The universal torsion-free image of a group*

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Abstract

We show that the universal torsion free homomorphic image of any group given by a sufficiently 'small' presentation is locally indicable, and give an application to a conjecture of Levin about equations over torsion free groups.

1 Introduction

Let G be a group. The set of normal subgroups $N \lhd G$ such that G/N is torsion-free is closed with respect to arbitrary intersections,

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so contains a unique minimal element $\rho(G)$, the **torsion-free radical** of G. The quotient group $\widehat{G} = G/\rho(G)$ is thus universal among all torsion-free homomorphic images of the group G. The purpose of the present paper is to show that, if G has a presentation that is 'small' in the sense that it has few relations, and they are short words in the generators, then this universal torsion-free homomorphic image $\widehat{G}$ is locally indicable. Recall that a group H is said to be **indicable** if there is an epimorphism from H to the infinite cyclic group; and H is said to be **locally indicable** if every nontrivial, finitely generated subgroup $K \subset H$, is indicable. Since much is known about one-relator products of locally indicable groups [**?**, **?**, **?**], we can then apply those results to one-relator products of torsion-free groups in general. Specifically, we prove the following results.

Lemma 1.1 *Let G be a 1-relator group. Then $\widehat{G}$ is locally indicable.*

Theorem 1.2 *Let G be a 2-relator group in which one relator has length at most 4. Then $\widehat{G}$ is locally indicable.*

Theorem 1.3 *Let G be a 2-relator group in which one relator has length 5 and the other has length at most 8. Then $\widehat{G}$ is locally indicable.*

Theorem 1.4 *Let G be a group with a presentation having at most 5 relators, each of length at most 3. Then $\widehat{G}$ is locally indicable.*

Let us define the **complexity** of a finite presentation $\mathcal{P}$ to be $c(\mathcal{P}) = \sum_r (\ell(r) - 2)$, where $\ell(r)$ denotes the length of a word r and the sum is over all relators that are *not* powers of generators.

Corollary 1.5 *Let G be a group given by a presentation of complexity at most 5. Then $\widehat{G}$ is locally indicable.*

Proof. Let $\mathcal{P}$ be a presentation for G of complexity at most 5. If $\mathcal{P}$ contains a relator of the form x^n for some generator x and some integer $n \neq 0$, then $x^n = 1$ in G so $x = 1$ in $\widehat{G}$. Hence this relator, together with x, can be omitted from $\mathcal{P}$ (deleting any occurrences of x in other relators) without changing $\widehat{G}$ or increasing the complexity. If

$\mathcal{P}$ has a relator of the form xy or xy^{-1} for two distinct generators x, y, then we can remove this relator and y from $\mathcal{P}$, replacing every other occurrence of y in other relators by x^{-1} or x, again without changing $\widehat{G}$ or increasing the complexity. Hence we may assume that every relator of $\mathcal{P}$ has length at least 3. Suppose $\mathcal{P}$ contains a relator xyW for some word W of length greater than 1. We may introduce a new generator z and replace the relator xyW by two relators xyz, $z^{-1}W$. This does not affect either G or $c(\mathcal{P})$. Repeating this argument, we can reduce $\mathcal{P}$ to a presentation for G with all relators of length exactly 3. Now apply Theorem **??**.

These results are best possible, as the following examples show.

Example. The Fibonacci group $G = F(2,6)$ has presentations

$$\langle x_1, \dots, x_6 \mid x_1x_2 = x_3, \dots, x_5x_6 = x_1, x_6x_1 = x_2 \rangle$$

and

$$\langle a, b \mid a^{-1}b^2ab^2 = b^{-1}a^2ba^2 = 1 \rangle.$$

The first of these has six relators, each of length exactly 3, while the second has two relators, each of length 6. Now G is the fundamental group of an aspherical 3-manifold [**?**], so torsion-free, and so $\widehat{G} = G$. However, G is finitely generated and non-indicable, so not locally indicable.

Example. The group $G = \langle a, b \mid abab^{-2} = a^{-6}bab = 1 \rangle$ is presented with two relators, of lengths 5 and 9 respectively. But G is isomorphic to the torsion-free centrally extended triangle group , $(2,3,7) = \langle a, b, c \mid a^7 = b^3 = c^2 = abc \rangle$ [**?**], §3. Indeed $G = [G, G] = \pi_1(M)$ for a certain aspherical 3-manifold M [**?**]. However, as G is perfect, it is not locally indicable.

Finally, we apply these results to the following conjecture of Levin [**?**].

Conjecture *Let A, B be torsion-free groups, and $w \in A \star B$ a cyclically reduced word of length at least 2. Let $N(w)$ denote the normal closure of w in $A \star B$. Then $A \cap N(w) = \{1\}$.*

The conjecture is known to hold for A, B locally indicable, but remains open in general. Combining this with our results on small presentations, we are able to prove the following.

Theorem 1.6 *Let A, B be torsion-free groups, and suppose $a \in A$ can be expressed in the form*

$$a = \prod_{i=1}^{n} v_i w^{\epsilon(i)} v_i^{-1}$$

with $n \leq 4$ and $\epsilon(i) = \pm 1$ for each i. Then $a = 1$.

Levin's conjecture is equivalent to the assertion of this theorem, without the restriction on n.

2 Pictures and norms

Let $G = (A \star B)/N(w)$ be a one-relator product of two groups A and B, that is, the quotient of their free product by the normal closure $N = N(w)$ of a single element $w \in A \star B$, assumed to be a cyclically reduced word of length at least 2, called the **relator**. If $u \in N(w)$ then u can be written as a product of conjugates of $w^{\pm 1}$:

$$u = \prod_{i=1}^{n} v_i w^{\epsilon(i)} v_i^{-1},$$

with $\epsilon(i) = \pm 1$ for all i. We define $\nu(u)$, the **norm** of u, to be the least value of n among all such expressions for u. For $u \in (A \star B) \setminus N(w)$, we define $\nu(u) = \infty$.

In terms of the norm, if $a \neq 1$ in A, then Theorem **??** says that $\nu(a) \geq 5$, while Levin's Conjecture says that $\nu(a) = \infty$.

