Free subgroups in certain generalized triangle groups of type $(2, m, 2)$

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Abstract

A generalized triangle group is a group that can be presented in the form $G = \langle\, x, y \mid x^p = y^q = w(x, y)^r = 1 \,\rangle$ where $p, q, r \geq 2$ and $w(x, y)$ is an element of the free product $\langle\, x, y \mid x^p = y^q = 1 \,\rangle$ involving both x and y. Rosenberger has conjectured that every generalized triangle group G satisfies the Tits alternative. It is known that the conjecture holds except possibly when the triple (p, q, r) is one of $(3, 3, 2)$, $(3, 4, 2)$, $(3, 5, 2)$, or $(2, m, 2)$ where $m = 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$. In this paper we show that the Tits alternative holds in the cases $(p, q, r) = (2, m, 2)$ where $m = 6, 10, 12, 15, 20, 30, 60$.

1 Introduction

A *generalized triangle group* is a group that can be presented in the form

$$G = \langle\, x, y \mid x^p = y^q = w(x, y)^r = 1 \,\rangle$$

where $p, q, r \geq 2$ and $w(x, y)$ is an element of the free product $\langle\, x, y \mid x^p = y^q = 1 \,\rangle$ involving both x and y that is not a proper power. It was conjectured by Rosenberger [11] that every generalized triangle group G satisfies the Tits alternative. That is, G either contains a non-abelian free subgroup or has a soluble subgroup of finite index.

It is now known that the Tits alternative holds for a generalized triangle group G except possibly when the triple (p, q, r) is one of $(3, 3, 2)$, $(3, 4, 2)$, $(3, 5, 2)$, or $(2, m, 2)$ where $m \geq 3$. (See [6] for a survey of these results.) In recent work Benyash-Krivets [3] considers the case $(2, m, 2)$. He has shown that if $m \geq 7$, $m \neq 10, 12, 15, 20, 30, 60$ then the Tits alternative holds for G. In this paper we augment that result to prove the following:

Main Theorem. *Let* $G = \langle\, x, y \mid x^2 = y^m = w(x, y)^2 = 1 \,\rangle$ *where* $w(x, y) = xy^{\alpha_1} \ldots xy^{\alpha_k}$, $1 \leq \alpha_i < m$, $m \geq 6$. *Then the Tits alternative holds for* G.

If $k = 1$ then the Tits alternative holds for G by [5]. If $m = 6$ and $k = 2$ or 3 then the Tits alternative holds for G by [11, 10] respectively. The Main Theorem then follows from Theorems 1, 2 and 3:

Theorem 1 *Let G be as defined in the Main Theorem. If $m = 6$ and $k > 3$, then G contains a non-abelian free subgroup.*

Theorem 2 *Let G be as defined in the Main Theorem. If $m = 5p$ where $p \neq 5$ is prime and $k > 1$, then G contains a non-abelian free subgroup.*

Theorem 3 *Let G be as defined in the Main Theorem. If $k > 1$ and $m = 12, 20, 30$, or 60 then G contains a non-abelian free subgroup.*

Theorem 1 has independently been obtained by Barkovich and Benyash-Krivets [1], and for this reason we do not give a complete proof. However, we require Theorem 1 in an essential way in the proofs of the other results, so in order to make our paper self-contained we have included a sketch proof in an Appendix.

2 Preliminaries

We first recall some definitions and well-known facts concerning generalized triangle groups; further details are available in (for example) [6]. Let G be as defined in the Main Theorem. A representation $\rho : G \to H$ (for some group H) is said to be *essential* if $\rho(x), \rho(y), \rho(w)$ are of orders $2, m, 2$ respectively. By [2] G admits an essential representation into $PSL(2, \mathbb{C})$.

A projective matrix $A \in PSL(2, \mathbb{C})$ is of order n if and only if $\mathrm{tr}(A) = 2\cos(q\pi/n)$ for some $(q, n) = 1$. Note that in $PSL(2, \mathbb{C})$ traces are only defined up to sign. A subgroup of $PSL(2, \mathbb{C})$ is said to be *elementary* if it has a soluble subgroup of finite index, and is said to be *non-elementary* otherwise.

Let $X, Y \in PSL(2, \mathbb{C})$ satisfy $\mathrm{tr}(X) = 0,\ \mathrm{tr}(Y) = 2\cos(\pi/m)$ (so that X, Y have orders 2 and m respectively) and let $\rho : \langle\, x, y \mid x^2 = y^m = 1 \,\rangle \to PSL(2, \mathbb{C})$ be given by $x \mapsto X,\ y \mapsto Y$. Now $\mathrm{tr}\rho(w)$ is a polynomial in $\lambda = \mathrm{tr}XY$ of degree k. We call this the *trace polynomial* of G and write $\tau = \tau(\lambda)$. The representation ρ induces an essential representation $G \to PSL(2, \mathbb{C})$ if and only if $\mathrm{tr}\rho(w) = 0$; that is, if and only if λ is a root of τ.

