$ *is generated by* $\mathcal{S}$ *together with dipoles, then the group* G *presented by* $\mathcal{P}$ *is infinite.*

Proof. In order to prove Lemma 4.2, we take $A=C_3*C_3=\langle a,c|a^3,c^3\rangle$, $B=A_4=\langle a,c|a^3,c^3,(ac)^2\rangle$ and $C=\langle a,b|a^3,b^n,(ab)^p\rangle$, the hyperbolic triangle group of type $(3,n,p)$ in the pushout (*). Here $f:A\to B$ is given by $f(a)=a$, $f(c)=c$ and $g:A\to C$ by $g(a)=a$, $g(c)=b^{-1}a^{-1}b$. The group D in (*) is then isomorphic to our group $G=\langle a,b|a^3,b^n,(ab)^p,[a,b]^2\rangle$. The noncollapsing hypothesis shows that the kernel of $A\to G$ is free, so of homological dimension at most 1, while that of $C\to G$ is torsion-free (since every torsion element of C is conjugate to a power of a,b or ab), so a hyperbolic surface group, of homological dimension at most 2. The hypothesis also shows that $B\to G$ is injective, since ac is contained in every nontrivial normal subgroup of B.

The above, together with the hypothesis that $\pi_2(\mathcal{P})$ is generated by $\mathcal{S}$ and dipoles, is enough to ensure that (*) is geometrically Meyer-Vietoris. To see this, take the above presentation for A as the 2-skeleton for an Eilenberg-MacLane space K_A, and similarly for B and C. Then the space X in the definition of geometrically Mayer-Vietoris has 2-skeleton homotopy equivalent to $\mathcal{P}$. Since $\mathcal{S}$ and all dipoles over $\mathcal{P}$ come from the presentations for B and C via the natural maps, they are nullhomotopic in K_B or K_C, hence in X. Thus $\pi_2(X) = 0$. It follows from [?], Theorem 4.2, that X is aspherical, so (*) is geometrically Mayer-Vietoris, as claimed.

Since A, B, C all have type $\mathrm{FP}_{\mathbb{Q}}$ it follows from Proposition 4.1 that so also does G, and that

$$\chi_{\mathbb{Q}}(G) = \chi_{\mathbb{Q}}(B) + \chi_{\mathbb{Q}}(C) - \chi_{\mathbb{Q}}(A) = \frac{1}{12} + (\frac{1}{3} + \frac{1}{n} + \frac{1}{p} - 1) - (\frac{1}{3} + \frac{1}{3} - 1).$$

Assume that G is finite, of order N. Then N is divisible by 12, by n, and by p, by the non-collapsing hypothesis. Moreover, $\chi_{\mathbb{Q}}(G) = \frac{1}{N}$, which yields the diophantine equation

$$\frac{1}{N} = \frac{1}{n} + \frac{1}{p} - \frac{1}{4}.$$

The only natural number solutions of this equation with $4 \leq n \leq p$ and N divisible by $\mathrm{lcm}(12, n, p)$ are as follows:

(i) $n = 4$, $N = p = 12k$;

(ii) $n = 5$, $p = 15$, $N = 60$;

(iii) $n = 5$, $p = 18$, $N = 180$;

(iv) $n = p = 6$, $N = 12$;

(v) $n = 6$, $p = 8$, $N = 24$;

(vi) $n = 6$, $p = 9$, $N = 36$;

(vii) $n = 6$, $p = 10$, $N = 60$;

(viii) $n = 6$, $p = 11$, $N = 132$;

(ix) $n = 7$, $p = 9$, $N = 252$.

In each case, we can quickly rule out the possiblity of G being finite of order N. In case (i) G is cyclic, since it contains an element of order $p = N$, but then G cannot contain A_4 as a subgroup. In case (ii) the commutator subgroup G' has index 5, so would have to be isomorphic to A_4. But A_4 has no automorphism of order 5, so $G = A_4 \times C_5$. But then any elements a, b of orders 3 and 5 in G commute, so $[a, b]$ cannot have order 2, a contradiction. In case (iii) G' is perfect, of order 60, so $G' \cong A_5$. But in that case G could contain no element of order 18, since A_5 has no elements of order 6. Again we have a contradiction. In case (iv) we would have $G \cong A_4$, which has no elements of order 6. In case (v) any element of order 8 in G would have to generate a cyclic Sylow 2-subgroup, and G could not contain A_4. In case (vi) $G/G' \cong C_3 \times C_3$, so if $N = 36$ there is no element of order 9 in G. In case (vii) the second commutator subgroup G'' has index 24 in G, contradicting $N = 60$. Finally, in cases (viii) and (ix) G' is perfect, of order 44 or 84 respectively, which is impossible.

Combining Lemmas 3.1, 3.2 and 4.2, we obtain the desired result.

Theorem *Let* $G = (3, n, p; 2) = \langle a, b \mid a^3 = b^n = (ab)^p = [a, b]^2 = 1\rangle$*, where either* $n = 4$ *and* $p \geq 17$ *or* $n \geq 5$ *and* $p \geq 11$*. Then* G *is infinite.*

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