We refer the reader to [**?**] for detailed definitions of **pictures** over the one-relator product $G = (A \star B)/N(w)$ on a surface Σ. In this paper we are interested only in the case $\Sigma = D^2$, and almost exclusively with **connected** pictures. A **picture** on D^2 over G consists of a properly embedded graph $\mathcal{P}$ in D^2, (except that some edges of the graph, instead of joining vertices to vertices, are allowed to join vertices to points on ∂D^2, or even join two points of ∂D^2). The components of $D^2 \setminus \mathcal{P}$ are known as **regions**, and are divided into A-regions and B-regions. Every edge separates an A-region from a B-region. To each **corner** of a region (either a point where the region meets a vertex, or component of region $\cap\, \partial D^2$) is associated a **label**, which is an element of X if the region is an X-region ($X = A, B$). The labels around a vertex, read counterclockwise, spell a word called the **vertex label**, which is required to be $w^{\pm 1}$ in cyclically reduced form (up to cyclic permutation). The clockwise label around ∂D^2 is called the **boundary label**. The clockwise labels around a simply connected A- (resp. B-) region spell a word which is required to be the identity in A (resp. B). For non-simply connected regions there is a more complicated condition, which need not concern us here. (For example the two boundary labels of an annular region are required to satisfy a conjugacy relation.) A picture is **connected** if it is connected as a graph.

From our point of view, the key fact about pictures is the following. There exists a picture on D^2 over G with boundary label $u \in A \star B$ if and only if $u \in N(w)$, and then $\nu(u)$ is the minimum number of vertices in such a picture. See for example [**?**] for details.

Proof of Theorem ??. Suppose that Levin's Conjecture is false. Choose $a \in (A \cup B) \setminus \{1\}$ of minimum norm n say. Then $1 < n = \nu(a) < \infty$. Without loss of generality, we may assume that $a \in A$.

Let $\mathcal{P}$ be a picture over G with n vertices and boundary label a. By the assumption of minimality of $\nu(a)$, it follows that $\mathcal{P}$ is connected. For otherwise there is a subpicture with fewer vertices and boundary label $b \in A \cup B$. By minimality we have $b = 1$, so this subpicture may be removed, contradicting $\nu(a) = n$.

Since $a \in A \cup B$, it also follows that no arc of $\mathcal{P}$ meets the boundary of D^2. Shrinking the boundary ∂D^2 to a point, we obtain a tessellation T of S^2 with $n \leq 4$ vertices. If, for each positive integer k, we let F_k denote the number of k-sided faces of T, then

$$\sum_{k=1}^{\infty}(k-2)F_k = 2n-4 \leq 4$$

by Euler's formula.

Define abstract groups A_0 and B_0 as follows. The generators of A_0 are the A-letters appearing in w, and the defining relators are the boundary labels of the disc A-regions of $\mathcal{P}$. Since these are identities in A, the group A_0 comes equipped with a natural homomorphism $A_0 \to A$, and $a \in A$ is the image of some $a_0 \in A_0$ under this homomorphism. The group B_0 and homomorphism $B_0 \to B$ are defined in an analogous way. Since A and B are torsion-free, these natural homomorphisms $A_0 \to A$ and $B_0 \to B$ factor through $\widehat{A}_0$ and $\widehat{B}_0$ respectively.

Note that no relator of A_0 or B_0 has the form x^t for any $t \in \mathbb{Z}$, since A, B are torsion-free and w is cyclically reduced. Moreover, each k-sided face of T represents a relator of A_0 or B_0 of length k, with the sole exception of the face arising from the shrinking of ∂D^2. Since that face has a positive number of sides, it follows from the above equation that $c(A_0) + c(B_0) \leq 5$, whence both $\widehat{A}_0$ and $\widehat{B}_0$ are locally indicable, by Corollary **??**. Since Levin's conjecture holds for locally indicable groups [**?**], it follows that the image of a_0 in $\widehat{A}_0$ vanishes, whence $a = 1$ in A, as claimed.

3 Proofs of the main results

We first prove a series of lemmas concerning groups whose universal torsion free images are locally indicable.

Lemma ?? *Let G be a 1-relator group. Then $\widehat{G}$ is locally indicable.*

Proof. Suppose $G = \langle x_\lambda (\lambda \in \Lambda) \mid s^m \rangle$, where $m \geq 1$ and s is not a proper power. Then $s = 1$ in $\widehat{G}$, so $\widehat{G}$ is a homomorphic image of the 1-relator group $G_0 = \langle x_\lambda \mid s \rangle$. But G_0 is torsion-free, since s is not a proper power, and so $\widehat{G} = G_0$. Finally, torsion-free one-relator groups are locally indicable [**?**], so $\widehat{G}$ is locally indicable, as claimed.

Lemma 3.1 *Let α be any integer, and let M_α denote the metabelian one-relator group $M_\alpha = \langle x, y \mid xyx^{-1}y^{-\alpha}\rangle$. Then every torsion-free homomorphic image of M_α is locally indicable.*

Proof. M_α is itself locally indicable, being a torsion-free one-relator group [**?**]. Suppose K is a normal subgroup of M_α such that $H = M_\alpha/K$ is torsion-free. If $y \in K$, then H is cyclic, either of order 1 or ∞ (since H is torsion free). In either case H is locally indicable. Now the normal closure A of y in M_α is locally cyclic, generated by $y_t = x^{-t}yx^t$ for $t \in \mathbb{Z}$, with $y_{t-1} = y_t^\alpha$. Hence every element of A is conjugate in M_α to a power of y. If some $a \neq 1 \in A \cap K$, then $y^k \in K$ for some $k \neq 0$, so $y \in K$ since H is torsion-free, so H is locally indicable. The only possibility remaining to consider is that K contains some element of $M_\alpha \backslash A$. Such an element has the form $x^k a$ for some $k \neq 0$ and $a \in A$. But then $y^{(\alpha^k - 1)} = [(x^k a)^{-1}, y] \in K$, so unless $\alpha^k = 1$ we deduce that $y \in K$ and H locally indicable, as before.

Finally, if $\alpha^k = 1$ then $\alpha = \pm 1$ and M_α has a free abelian subgroup of rank 2 and index 2. It follows that H has a cyclic subgroup of finite index, and since H is torsion-free it must be infinite cyclic.