Now if X, Y generate a non-elementary subgroup of $PSL(2, \mathbb{C})$ then $\rho(G)$ (and hence G) contains a non-abelian free subgroup. Thus in proving that G contains a non-abelian free subgroup we may assume that X, Y generate an elementary subgroup of $PSL(2, \mathbb{C})$. By Corollary 2.4 of [11] there are then three possibilities: (i) X, Y generate a finite subgroup of $PSL(2, \mathbb{C})$; (ii) $\mathrm{tr}[X, Y] = 2$; or (iii) $\mathrm{tr}XY = 0$. Manipulation using trace identities shows that (ii) is equivalent to $\mathrm{tr}XY = \pm 2\sin(\pi/m)$; in this case G also admits an essential cyclic representation. In (iii) ρ is an essential dihedral representation of G. Moreover, $\tau(\lambda) = \pm\tau(-\lambda)$ so roots $\lambda, -\lambda$ occur with equal multiplicity. The leading coefficient of τ is given by

$$c = \frac{1}{(\sin(\pi/m))^k} \prod_{i=1}^{k} \sin\left(\frac{\pi\alpha_i}{m}\right).$$

For each $1 \leq j \leq m/2$ we shall let $t_j = \sin(j\pi/m)$ and let k_j denote the number of times $\alpha_i = j$ or $(m-j)$ in the word $w(x,y)$ (so that $k = k_1 + \ldots + k_{\lfloor m/2 \rfloor}$). The above formula then becomes $c = (t_1^{k_1} \ldots t_{\lfloor m/2 \rfloor}^{k_{\lfloor m/2 \rfloor}})/(\sin(\pi/m)^k)$. The other coefficients of $\tau(\lambda)$ can also be calculated – see Lemma 9 in the Appendix.

3 The case $m = 4$

Lemma 4 *Let $G = \langle\ x, y \mid x^2 = y^4 = (xy^{\alpha_1} \ldots xy^{\alpha_k})^2 = 1\ \rangle$ and let k_2 denote the number of values of i for which $\alpha_i = 2$. Then G contains a non-abelian free subgroup unless one of the following holds:*

1. *k is odd and one of the following holds:*
   (a) $\sum_{i=1}^{k} \alpha_i = 0$ *mod* 4;
   (b) $\sum_{i=1}^{k} \alpha_i = 2$ *mod* 4 *and* $k_2 = 1$;
   (c) $\sum_{i=1}^{k} \alpha_i = 1, 3$ *mod* 4 *and* $k_2 = 0$;
2. *k is even and one of the following holds:*
   (a) $\sum_{i=1}^{k} \alpha_i = 2$ *mod* 4;
   (b) $\sum_{i=1}^{k} \alpha_i = 0$ *mod* 4 *and either*
      (i). $k_2 = 0$ *and* $k = 2$ *mod* 4; *or*
      (ii). $k_2 = 2$;
   (c) $\sum_{i=1}^{k} \alpha_i = 1, 3$ *mod* 4 *and* $k_2 = 1$.

Proof
The only finite subgroups of $PSL(2,\mathbb{C})$ generated by projective matrices X, Y of orders $2, 4$ respectively are $\mathbb{Z}_4, D_8$, and S_4. In these groups the product XY has orders $4, 2, 3$ respectively. If τ denotes the trace polynomial for G we may therefore assume that the roots of τ are $\pm\sqrt{2}, 0, \pm 1$. Thus

$$\tau(\lambda) = c\lambda^s(\lambda^2 - 1)^t(\lambda^2 - 2)^u$$

where

$$c = \frac{1}{(\sin(\pi/4))^k}(\sin(\pi/4))^{k_1}(\sin(2\pi/4))^{k_2} = \sqrt{2}^{k_2},$$

and where k_1, k_2 denote the number of times α_i takes the values $\pm 1, 2$ respectively. Let

$$A = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}, \quad B = \begin{pmatrix} (1+i)/\sqrt{2} & z \\ 0 & (1-i)/\sqrt{2} \end{pmatrix}$$

be elements of $PSL(2,\mathbb{C})$ so that $\mathrm{tr}A = 0,\ \mathrm{tr}B = \sqrt{2},\ \mathrm{tr}AB = z - \sqrt{2}$.

Consider the representation $\rho : \langle\ x, y \mid x^2 = y^4 = 1\ \rangle \rightarrow PSL(2, \mathbb{C})$ given by $x \mapsto A,\ y \mapsto B$ then

$$\begin{aligned} \mathrm{tr}\rho(xy^{\alpha_1} \ldots xy^{\alpha_k}) &= \tau(z - \sqrt{2}) \\ &= \pm(\sqrt{2})^{k_2}(z-\sqrt{2})^s(z^2 - 2\sqrt{2}z + 1)^t(z - 2\sqrt{2})^u z^u \end{aligned}$$

whose constant term is 0 if $u > 0$, and $\pm(\sqrt{2})^{k_2+s}$ if $u = 0$. Now the constant term in $\mathrm{tr}(AB^{\alpha_1} \ldots AB^{\alpha_k})$ is given by $2\cos((2k + \sum_{i=1}^k \alpha_i)\pi/4)$. If $u > 0$ we have that $2k + \sum_{i=1}^k \alpha_i = 2 \bmod 4$, and one of the conclusions 1(a) or 2(a) holds. Thus we may assume $u = 0$, and therefore $k_2 + s = 1$ or 2.

Suppose k is odd. Then s is odd. Since $2k + \sum_{i=1}^k \alpha_i \neq 2 \bmod 4$ we have $\sum_{i=1}^k \alpha_i = 1, 2,$ or $3 \bmod 4$. If $\sum_{i=1}^k \alpha_i = 2 \bmod 4$ then k_2 is odd so $k_2 = 1$, $s = 1$ and we are in case 1(b). If $\sum_{i=1}^k \alpha_i = 1, 3 \bmod 4$ then k_2 is even so $k_2 = 0$, $s = 1$ and we are in case 1(c).

Suppose k is even. Then s is even. Since $2k + \sum_{i=1}^k \alpha_i \neq 2 \bmod 4$ we have $\sum_{i=1}^k \alpha_i = 0, 1,$ or $3 \bmod 4$. If $\sum_{i=1}^k = 1$ or $3 \bmod 4$ then k_2 is odd so $k_2 = 1, s = 0$ and we are in case 2(c). If $\sum_{i=1}^k \alpha_i = 0 \bmod 4$ then k_2 is even so either $k_2 = 0, s = 2$ or $k_2 = 2, s = 0$. In the latter option we are in case 2(b)(ii). In the former 0 is a root of $\tau(\lambda)$ so G admits an essential dihedral representation. Thus $\sum_{i=1}^k (-1)^i \alpha_i = 2 \bmod 4$. Combining this with $\sum_{i=1}^k \alpha_i = 0$ and the fact that each α_i is odd, we obtain $k = 2 \bmod 4$ and we are in case 2(b)(i). □

4 The cases $m = 10, 15$

In this section we consider the following situation. Let G be as defined in the Main Theorem where $m = 5p$ for some prime p.

If k is even and $p = 2$ then G contains a non-abelian free subgroup by [12, Theorem A]. If k is even and $p > 2$ then G admits no essential cyclic or dihedral representations and G contains a non-abelian free subgroup. Thus we may assume that k is odd. We do so throughout this section without further comment.

Now G maps homomorphically onto the group

$$\overline{G} = \langle\ x, y \mid x^2 = y^5 = \overline{w}(x, y)^2 = 1\ \rangle \qquad (1)$$

where $\overline{w} \in \langle\ x, y \mid x^2 = y^5 = 1\ \rangle$ is given by $\overline{w} = xy^{\beta_1} \ldots xy^{\beta_k}$ where $\beta_i = \alpha_i \bmod 5$ $(1 \leq i \leq k)$. Now $\overline{w} \neq y^\beta$ for any β, since k is odd. If $\overline{w} = x$ then $\overline{G} \cong \mathbb{Z}_2 * \mathbb{Z}_5$ and so $\overline{G}$, and hence G, contains a non-abelian free subgroup. If $\overline{w}$ is a proper power then $\overline{G}$, and hence G, contains a non-abelian free subgroup by [2].

Thus we will assume that $\overline{w}$ can be freely reduced to a word of the form $\overline{w} = xy^{\gamma_1} \ldots xy^{\gamma_\ell}$ that is not a proper power, where $1 \leq \gamma_i \leq 4$ $(1 \leq i \leq \ell)$, $\ell \geq 1$. Hence the corresponding presentation (1) is a presentation of $\overline{G}$ as a generalized triangle group. We let $\tau(\lambda)$, $\sigma(\mu)$ denote the trace polynomials of G and $\overline{G}$ respectively.

Lemma 5 *If* 1 *is a repeated root of* $\sigma(\mu)$ *then* G *contains a non-abelian free subgroup.*

Proof

Let $q : G \to \overline{G}$ denote the canonical epimorphism. By hypothesis, there is an essential representation $\rho : \overline{G} \to PSL_2(\mathbb{C}[\mu]/(\mu-1)^2)$. Indeed, we can construct ρ explicitly via:

$$\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(y) = \begin{pmatrix} e^{i\pi/5} & \mu \\ 0 & e^{-i\pi/5} \end{pmatrix}.$$

Composing this with the canonical epimorphism

$$\psi : PSL_2(\mathbb{C}[\mu]/(\mu-1)^2) \to PSL_2(\mathbb{C}[\mu]/(\mu-1)) \cong PSL_2(\mathbb{C})$$

gives an essential representation $\tilde{\rho} = \psi \circ \rho : \overline{G} \to PSL_2(\mathbb{C})$ with image A_5, corresponding to the root 1 of the trace polynomial.

Let $\overline{K}$ denote the kernel of $\tilde{\rho}$, V the kernel of ψ, and K the kernel of the composite map $\tilde{\rho} \circ q : G \to PSL_2(\mathbb{C})$. Then V is a complex vector space, since its elements have the form $\pm(I + (\mu-1)A)$ for various 2×2 matrices A, with multiplication

$$[\pm(I + (\mu-1)A)][\pm(I + (\mu-1)B)] = \pm(I + (\mu-1)(A+B)).$$

Now $\overline{K}$ is generated by conjugates of $c := (xy)^3$. Consider four such conjugates: $c_1 = c$, $c_2 = xcx$, $c_3 = yxy^3cy^2xy^4$, and $c_4 = yxy^4cyxy^4$. Then an easy calculation shows that $\rho(c_i) = -I + (\mu-1)M_i$ where M_1, M_2, M_3, M_4 are linearly independent over $\mathbb{Q}$. The group A_5 acts on V via conjugation and since $\tilde{\rho}(x)$ is of order 2, the action of $\tilde{\rho}(x)$ on V is diagonalizable. Moreover, the only possible eigenvalues are ± 1. Thus V splits as a $\mathbb{Q}$-direct sum $V_+ \oplus V_-$, where $\tilde{\rho}(x)$ acts as the identity on V_+ and as the antipodal map $v \mapsto -v$ on V_-. The canonical projection $V \to V_-$ with kernel V_+ is $\tilde{\rho}(x)$-equivariant.

An easy calculation shows that $\rho(xc_jx) = \rho(c_j^{-1})$ for $j = 3, 4$, while $xc_1x = c_2$ and $xc_2x = c_1$. Thus $\rho(c_1c_2^{-1}), \rho(c_3), \rho(c_4) \in V_-$ and $\rho(c_1c_2) \in V_+$. Let N be the pre-image of V_+ in K. Then N is normal in K and is invariant under conjugation by x. It follows that K/N is free abelian of rank at least 3 and that $\tilde{\rho}(x)$ acts on K/N as the antipodal map.