Lemma 3.2 *Let G be a torsion-free group containing a free abelian subgroup A of rank $r \leq 2$ and of finite index in G. Then G is locally indicable.*

Proof. Without loss of generality, we assume that A is normal in G. Then the quotient group , $= G/A$ acts (linearly) on $A \cong \mathbb{Z}^r$ via conjugation in G. We consider first the case where this action is orientation-preserving, in other words by matrices of determinant 1. In this case we will show that G is itself free abelian, arguing by induction on the order of , . In the initial case, $G = A$ and there is nothing to prove. For the inductive step we may assume that , is simple.

Suppose that $1 \neq g \in G$ acts via a matrix $B \in SL(r, \mathbb{Z})$. For some $k \geq 1$, $1 \neq g^k \in A$, since G is torsion-free. Since g commutes with g^k, at least one of the eigenvalues of B is 1. But $r \leq 2$ and det(B)=1, so all eigenvalues of B are equal to 1, and B is parabolic. Moreover, $B^k = I$, so $B = I$. Hence A is central in G.

If , is nonabelian, then it is perfect, so

$$H^2(, , A) \cong \mathrm{Hom}(H_2(,), A) \times \mathrm{Ext}(H_1(,), A),$$

by the universal coefficient theorem (see e.g. [**?**], p. 49 or [**?**], p. 8). But the right hand side vanishes, because $H_2(,)$ is finite and $H_1(,) = 0$. Hence every central extension of A by , splits - in particular $G \cong A \times ,$, contradicting the fact that G is torsion-free. Hence , is cyclic of prime order. Since A is central in G, it follows that G is abelian, and hence free abelian of rank r.

Finally, suppose that the action of , on A is not orientation-preserving. There is a subgroup Δ of index 2 in , such that the restriction of the action to Δ is orientation-preserving; and by the above the corresponding subgroup H of index 2 in G is free abelian. We may therefore assume that $A = H$. Choose $x \in G \backslash H$, and let B be the corresponding matrix in $GL(r, \mathbb{Z})$. As before, one of the eigenvalues of B is 1, but det(B)= -1, so $r = 2$ and the second eigenvalue is -1. Moreover the eigenspace N of -1 can readily be seen to be an infinite cyclic normal subgroup of G, and G is a semidirect product of N with the centraliser C of x in G. By the orientation-preserving case, C is also infinite cyclic. Hence G is isomorphic to $\langle x, y \mid xyx^{-1}y \rangle$, the fundamental group of the Klein bottle. In particular G is locally indicable.

Definition. A **one-relator extension** of a group G is a one-relator product of G with a free group.

Note that a one-relator extension H of a locally indicable group G is locally indicable, by [**?**], provided the relator is not a proper power. On the other hand, if the relator has the form s^m where s is not a proper power, then $\widehat{H}$ is the one-relator product with relator s, and so $\widehat{H}$ is locally indicable.

Theorem ?? *Let G be a 2-relator group in which one relator has length at most 4. Then $\widehat{G}$ is locally indicable.*

Proof. Let $G = \langle x_1, \dots \mid r, s \rangle$, where r has length at most 4. If some generator occurs exactly once in r, then we may use r to rewrite that generator in terms of the others, obtaining a one-relator presentation of G with fewer generators. In this case the result follows from Lemma **??**.

Moreover, if either relator is a proper power, we may replace it by its root without affecting $\widehat{G}$, so we may assume that neither relator is a power. In particular, we may assume that r has one of the forms $x_1x_2x_1^{-1}x_2^{\pm1}$ or $x_1^2x_2^2$. Note that $H = \langle x_1, x_2 \mid r\rangle$ is either free abelian of rank 2 or the Klein bottle group. In either case every torsion-free homomorphic image of H is locally indicable. If s is equivalent, modulo r, to a word in x_1, x_2, then G is a free product of a homomorphic image of H with a free group, and so $\widehat{G}$ is locally indicable. Otherwise G is a one-relator extension of H, and so again $\widehat{G}$ is locally indicable.

Theorem ?? *Let G be a 2-relator group in which one relator has length 5 and the other has length at most 8. Then $\widehat{G}$ is locally indicable.*

Proof. Let $G = \langle x_1, \dots \mid r, s\rangle$, where r has length 5. As in the proof of Theorem **??**, we may assume that r involves precisely two generators, say x_1, x_2, each at least twice. Without loss of generality, r has one of the forms (i) $x_1^2x_2x_1x_2$, (ii) $x_1^2x_2^{-1}x_1x_2$, (iii) $x_1^{-2}x_2^{-1}x_1x_2$, (iv) $x_1^{-2}x_2x_1x_2$, or (v) $x_1^3x_2^2$.

In case (i) we may replace the generator x_2 by $y = x_1x_2$. Then r is a word of length 3 in x_1, y, and so the result follows from Theorem **??**. In case (ii) the group $H = \langle x_1, x_2 \mid r\rangle$ is isomorphic to the metabelian group M_{-2}, and in case (iii) H is isomorphic to M_2. In either case every torsion-free homomorphic image of H is locally indicable, by Lemma **??**. Now G is either a free product of a free group with a homomorphic image of H, or a one-relator extension of H. In either case, $\widehat{G}$ is locally indicable.

In cases (iv) and (v), if s is not equivalent (mod r) to a word in x_1, x_2, then G is a one-relator extension of the one-relator group H, so $\widehat{G}$ is locally indicable. Hence we may assume that s is equivalent (mod r) to such a word s', say. Note also that s' may be chosen to be no longer then the word s. Then G is a free product of a free group with $G' = \langle x_1, x_2 \mid r, s'\rangle$. It therefore suffices to show that $\widehat{G}'$

is locally indicable. Let C be the subgroup of G' generated by x_1^3. Then C is central in G', and $G'' = G'/C$ is either a free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, or a one-relator product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, with relator s'' of length at most 8. It follows also that G' is a central extension of G''.

Suppose first that $G'' \cong \mathbb{Z}_2 \star \mathbb{Z}_3$. Then either $\widehat{G}' = \{1\}$ (if C is finite), or $\widehat{G}' = G' = \langle x_1, x_2 \mid r \rangle$, the trefoil knot group, which is locally indicable, being a torsion-free one-relator group.

Hence we may assume that G'' is a one-relator product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. In particular, if we can show that G'' is finite, then so is G', so $\widehat{G}'$ is trivial.

In case (v) this is automatic, since the free product length of s'' is at most 8, and any such one-relator product of the modular group is finite (see for example [**?**]).

In case (iv) we can replace x_2 by $y = x_1x_2$, and r becomes $x_1^{-3}y^2$, as in case (v). In this case, however, the word s' may have become extended in length by the rewriting process. Specifically, s' is a word of length at most 8 in x_1, x_2, which we may assume involves x_1 at least twice, so when rewritten in terms of x_1, y the length of s' may increase to (at most) 14, with (at most) 6 occurrences of y, and hence the resulting word $s'' \in \mathbb{Z}_2 \star \mathbb{Z}_3$ has free product length at most 12 in $\mathbb{Z}_2 \star \mathbb{Z}_3$. By [**?**], we can argue as above unless s'' is one of $(x_1y)^6$ or $[x_1, y]^3$ (up to cyclic permutation and inversion). Let us examine how such words can arise as s''.

If s'' involves 6 occurrences of y, then s', written in terms of x_1, x_2, involves precisely 6 occurrences of x_2 and 2 of x_1. In other words, we may assume that $s' = x_1x_2^ax_1^\delta x_2^b$, where $\delta = \pm 1$, $a \neq 0 \neq b$ and $|a| + |b| = 6$; or $s' = x_1^2x_2^{\pm 6}$. Substituting $x_2 = x_1^{-1}y$ gives $s' = x_1(x_1^{-1}y)^ax_1^\delta(x_1^{-1}y)^b$; or $s' = x_1^2(x_1^{-1}y)^{\pm 6}$. We can rule out the second form, as it gives $s'' = x_1y(x_1^2y)^5$ or $s'' = x_1^2y(x_1y)^5$. Hence only the first form can occur. Since at least one of $|a|, |b|$ is greater than 2, there is a subword $(yx_1yx_1y)^{\pm 1}$ in s'', which rules out the possibility that $s'' = [x_1, y]^3$. Hence we may assume that $s'' = (x_1y)^6$. If no cancellation occurs in rewriting s', then $a, b < 0$ and $\delta = 1$, so $s' = (x_1y^{-1})^{-a}x_1^2(y^{-1}x_1)^{-1-b}y^{-1}x_1$, and the cyclically reduced form of s'' has precisely two x_1^2 and four x_1 letters, a contradiction. Hence cancellation does occur. This cancellation must involve precisely one

x_1 symbol with one x_1^{-1} symbol, and all other occurrences of x_1 must have the same sign. There are two ways in which this can happen. Firstly, $\delta = -1$ and a, b have the same sign (which we may assume is positive). But then y^2 appears in the cyclically reduced form of s', and s'' has free product length less than 12, a contradiction. Secondly, $\delta = 1 = b$, and $a = -5$. In this case $s' = x_1(y^{-1}x_1)^5.x_1.x_1^{-1}y$, so $s'' = (x_1y)^6$, as required. This last case is therefore the only possibility.

We are thus reduced to the case where

$$G = \langle x_1, x_2 \mid x_1^{-2}x_2x_1x_2, x_2^{-5}x_1x_2x_1\rangle.$$

But then $G/[G,G]$ is infinite cyclic, while a calculation using the Reidemeister-Schreier rewriting process shows that $[G,G]$ is free abelian of rank 2. Hence G is locally indicable.

We are now ready to study presentations with few relations, all of length 3.

Lemma 3.3 *Let G be given by a presentation with at most 3 generators, and 2 non-equivalent relations of length 3. Then G has a free abelian subgroup of finite index, of rank at most two, and hence every torsion-free homomorphic image of G is locally indicable.*

Proof. Suppose first that some relator has the form $x_i^{\pm 3}$ for some generator x_i. Then $x_i = 1$ in every torsion-free homomorphic image of G, so we may replace G by a 1- or 2-generator group with a single relator of length 1, 2, or 3. The result is immediate for such a group. A similar argument applies if some generator occurs exactly once in the relators. We assume that neither of these happens.

Next note that if G has only two generators x_1, x_2, then each relator has the form $x_i^{\pm 2}x_j^{\pm 1}$ with $i \neq j$, so G is cyclic of finite order, and the only torsion-free homomorphic image of G is the trivial group. Hence also if G has three generators, but only two of them are involved in relators, then $\widehat{G}$ is infinite cyclic, and the result follows.

We are reduced to the case where G has precisely three generators, each occurring exactly twice in the relators. We may rewrite this as a

2-generator presentation with a single relation of length 4, involving each generator exactly twice. Hence G is either free abelian of rank 2, or isomorphic to M_{-1}, the fundamental group of the Klein bottle (and so has a free abelian subgroup of rank 2 and index 2). In either case the result follows from lemma **??**.

Lemma 3.4 *Let G be given by a presentation with at most 4 generators, in which every relator has length at most 3. Then $\widehat{G}$ is locally indicable.*

Proof. The result is immediate from Lemmas **??** and **??** if there are fewer than four generators, so suppose there are precisely four generators, x_1, x_2, x_3, x_4 say. We may assume that each generator occurs at least twice in relators, so there at least three relators. We may also assume that the relators are pairwise inequivalent.

Suppose first that some pair of relators, say r_1, r_2, involves only three generators, say x_1, x_2, x_3, and consider the relations that involve x_4. If some relation contains a single occurrence of x_4, then G is a homomorphic image of $\langle x_1, x_2, x_3 \mid r_1, r_2\rangle$, and so $\widehat{G}$ is locally indicable, by Lemma **??**. Hence any relator involving x_4 can be assumed to be of the form $x_4^2 x_i^{\mp 1}$ for some $i \leq 3$. If two such relators occur, we may combine them to form a relator of length 2, which can be eliminated to obtain a 3-generator presentation, and so again $\widehat{G}$ is locally indicable. Hence we may assume that only one such relator occurs. Then $\widehat{G}$ is a one-relator extension of $\widehat{H}$, where H is the subgroup of G generated by x_1, x_2, x_3. Since $\widehat{H}$ is locally indicable, so is $\widehat{G}$.