Note that K is the fundamental group of a 2-dimensional CW-complex X arising from the given presentation of G. This complex X has 60 cells of dimension 0, 120 cells of dimension 1, and $60(\frac{1}{2}+\frac{1}{5}+\frac{1}{2}) = 72$ cells of dimension 2. Here, $60/5 = 12$ of the 2-cells (call them $\alpha_1, \ldots, \alpha_{12}$, say) arise from the relator y^{5p}, $60/2 = 30$ ($\alpha_{13}, \ldots, \alpha_{42}$, say) arise from the relator x^2, and $60/2 = 30$ ($\alpha_{43}, \ldots, \alpha_{72}$, say) arise from the relator $w(x,y)^2$. Moreover, $\alpha_1, \ldots, \alpha_{12}$ are attached by maps which are pth powers. Let $\widehat{X}$ be the regular covering complex of X corresponding to the normal subgroup N of K and let $\widehat{\alpha}_i$ denote a lift of the 2-cell α_i. Then each of $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$ is a 2-cell attached by a map which is a pth power.

Let GF_p denote the field with p elements. Now $H_2(\widehat{X}, GF_p)$ is a subgroup of the 2-chain group $C_2(\widehat{X}, GF_p)$ and since K/N freely permutes the cells of $\widehat{X}$, $C_2(\widehat{X}, GF_p)$ is a free $GF_p(K/N)$-module on the basis $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{72}$. Let Q be the free $GF_p(K/N)$-submodule of $C_2(\widehat{X}, GF_p)$ of rank 12 generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$. Since these 2-cells are attached by maps which are pth powers, their boundaries in the 1-chain group $C_1(\widehat{X}, GF_p)$ are zero. Thus Q is a subgroup of $H_2(\widehat{X}, GF_p)$.

Suppose $Q \neq H_2(\widehat{X}, GF_p)$, and let $\widehat{\beta} \in H_2(\widehat{X}, GF_p)\backslash Q$. Then $\widehat{\beta} = \sum_{i=1}^{72} \mu_i \widehat{\alpha}_i$ where $\mu_i \in GF_p(K/N)$ $(1 \leq i \leq 72)$ and $\mu_q \neq 0$ for some $q > 12$. Let L be the submodule of $H_2(\widehat{X}, GF_p)$ generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}, \widehat{\beta}$. Let $\pi_q : C_2(\widehat{X}, GF_p) \to GF_p(K/N)$ denote the projection map on the basis element $\widehat{\alpha}_q$ and suppose $\lambda, \lambda_1, \ldots, \lambda_{12} \in GF_p(K/N)$ satisfy

$$v := \lambda\widehat{\beta} + \lambda_1\widehat{\alpha}_1 + \ldots + \lambda_{12}\widehat{\alpha}_{12} = 0$$

in $C_2(\widehat{X}, GF_p)$. Then $0 = \pi_q(v) = \lambda\mu_q$, and since $GF_p(K/N)$ is an integral domain we have that $\lambda = 0$ so $\lambda_1\widehat{\alpha}_1 + \ldots + \lambda_{12}\widehat{\alpha}_{12} = 0$ in Q. But $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$ form a $GF_p(K/N)$-basis for Q so $\lambda_1 = \cdots = \lambda_{12} = 0$ and hence L is free on $\{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}, \widehat{\beta}\}$. Thus $H_2(\widehat{X}, GF_p)$ contains a free $GF_p(K/N)$-submodule of rank $13 = 1 + \chi(X)$ so by [9, Proposition 2.1 and Theorem 2.2], $K = \pi_1(X)$ contains a non-abelian free subgroup.

Suppose then that $H_2(\widehat{X}, GF_p) = Q$. We argue as in the proof of [9, Corollary 3.2]. The element $c_1c_2 \in N$ is mapped to the element $-I + (\mu - 1)(M_1 + M_2)$ of infinite order in V_+ so N^{ab} has torsion-free rank at least 1. Thus $H_1(\widehat{X}, GF_p) \cong N^{ab}/pN^{ab} \neq 0$. We also have that $H_2(\widehat{X}, GF_p)$ is a free $GF_p(K/N)$-module and K/N is a free abelian group of rank at least 3, so by [9, Theorem D] there is a subgroup J/N of K/N such that $(K/N)/(J/N) \cong K/J \cong \mathbb{Z}^2$ and $H_1(\widehat{X}, GF_p)$ contains a non-zero free $GF_p(J/N)$-submodule. Moreover, J/N is infinite so this module is of infinite GF_p-dimension.

Thus, by definition, the Bieri-Strebel invariant ([4]) Σ of the $GF_p(K/N)$-module $H_1(\widehat{X}, GF_p)$ is a proper subset of the sphere S^{d-1} (where d is the rank of the free abelian group K/N). But $\Sigma = -\Sigma$, since $\tilde{\rho}(x)$ acts as the antipodal map on K/N. Hence $\Sigma \cup -\Sigma \neq S^{d-1}$, and so N has a non-abelian free subgroup by [4, Theorem 4.1]. □

Lemma 6 *If $\overline{G}$ has an essential cyclic representation then G contains a non-abelian free subgroup.*

Proof

If $\overline{G}$ has an essential cyclic representation, then it also has an essential representation into $PSL_2(\mathbb{C})$ whose image is an extension of $\mathbb{Z}^4$ by $\mathbb{Z}_{10}$. Indeed, one can construct such a representation ρ explicitly by setting

$$\rho(x) = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, \qquad \rho(y) = \begin{pmatrix} e^{i\pi/5} & 0 \\ 0 & e^{-i\pi/5} \end{pmatrix},$$

and noting that the elements $\rho(y^t[x,y]y^{-t})$ $(t = 0, 1, 2, 3)$ commute and have upper right entries that are linearly independent over $\mathbb{Q}$.