Hence we may suppose that any two relators involve, between them, all four generators. In particular there are at most four relators, and at least two of them involve three generators each. Suppose that r_1 involves x_1, x_2, x_3 and r_2 involves x_2, x_3, x_4. Then any other relator involves both x_1, x_4, and if there are two other relators then each also involves one of x_2, x_3. Using r_1, r_2 to rewrite x_1, x_4 in terms of x_2, x_3, we obtain a 2-generator presentation for G which either has a single relator (of length at most 6), or two relators, each of length 5. The result follows by Lemma **??** and Theorem **??**.

For the rest of this section we assume that our presentation has at least 5 generators, and either 4 or 5 relators, each of length 3. We also assume that each generator occurs at least twice in the relators (so there are at most 7 generators). If some generator (say x_1) occurs only in one relator r_1, then G is a one-relator extension of the group $G' = \langle x_2, \ldots \mid r_1 \ldots \rangle$. If we assume inductively that $\widehat{G'}$ is locally indicable, then so is $\widehat{G}$. Hence we may in fact assume that every generator occurs in at least two distinct relators.

Our next method of attack is to try to merge relators to obtain a presentation with fewer relators. This can readily be done when a generator occurs in only two relators. Suppose for example that x_1 occurs in r_1 and r_2. If x_1 occurs twice in each, then we have $r_1 = x_1^2 x_a^s$, $r_2 = x_1^2 x_b^t$ with $s, t = \pm 1$. We can then replace r_2 by the shorter relator $x_a^s x_b^{-t}$. Arguing inductively once again, we may assume that this does not happen. Assume then that x_1 occurs only once in r_2. We may then remove x_1 and r_2 from the presentation, at the expense of replacing r_1 by a relator of length 4 (if x_1 occurred once in r_1) or 5 (if it occurred twice).

To organize this approach, we encode the information concerning generators occurring in only two relators in the form of a graph Γ with a partial orientation. The vertices of Γ are the relators r_i, the edges are those generators that occur only in two relators. Thus if a generator x_1 occurs in r_1 and r_2, then there will be an edge labelled x_1 joining r_1 to r_2. An edge is oriented towards any relator in which the corresponding generator occurs twice. By the above remarks, this makes sense in that no edge is simultaneously oriented in both directions. However an edge has no orientation if the corresponding generator occurs once only in each of two relators. Note also that no vertex of Γ has more than three incident edges, and if one incident edge is oriented towards the vertex, there is at most one other incident edge, which cannot be oriented towards the vertex. We call a path in Γ **semi-directed** if all the directed edges in it are oriented in the direction of the path.

Lemma 3.5 *If Γ has a semi-directed cycle, then $\widehat{G}$ is locally indicable.*

Proof. Suppose , has a semi-directed cycle of length k. Note that $k \geq 3$. Without loss of generality we may assume that this cycle involves generators $x_1, \ldots, x_k$, and that x_1 joins r_1 to r_2, and so on. Let $H = \langle x_{k+1}, \ldots \mid r_{k+1} \ldots \rangle$. Since $k \geq 3$ and the original presentation of G has at most 7 generators, this presentation for H has at most 4 generators. Hence $\widehat{H}$ is locally indicable, by Lemma **??**. But then we may use $r_2, \ldots, r_k$ to express $x_2, \ldots, x_k$ in terms of $x_1, x_{k+1}, \ldots$, so G is a one-relator extension of H, whence $\widehat{G}$ is locally indicable.

Lemma 3.6 *If , has a cycle, then* $\widehat{G}$ *is locally indicable.*

Proof. By Lemma **??** we may assume that this cycle is not semi-directed. Assuming the cycle is as small as possible, it has length at most 5 (since , has at most 5 vertices). Since no two oriented edges have the same terminal vertex, such a cycle must consist of two semi-directed paths with the same initial and terminal vertices. (Otherwise, there are at least four changes of direction of oriented edges as we travel around the cycle. Each time we pass from a positively oriented edge to a negatively oriented edge, we must cross an oriented edge between them, so the total number of edges in the cycle would be at least 6.) Suppose the first path consists of edges x_1 joining r_1 to r_2, $\ldots$, and x_{k-1} joining r_{k-1} to r_k; while the second consists of edges x_k joining r_1 to r_{k+1}, $\ldots$, and x_m joining r_m to r_k. Let $H = \langle x_{m+1}, \ldots \mid r_{m+1}, \ldots \rangle$. Since there are at least three edges in our cycle, we have $m \geq 3$. Since at least two of the edges are oriented (because otherwise the cycle would be semi-directed), we can deduce that at least two of the generators occur more than twice in relators. Since the total number of occurrences of generators in relators is at most 15, and every generator occurs at least twice, it follows that there are at most 6 generators. Hence H has at most three generators, and at most one more generator than relator. By Lemma **??** every torsion-free homomorphic image of H is locally indicable.

Without loss of generality, we may assume that the edge x_{k-1} is oriented, for otherwise we could consider instead the semi-directed paths $(r_1, r_2, \ldots, r_{k-1})$ and $(r_1, r_{k+1}, \ldots, r_m, r_k, r_{k-1})$. Hence r_k contains precisely two occurrences of x_{k-1} and one of x_m. Now we can use the relators $r_2, \ldots, r_{k-1}$ to rewrite $x_2, \ldots, x_{k-1}$ as words in $Y = \{x_1, x_{m+1}, \ldots\}$; and the relators $r_k, \ldots, r_m$ to write $x_k, \ldots, x_m$ in terms of Y. The group G is then a one-relator product of H and $\langle x_1 \rangle$ with relator (a rewritten form of) r_1. If r_1 is conjugate (in $H * \langle x_1 \rangle$) to an element of H, then G is a free product of an infinite cyclic group and a homomorphic image of H. Otherwise G is a one-relator extension of H. Since every torsion free homomorphic image of H is locally indicable, it follows that $\widehat{G}$ is locally indicable.