Note also that conjugation by $\rho(x)$ inverts any element of $\rho([\overline{G}, \overline{G}])$. Hence G has normal subgroups $N \subset K$ with $G/K \cong \mathbb{Z}_{10}$, $K/N \cong \mathbb{Z}^4$, and $\rho(x)$ acts as the antipodal map on K/N. Moreover, K is the fundamental group of a 2-dimensional CW-complex with 10 0-cells, 20 1-cells and 12 2-cells, 2 of which correspond to the relator y^{5p}, and so are attached by pth powers. The argument given in the proof of Lemma 5 shows that K has a non-abelian free subgroup. □

Lemma 7 *Suppose that ℓ is odd and that $\overline{G}$ admits no essential cyclic representation. If 0 is a repeated root of $\sigma(\mu)$ then $\overline{G}$ (and hence G) contains a non-abelian free subgroup.*

Proof

Let $X, Y \in PSL(2,\mathbb{C})$ be elements of orders 2 and 5 respectively. We may assume $\mathrm{tr}X = 0$, $\mathrm{tr}Y = \eta$ where $\eta = 2\cos(\pi/5)$. (Note that $\eta^4 - 3\eta^2 + 1 = 0$.) The only finite subgroups of $PSL(2,\mathbb{C})$ that the pair X, Y can generate are the groups $\mathbb{Z}_{10}, D_{10}$, and A_5. If X, Y generate A_5 then XY is of order 3 or 5. Moreover if XY has order 5 then it is conjugate to Y^2 so $\mathrm{tr}XY = \pm\mathrm{tr}Y^2 = \pm((\mathrm{tr}Y)^2 - 2) = \pm\eta^{-1}$. If X, Y generate $\mathbb{Z}_{10}$ then $\mathrm{tr}XY = \pm\sqrt{4-\eta^2}$; if X, Y generate D_{10} then $\mathrm{tr}XY = 0$. The leading coefficient of $\sigma(\mu)$ is given by $c = \eta^{k_2}$. Thus $\sigma(\mu)$ takes the form

$$\sigma(\mu) = \eta^{k_2}\mu^s(\mu^2-1)^t(\mu^2-\eta^{-2})^u(\mu^2-(4-\eta^2))^v.$$

Let $A, B \in PSL(2,\mathbb{C})$ be defined as follows:

$$A = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\pi/5} & z \\ 0 & e^{-i\pi/5} \end{pmatrix}.$$

Then $\mathrm{tr}A = 0$, $\mathrm{tr}B = \eta$, $\mathrm{tr}AB = z - \sqrt{4-\eta^2}$.

Consider the representation $\rho : \langle\, x, y \mid x^2 = y^5 = 1 \,\rangle \to PSL(2,\mathbb{C})$ given by $x \mapsto A$, $y \mapsto B$, then

$$\begin{aligned}\mathrm{tr}\rho(xy^{\gamma_1}\ldots xy^{\gamma_\ell}) &= \sigma(z - \sqrt{4-\eta^2}) \\ &= \eta^{k_2}(z-\sqrt{4-\eta^2})^s(z^2 - 2z\sqrt{4-\eta^2} + \eta^{-2})^t \\ &\qquad \cdot (z^2 - 2z\sqrt{4-\eta^2}+1)^u(z - 2\sqrt{4-\eta^2})^v z^v\end{aligned}$$

whose constant term is 0 if $v > 0$ and is $\eta^{k_2-2t}(\sqrt{4-\eta^2})^s$ if $v = 0$. Now the constant term in $\mathrm{tr}(AB^{\gamma_1}\ldots AB^{\gamma_\ell})$ is $2\cos((5\ell + 2\sum_{i=1}^{\ell}\gamma_i)\pi/10)$. Since ℓ is odd and $\overline{G}$ admits no essential cyclic representation, this constant term is either $\pm 2\cos(\pi/10) = \pm\eta\sqrt{4-\eta^2}$ or $\pm 2\cos(3\pi/10) = \pm\sqrt{4-\eta^2}$. Thus we can conclude that $v = 0$, that

$$\eta^{k_2-2t}(\sqrt{4-\eta^2})^s = \eta\sqrt{4-\eta^2} \quad \text{or} \quad \sqrt{4-\eta^2},$$

and therefore that $s = 1$ and $t = k_2/2$ or $t = (k_2-1)/2$. Hence 0 is not a repeated root of $\sigma(\mu)$, contrary to hypothesis. □

For the proof of Theorem 2 we shall require the following proposition.

Proposition 8 *Let r, n be natural numbers. Then*

$$\prod_{\substack{1\le r\le n-1 \\ (r,n)=1}} 2\sin(r\pi/n) = \begin{cases} p & \text{if } n \text{ is a power of a prime } p \\ 1 & \text{otherwise} \end{cases}$$

Proof

By identity 1.392(1) of [8] we have that for all real numbers x

$$\sin(x)\prod_{r=1}^{n-1} 2\sin(x + r\pi/n) = \sin(nx).$$

Differentiating and substituting $x = 0$ we obtain $\prod_{r=1}^{n-1} 2\sin(r\pi/n) = n$. Applying this identity to all divisors of n yields the result. □

Proof of Theorem 2

We will consider the homomorphic image $\overline{G}$ of G defined by the presentation (1). As explained at the start of this section we will assume that $\overline{w}(x, y)$ is not a proper power and can be freely reduced to the form $\overline{w}(x, y) = xy^{\gamma_1} \ldots xy^{\gamma_\ell}$ where $1 \le \gamma_i \le 4$ $(1 \le i \le \ell - 1)$, $\ell \ge 1$.

We may assume that neither G nor $\overline{G}$ admits an essential cyclic representation, for otherwise by [9, Theorem E] and Lemma 6 above G contains a non-abelian free subgroup. Thus if $a = \sum_{i=1}^{k} \alpha_i$ then $(a, 5p) = 1$ or p. Moreover the only finite subgroups of $PSL(2, \mathbb{C})$ generated by an element of order 2 and an element of order $5p$ are cyclic and dihedral groups. Thus if G contains no non-abelian free subgroup we may assume that all its essential representations to $PSL(2, \mathbb{C})$ are to dihedral groups. Thus the trace polynomial $\tau(\lambda)$ for G has the form $\tau(\lambda) = c\lambda^k$ where

$$c = \frac{1}{(\sin(\pi/5p))^k}\prod_{i=1}^{k} \sin\left(\frac{\pi\alpha_i}{5p}\right).$$

Let $X, Y \in PSL(2, \mathbb{C})$ be elements of orders $2, 5p$ that generate a cyclic subgroup of $PSL(2, \mathbb{C})$. We may assume that $\mathrm{tr}X = 0$, $\mathrm{tr}Y = 2\cos(\pi/5p)$, so that $\mathrm{tr}XY = 2\sin(\pi/5p)$. Let $\rho : \langle\ x, y \mid x^2 = y^{5p} = 1\ \rangle \to PSL(2, \mathbb{C})$ be given by $x \mapsto X,\ y \mapsto Y$. Then $\mathrm{tr}\rho(w) = 2\cos((5pk + 2a)\pi/10p) = \pm 2\sin(a\pi/5p)$. On the other hand $\mathrm{tr}\rho(w) = \tau(2\sin(\pi/5p)) = \prod_{i=1}^{k} 2\sin(\alpha_i\pi/5p)$. Thus $2\sin(a\pi/5p) = \prod_{i=1}^{k} 2\sin(\alpha_i\pi/5p)$.

Let $A = \mathrm{Aut}(\mathbb{Z}_{5p}) \cong \mathbb{Z}_4 \times \mathbb{Z}_{p-1}$. For each $1 \le i \le k$, Proposition 8 gives

$$\prod_{\psi\in A} 2\sin(\psi(\alpha_i)\pi/5p) = \begin{cases} p^4 & \text{if } 5 \mid \alpha_i\ , \\ 5^{p-1} & \text{if } p \mid \alpha_i\ , \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, by Proposition 8 and the fact that $5 \not| \, a$, we have

$$\prod_{\psi\in A}\prod_{i=1}^{k} 2\sin(\psi(\alpha_i)\pi/5p) = \prod_{\psi\in A} 2\sin(\psi(a)\pi/5p) \in \{1, 5^{p-1}\}.$$

Hence $5 \not| \, \alpha_i$ for each i.

Hence $\ell = k$, and thus the trace polynomial $\sigma(\mu)$ of $\overline{G}$ is of degree $k \ge 3$. As explained in the proof of Lemma 7 we may assume that $\sigma(\mu)$ is of the form $\sigma(\mu) = c'\mu^s(\mu^2 - 1)^t(\mu^2 - \eta^{-2})^u$ where $\eta = 2\cos(\pi/5)$ and s is odd. By Lemma 7 we may assume $s = 1$, and by Lemma 5 we may assume $t \leq 1$. The automorphism ψ of $\mathbb{Z}_5$ generated by the map $1 \mapsto 2$ yields the alternative presentation $\overline{G} = \langle\ x, y \mid x^2 = y^5 = (xy^{\psi(\beta_1)} \dots xy^{\psi(\beta_k)})^2 = 1\ \rangle$ with trace polynomial of the form $\sigma'(\mu) = c''\mu^s(\mu^2 - \eta^{-2})^t(\mu^2 - 1)^u$. By another application of Lemma 5 we may assume $u \leq 1$. Since $k = s + 2t + 2u > 1$ we are reduced to the cases $k = 3, 5$.

If $k = 3$ then G contains a non-abelian free subgroup by [10, Theorem 1].

Suppose that $k = 5$. Then $\sigma(\mu) = c'\mu(\mu^2 - 1)(\mu^2 - \eta^{-2})$. A computer search reveals that the only words $w(x, y)$ (up to cyclic permutation, inversion, and automorphisms of $\langle\ y \mid y^5 = 1\ \rangle$) with trace polynomial of that form are $xyxy^3xy^2xy^4xy^t$ with $t \in \{1, 2\}$. In each case, a GAP [7] calculation shows that $\overline{G}$ (and hence also G) has a subgroup of index 11 admitting the free group of rank 2 as a homomorphic image. □