Theorem 3.7 *If G has more generators than relators, then $\widehat{G}$ is locally indicable.*

Proof. If d is the deficiency of the presentation, then at least $3d$ generators occur only twice in the relators, so , has at least $3d$ unoriented edges. By Lemma **??** we may assume that , is a forest, and since , has at most five vertices, we must have $1 \leq d \leq \frac{5-1}{3}$, so $d = 1$. If there are four relators, then there are five relators, of total length 15. Hence precisely three generators occur only twice, in other words , has precisely three edges. Hence , is a tree consisting of three unoriented edges (corresponding to generators x_3, x_4, x_5, say). We may use three of the relators to write x_3, x_4, x_5 in terms of x_1, x_2. Rewriting the fourth relation gives a 2-generator, 1-relator presentation for G, and the result follows from Theorem **??**.

Similarly, if there are five relators, then , has either three or four edges, at least three of which are unoriented. Using the relators corresponding to any three unoriented edges to write the corresponding generators in terms of the others, we obtain a 3-generator, 2-relator presentation for G in which the sum of the relator lengths is 9. The result follows from Theorem **??**.

We are now reduced to the case of a five-generator, five-relator presentation

$$G = \langle x_1, \ldots, x_5 | \, r_1, \ldots, r_5 \rangle.$$

We will use this notation consistently from now on. We continue to analyse the structure of the graph , .

Lemma 3.8 *If more than one edge of , is incident at a vertex r_1, then $\widehat{G}$ is locally indicable.*

Proof. Assume that x_4, x_5 are edges joining r_1 to r_4, r_5 respectively. Then r_2, r_3 are words in x_1, x_2, x_3. The group $H = \langle x_1, x_2, x_3 | \, r_2, r_3 \rangle$ has the property that each of its torsion free homomorphic images is locally indicable, by Lemma **??**. Hence it would suffice to show that G is a homomorphic image of H.

Suppose that one of the edges concerned, say x_5, is not oriented away from r_1. Then x_5 occurs only once in r_5, so r_5 can be used to write x_5 as a word in x_1, x_2, x_3. Then at least one of r_1, r_4 can be used to write x_4 as a word in x_1, x_2, x_3, and G is a homomorphic image of H, as required.

If both edges are oriented away from r_1, then without loss of generality $r_4 = x_4^2 x_1$, $r_5 = x_5^2 x_2$ and $r_1 = x_4 x_5 x_a^t$ for some $a \in \{1, 2, 3\}$ and $t = \pm 1$. Since $x_4^2, x_5^2, x_4 x_5$ generate a subgroup of index 2 in the free group $\langle x_4, x_5 \rangle$, it follows that G contains some homomorphic image of H as a subgroup of index at most 2. By Lemma **??** G has a free abelian subgroup of finite index and of rank at most 2, so every homomorphic image of G is locally indicable.

We can now assume that no component of , contains more than one edge. Since , has precisely five vertices, it can have at most two edges.

Lemma 3.9 *If , has more than one edge, then* $\widehat{G}$ *is locally indicable.*

By Lemma **??** we may assume that , has precisely two edges, say x_1 joining r_1 and r_3, x_2 joining r_2 and r_4. Suppose first that x_1 is oriented towards r_3, and x_2 towards r_4. Then each generator occurs exactly three times in the relators, so any relator involving two occurrences of some generator has to be the terminal vertex of an oriented edge of , . In particular r_5 must involve all three of x_3, x_4, x_5, and one of these three generators, say x_5, occurs in each of r_1, r_2.

If r_3 and r_4 have a common generator (say x_3), then we may eliminate x_3 from r_3 and r_4 to obtain a relator $x_1^2 x_2^{\pm 2}$. We may also use r_1 and r_2 to write x_2 and x_4 in terms of x_1 and x_5. These leaves us with a 2-generator presentation with two relators of length 5 and (at most) 8. By Theorem **??**, $\widehat{G}$ is locally indicable.

If one of x_3, x_4, say x_3, occurs in r_1 and r_3, then we may proceed as follows. Use r_1, r_2 to write x_1, x_2 in terms of x_3, x_4, x_5, replacing r_3 and r_4 by words of length 5, such that r_3 involves only x_3, x_5, and r_4 involves precisely three occurrences of x_4. Now use r_5 to write x_4 as a word of length 2 in x_3, x_5. Then $G = \langle x_3, x_5 | \ r_3, r_4 \rangle$ is a 2-relator presentation in which the relator r_3 has length 5 and the relator r_4 has length 8 (as a word in x_3, x_5). It follows from Theorem **??** that $\widehat{G}$ is locally indicable.

Hence suppose that x_3 occurs in r_2 and r_3, while x_4 occurs in r_1 and r_4. Without loss of generality we have $r_1 = x_1^{\alpha} x_4^{\beta} x_5^{\gamma}$, $r_2 = x_2^{\delta} x_3^{\epsilon} x_5^{\zeta}$ and $r_5 = x_3 x_4 x_5$ for some $\alpha, \beta, \gamma, \delta, \epsilon, \zeta = \pm 1$. We can use r_3 and r_4 to replace x_3, x_4 by x_1^{-2}, x_2^{-2} respectively. If $\gamma = 1$ then we can use r_5 to rewrite r_1 as $x_1^{2+\alpha} x_2^{2-2\beta}$, and $\widehat{G}$ is cyclic except possibly if $\alpha = 1 = -\beta$. But in this case G is a central extension of a one-relator product of $\mathbb{Z}_3 * \mathbb{Z}_4$ in which the relator has free product length 2 or 4. Since any such group is finite, so is G and we are finished.

Similar arguments hold if $\zeta = -1$ (using r_5 to rewrite r_2) or if $\zeta = -\gamma$ (using r_2 to rewrite r_1). In all cases $\widehat{G}$ is locally indicable, as required.

Secondly, suppose that x_1 is oriented towards r_3, and that x_2 is unoriented. Now every generator that occurs in r_5 occurs in three distinct relators. At least one such generator, x_5 say, occurs precisely once in each of three distinct relators. Now use r_1, r_2, r_5 to write x_1, x_2, x_5 in terms of x_3, x_4. Rewriting r_3, r_4 as words in x_3, x_4, we get two relators of lengths 4 and 8 (if x_5 occurs in r_1 and r_3) or 5 and at most 7 (otherwise). The result then follows from Theorems **??** and **??**.