5 The cases $m = 12, 20, 30, 60$

Proof of Theorem 3
We shall consider alternative presentations for G:

$$G = \langle\ x, y \mid x^2 = y^m = (xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})^2 = 1\ \rangle$$

where ψ is an automorphism of $\mathbb{Z}_m$. By [10, Theorem 5] we may assume that k is odd. By [9, Theorem E] we may assume that G admits no essential cyclic representation, and hence that all essential representations to $PSL(2, \mathbb{C})$ are dihedral. It follows that the trace polynomial for G takes the form $\tau(\lambda) = c\lambda^k$ where $c = (t_1^{k_1} \dots t_{m/2}^{k_{m/2}})/(\sin(\pi/m))^k$. Let $X, Y \in PSL(2, \mathbb{C})$ have orders 2 and m respectively that generate a cyclic group of order m. We may assume $\mathrm{tr}(XY) = 2\sin(\pi/m)$. Fix ρ to be the representation $\rho : \langle\ x, y \mid x^2 = y^m = 1\ \rangle \to PSL(2, \mathbb{C})$ given by $x \mapsto X,\ y \mapsto Y$. Then

$$\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)}) = \pm 2\cos(q\pi/m) \quad \text{for some} \quad 1 \leq q < m/2. \qquad (2)$$

(Note that if $q = m/2$ then ρ induces an essential cyclic representation of G, contrary to our earlier assumption.) In particular,

$$-1 \leq \prod_{\psi \in A} \frac{\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})}{2} \leq 1 \qquad (3)$$

for any group A of automorphisms of $\mathbb{Z}_m$.

Now

$$\begin{aligned} \mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)}) &= \tau(2\sin(\pi/m)) \\ &= 2^k \prod_{i=1}^{k} \sin\left(\frac{\pi\psi(\alpha_i)}{m}\right) \end{aligned}$$

so

$$\frac{\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})}{2} = 2^{k-1} \cdot t_1^{k_{\psi(1)}} \cdots t_{m/2}^{k_{\psi(m/2)}}. \qquad (4)$$

We now consider each value of m separately.

The case $m = 12$.
Let ψ be the automorphism of $\mathbb{Z}_{12}$ generated by the map $1 \mapsto 5$ and let $A = \langle\psi\rangle$. Then using (3) and (4) we obtain

$$2^{2(k-1)}(t_1 t_5)^{k_1+k_5} \cdot (t_2)^{2k_2} \cdot (t_3)^{2k_3} \cdot (t_4)^{2k_4} \cdot (t_6)^{2k_6} \leq 1$$

which (using Proposition 8) simplifies to

$$2^{k_3+2k_6-2} \cdot 3^{k_4} \leq 1.$$

We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\, x, y \mid x^2 = y^6 = (xy^{\beta_1} \ldots xy^{\beta_k})^2 = 1 \,\rangle, \\ L &= \langle\, x, y \mid x^2 = y^4 = (xy^{\gamma_1} \ldots xy^{\gamma_k})^2 = 1 \,\rangle, \end{aligned}$$

where $\beta_i = \alpha_i \bmod 6$ and $\gamma_i = \alpha_i \bmod 4$ for each $1 \leq i \leq k$. Suppose $k_6 = 0$. Then each β_i is non-zero. If $k > 3$ then by Theorem 1 H, and hence G, contains a non-abelian free subgroup. If $k = 3$ then by [10, Theorem 1] G contains a non-abelian free subgroup. Thus we may assume $k_6 \geq 1$ and hence $k_6 = 1, k_3 = k_4 = 0$. Moreover we may assume

$$\mathrm{tr}\rho(xy^{\alpha_1} \ldots xy^{\alpha_k}) = \pm 2 \tag{5}$$

for otherwise one of $\rho(xy^{\alpha_1} \ldots xy^{\alpha_k})$ or $\rho(xy^{\psi(\alpha_1)} \ldots xy^{\psi(\alpha_k)})$ provides a contradiction to (2). Using (4) equation (5) simplifies to

$$\begin{aligned} 2 &= 2^{k_1+k_2+k_5+1} \cdot t_1^{k_1} t_2^{k_2} t_5^{k_5} t_6^{1} \\ &= 2\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^{k_1-k_5} \end{aligned}$$

so $k_1 = k_5$. Since the image of ρ is isomorphic to $\mathbb{Z}_{12}$ and by equation (5) $\rho(w)$ is the zero of this group we have that $6k + \sum_{i=1}^{k} \alpha_i = 0 \bmod 12$, and k is odd so

$$\sum_{i=1}^{k} \alpha_i = 6 \bmod 12, \tag{6}$$

which implies $\sum_{i=1}^{k} \gamma_i = 2 \bmod 4$. By Lemma 4 L (and hence G) contains a non-abelian free subgroup unless precisely one $\gamma_i = 2$. This implies that $k_2 + k_6 = 1$, but $k_6 = 1$ so $k_2 = 0$.

Let $\overline{w}(x, y) = xy^{\beta_1} \ldots xy^{\beta_k}$. Using the relations $x^2 = 1, y^6 = 1$ of H we can cyclically reduce $\overline{w}(x, y)$ to x (in which case $H \cong \mathbb{Z}_2 * \mathbb{Z}_6$, so G contains a non-abelian free subgroup) or to the form $\overline{w}(x, y) = xy^{\delta_1} \ldots xy^{\delta_\ell}$ where ℓ is odd and $1 \leq \delta_i \leq 5$ for each $1 \leq i \leq \ell$. If $\ell > 3$ then by Theorem 1 H, and hence G, contains a non-abelian free subgroup. Thus we may assume $\ell = 1$ or 3. The words $w, \overline{w}$ then take the following forms:

$$\begin{array}{lll} \ell = 1: & w = xy^{\xi_1}xy^{\xi_2}u(x,y)xy^6v(x,y) & \overline{w} = xy^{\xi_1+\xi_2} \\ \ell = 3: & w = xy^{\xi_1}xy^{\xi_2}xy^{\xi_3}xy^{\xi_4}u(x,y)xy^6v(x,y) & \overline{w} = xy^{\xi_1+\xi_4}xy^{\xi_2}xy^{\xi_3} \end{array}$$

where $\xi_1, \xi_2, \xi_3, \xi_4 \in \{1, 5\}$ and

$$u(x,y) = xy^{a_1} \dots xy^{a_n}$$
$$v(x,y) = xy^{b_n} \dots xy^{b_1}$$

with $a_i + b_i = 0 \bmod 6$ for each $1 \leq i \leq n$.