Finally, suppose that neither edge is oriented. Using r_1, r_2 to eliminate x_1, x_2 as above, we obtain a 3-generator, 3-relator presentation in which the relators have lengths 4,4,3 respectively. Hence one generator (x_3 say) occurs at most (hence precisely) three times. In particular there is a relator containing precisely one occurrence of x_3. Using that relator to eliminate r_3, we obtain a 2-relator presentation in which either one relator has length at most 4, or the relator lengths are 5 and (at most) 6. By Theorems **??** and **??** again the result follows.

Theorem 3.10 *If* $G = \langle x_1, \dots, x_5 | \; r_1, \dots, r_5 \rangle$ *where each* r_i *has length 3, and each of* $r_1, \dots, r_4$ *involves three distinct generators, then* $\widehat{G}$ *is locally indicable.*

Proof. Firstly, suppose that some generator, say x_5, occurs only once in $r_1, \dots, r_4$ - say in r_4. In particular, G is a homomorphic image of $G_1 = \langle x_1, \dots x_4 | \; r_1, r_2, r_3 \rangle$.

If in addition some other generator (say x_4) occurs at most once in r_1, r_2, r_3 (say in r_3), then either G_1 is isomorphic to

$$G_2 = \langle x_1, x_2, x_3 | \; r_1, r_2 \rangle$$

(if x_4 occurs once), or G_1 is a free product of a homomorphic image of G_2 with an infinite cyclic group. Thus G is either a homomorphic image of G_2 or a one-relator extension of such a homomorphic image. By Lemma **??** all torsion-free homomorphic images of G_2 are locally indicable, and it follows that $\widehat{G}$ is locally indicable.

Suppose then that x_5 occurs precisely once in r_4 and not at all in r_1, r_2, r_3, while each of $x_1, \dots, x_4$ occurs at least twice in r_1, r_2, r_3. Then one generator (say x_4) occurs in all three of r_1, r_2, r_3, while each of x_1, x_2, x_3 occurs in precisely two of r_1, r_2, r_3. Without loss of generality x_i occurs in r_j (for $i, j \in \{1, 2, 3\}$) if and only if $i \neq j$.

Now r_4 involves x_5 and precisely two of $x_1, \dots, x_4$. Without loss of generality x_1 occurs in r_4. We can use r_2, r_3 to write each of x_3, x_2 respectively as a word of length 2 in x_1, x_4. This allows us to rewrite r_1 as a word of length 5 in x_1, x_4. Use r_4 to write x_5 as a word of length 2 in x_1, x_2, x_3, x_4 that definitely involves x_1, and hence as a word of length at most 3 in x_1, x_4. Finally, r_5 involves x_5 at most twice, so can be rewritten as a word of length at most 8 in x_1, x_4. Thus G has a 2-relator presentation with one relator of length 5 and the other of length at most 8, so $\widehat{G}$ is locally indicable, by Theorem **??**.

Secondly, suppose that each generator occurs in at least two of $r_1, \dots, r_4$; and that the generator x_5 occurs in all four of them. Then each of $x_1, \ldots, x_4$ occurs precisely twice in $r_1, \ldots, r_4$. By Lemma **??** we may assume that , has at most one edge, so there is at most one generator that occurs only twice in $r_1, \ldots, r_5$. Hence we may assume that r_5 involves three of $x_1, \ldots, x_4$. Assume that r_5 involves x_2, x_3, x_4. Then one relator (say r_1) involves x_1. But we are now in the same circumstances as in the first case of the proof, for each of $r_2, \ldots, r_5$ involves three distinct generators, and the generator x_1 is involved only once in $r_2, \ldots, r_5$. As before, $\widehat{G}$ is locally indicable.

Finally, let us suppose that each of x_1, x_2, x_3 occurs twice in the relators $r_1, \ldots, r_4$, whilst each of x_4, x_5 occurs three times. Since each of $r_1, \ldots, r_4$ involves at least one of x_1, x_2, x_3 there are essentially only three possibilities (up to re-numbering):

(i) r_1, r_2 involve both x_1, x_2; r_3, r_4 involve x_3;

(ii) x_i occurs in r_i and r_4 $(i = 1, 2, 3)$;

(iii) x_i occurs in r_i and r_{i+1} $(i = 1, 2, 3)$.

We treat each of these cases separately. Note that at least two of x_1, x_2, x_3 occur in r_5. If also x_4 or x_5 occurs in r_5, then r_5 also involves three distinct generators, and some generator (say x_1) occurs only twice. We may then argue as in the first part of the proof to show that $\widehat{G}$ is locally indicable. Hence we will assume for the remainder of the proof that r_5 is a word in x_1, x_2, x_3.

Case (i). Note that G is generated by x_1 and x_2. If r_5 involves only x_1, x_2, then G is cyclic, and the result follows. If x_1, x_2, x_3 each occur in r_5, then G contains as a subgroup of index at most 2 some homomorphic image of $H = \langle x_3, x_4, x_5 | \ r_3, r_4 \rangle$. The result then follows from Lemmas **??** and **??**.

Assume then that only x_1 and x_3 occur in r_5. If $r_5 = x_1 x_3^2$ then we use r_5, r_4 and r_2 to write x_1, x_5 and x_2 as words in x_3, x_4. Then r_3 becomes a relator of length 4 and r_1 a second relator. The result follows from Theorem **??**. Finally, suppose that $r_5 = x_1^2 x_3$. Using r_2, r_3, r_5 to eliminate x_2, x_3, x_5, we obtain a two generator presentation with generators x_1, x_4 and two relators r_1, r_4, where r_1 has one of the forms $x_1^4x_4^{\pm 2}$ or $x_1^3x_4^{\pm 1}x_1^{\pm 1}x_4^{\pm 1}$; and r_4 has one of the forms $x_1^4x_4^{\pm 2}$ or $x_1^2x_4^{\pm 1}x_1^{\pm 2}x_4^{\pm 1}$. A case-by-case analysis verifies that $\widehat{G}$ is locally indicable in all cases.

Case (ii). Note that x_4, x_5 occur in each of r_1, r_2, r_3, so we can use these relators to write x_1, x_2, x_3 as words in x_4, x_5, each of which contains one occurrence each of x_4 and of x_5. Moreover, these words and their inverses are mutually distinct (for otherwise we could combine two of the relators to obtain an identity $x_i = x_j^{\pm 1}$ for some $i, j \in \{1, 2, 3\}$).

In particular G is generated by x_4, x_5, and the subgroup generated by x_1, x_2, x_3 has index at most 2. Hence G has a subgroup of finite index that is a homomorphic image of $\langle x_1, x_2, x_3 |\, r_4, r_5\rangle$. Hence $\widehat{G}$ is locally indicable, by Lemmas **??** and **??**.

Case (iii). Suppose first that x_3 does not occur in r_5. Using r_1, r_2, r_4 to write x_3, x_4, x_5 in terms of x_1, x_2, we see that G is generated by x_1, x_2, and hence cyclic, since r_5 is a word in x_1, x_2. Similarly G is cyclic if x_1 does not occur in r_5, so we may assume that both x_1, x_3 occur in r_5.

If x_2 does not occur in r_5, then there is no loss of generality in assuming that $r_5 = x_1^2x_3$. Use r_1, r_2, r_4 to write x_1, x_2, x_3 in terms of x_4, x_5. Then r_3, r_5 each become words of length 6 in x_4, x_5, and moreover each contains precisely 3 occurrences each of x_4, x_5. We may also assume that these words are cyclically reduced, or else the result follows from Theorem **??**. Without loss of generality r_1 has the form $x_1^{-1}x_4x_5$, so the rewrite of r_5 has one of the forms $x_4x_5x_4x_5x_4^{\alpha}x_5^{\beta}$ or $x_4x_5x_4x_5^2x_4$. In the latter case G is generated by $x_1 = x_4x_5$ and x_4, which satisfy $x_1^2x_4^{-1}x_1x_4$, and the result follows from Lemma **??**. In the former case, if $\alpha = \beta = 1$ then $x_4x_5 = 1$ in $\widehat{G}$, so $\widehat{G}$ is cyclic. If $\alpha = -\beta$ then G has a presentation $\langle a, b |\, a^3 = b^2, w(a, b) = 1\rangle$, where w contains at most 3 occurrences of a. In particular G is a finite extension of its central subgroup $\langle b^2\rangle$, so finite.

We are thus reduced to the case where $r_5 = x_4x_5x_4x_5x_4^{-1}x_5^{-1}$. Recall that r_3 also rewrites to a word of length 6 involving exactly three occurrences of each of x_4, x_5. Considering all possible such words, and performing coset enumerations, we see that G is finite except for those cases where G has infinite abelianisation (in other words, where the exponent sums of x_4 and x_5 in r_3 are equal). But in those cases we can verify by Reidemeister-Schreier rewriting that the commutator subgroup of G is cyclic, and hence $\widehat{G}$ is locally indicable, as required.

Finally, suppose that each of x_1, x_2, x_3 occurs in r_5. Then, after replacing some generators and/or relators by their inverses if necessary, and possibly interchanging x_4 and x_5, we have $r_1 = x_1 x_4^{a(1)} x_5^{b(1)}$, $r_2 = x_2 x_5^{a(2)} x_1^{b(2)}$, $r_5 = x_3 x_1^{a(3)} x_2^{b(3)}$, $r_3 = x_4 x_2^{a(4)} x_3^{b(4)}$, and $r_4 = x_5 x_3^{a(5)} x_4^{b(5)}$, where $a(i), b(i) = \pm 1$ for all i. A computer search through all 2^{10} possible values of the $a(i)$ and $b(i)$, using coset enumeration, verifies that in all cases G is finite.

Proof of Theorem ??. Suppose $G = \langle x_1, \dots \mid r_1, \dots, r_k \rangle$, where $k \leq 5$ and each relator has length (at most) 3. Any relators of length less than 3 may be eliminated (along with a generator in each case) without affecting $\widehat{G}$, so we may assume that all relators have length 3. We may also assume that each generator occurs in at least two relators. By Lemma **??** we may assume that there are no more generators than relators, and by Lemma **??** we may assume that there are at least 5 generators, so we assume that there are precisely five generators and five relators.

If the graph , has more than one edge, then the result follows from Lemma **??**, so assume that , has at most one edge. If some relator involves a generator x_i twice, then either there is an edge labelled x_i oriented towards that relator, or the generator x_i occurs more than three times, in which case to compensate there must be another generator x_j occurring fewer than three times, and hence an unoriented edge x_j in , . Since , has only one edge, there can be at most one relator of this form. In other words there at least 4 relators $r_1, \dots, r_4$ say, each of which involves three distinct generators. The result now follows from Theorem **??**.

Proof of Corollary ??. If some relator has the form $r = x^t$ for some generator x and integer $t \neq 0$, then $x = 1$ in $\widehat{G}$, so we may omit x and r from the presentation, deleting all occurrences of x from other relators as we go. This reduces the number of relators, without changing $\widehat{G}$ or increasing complexity. Without loss of generality, we may assume there are no such relators.

Next, any relator of length 2 has the form $r = x^\alpha y^\beta$ for distinct generators x, y, where $\alpha, \beta \in \{\pm 1\}$. We may remove r and y, replacing every occurrence of y in other relators by $x^{-\alpha\beta}$, without changing G or increasing complexity. Hence we may assume that every relator has length at least 3.

Finally, if $r = x_1^{a(1)} x_2^{a(2)} \dots x_k^{a(k)}$ is a relator with $k \geq 4$, we may introduce $k-3$ new generators $y_4, \dots, y_k$ and replace r by $k-2$ relators $x_1^{a(1)} x_2^{a(2)} y_4$, $y_4^{-1} x_3^{a(3)} y_5$, $\dots$, $y_k^{-1} x_{k-1}^{a(k-1)} x_k^{a(k)}$, without changing G or the complexity. Repeating for all relators, we obtain a presentation in which all relators have length 3. The number of relators is then equal to the complexity, which is at most 5 by hypothesis, so we may apply Theorem **??**.

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