In the case $\ell = 1$ equation (6) implies $\sum_{i=1}^{k} \alpha_i = 0 \bmod 6$ so

$$\xi_1 + \xi_2 + (a_1 + \dots + a_n) + 6 + (b_n + \dots + b_1) = 0 \bmod 6$$

which implies $\xi_1 + \xi_2 = 0 \bmod 6$ contradicting our assumption that the exponents of y in $\overline{w}$ are non-zero. In the case $\ell = 3$, since $\xi_1 + \xi_2 + \xi_3 + \xi_4$ is even, Theorem 1 of [10] implies that H, and hence G, contains a non-abelian free subgroup.

The case $m = 20$.
We shall consider the following homomorphic image of G:

$$H = \langle\ x, y \mid x^2 = y^{10} = (xy^{\beta_1} \dots xy^{\beta_k})^2 = 1\ \rangle$$

where $\beta_i = \alpha_i \bmod 10$ for each $1 \leq i \leq k$.

Let ψ be the automorphism of $\mathbb{Z}_{20}$ generated by the map $1 \mapsto 3$ and let $A = \langle \psi \rangle$. Then using (3) and (4) we obtain

$$2^{4(k-1)}(t_1t_3t_7t_9)^{k_1+k_3+k_7+k_9}(t_2t_6)^{2(k_2+k_6)}(t_4t_8)^{2(k_4+k_8)}t_5^{4k_5}t_{10}^{4k_{10}} \leq 1$$

which (using Proposition 8) simplifies to

$$2^{2k_5+4k_{10}-4} \cdot 5^{k_4+k_8} \leq 1.$$

If $k_{10} = 0$ then each β_i is non-zero so H contains a non-abelian free subgroup by Theorem 2. Thus we may assume that $k_{10} \geq 1$ and hence $k_{10} = 1$, $k_5 = k_4 = k_8 = 0$. Moreover we may assume

$$\mathrm{tr}\rho(xy^{\alpha_1} \dots xy^{\alpha_k}) = \pm 2 \tag{7}$$

for otherwise for some $\phi \in A$ the element $\rho(xy^{\phi(\alpha_1)} \dots xy^{\phi(\alpha_k)})$ provides a contradiction to (2). The image of ρ is isomorphic to $\mathbb{Z}_{20}$ and by equation (7) $\rho(w)$ is the zero of this group so we have that $\sum_{i=1}^{k} \alpha_i = 10 \bmod 20$ (since k is odd). Thus $\sum_{i=1}^{k} \beta_i = 0 \bmod 10$ so H admits an essential cyclic representation, and the result follows from [9, Theorem E].

The case $m = 30$.
We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\, x, y \mid x^2 = y^{10} = (xy^{\beta_1} \dots xy^{\beta_k})^2 = 1 \,\rangle, \\ L &= \langle\, x, y \mid x^2 = y^{15} = (xy^{\gamma_1} \dots xy^{\gamma_k})^2 = 1 \,\rangle, \end{aligned}$$

where $\beta_i = \alpha_i \bmod 10$, $\gamma_i = \alpha_i \bmod 15$ for each $1 \le i \le k$.

Let ψ be the automorphism of $\mathbb{Z}_{30}$ generated by the map $1 \mapsto 7$ and let $A = \langle \psi \rangle$. Then using (3) and (4) we obtain

$$\begin{aligned} 2^{4(k-1)}(t_1t_7t_{11}t_{13})^{k_1+k_7+k_{11}+k_{13}}(t_2t_{14}t_8t_4)^{k_2+k_{14}+k_8+k_4} \\ \cdot(t_3t_9)^{2(k_3+k_9)}(t_5)^{4k_5}(t_6t_{12})^{2(k_6+k_{12})}t_{10}^{4k_{10}}t_{15}^{4k_{15}} \\ \le 1 \end{aligned}$$

which (using Proposition 8) simplifies to

$$2^{4k_{15}-4} \cdot 5^{k_6+k_{12}} \cdot 9^{k_{10}} \le 1.$$

If $k_{15} = 0$ then each γ_i is non-zero which implies that L, and hence G, contains a non-abelian free subgroup by Theorem 2. If $k_{15} > 0$ then $k_{10} = 0$, so H, and hence G, contains a non-abelian free subgroup by Theorem 2.

The case $m = 60$.
We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\, x, y \mid x^2 = y^{20} = (xy^{\beta_1} \dots xy^{\beta_k})^2 = 1 \,\rangle, \\ L &= \langle\, x, y \mid x^2 = y^{30} = (xy^{\gamma_1} \dots xy^{\gamma_k})^2 = 1 \,\rangle, \end{aligned}$$

where $\beta_i = \alpha_i \bmod 20$, $\gamma_i = \alpha_1 \bmod 30$ for each $1 \le i \le k$.

Consider the group $A \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ of automorphisms of $\mathbb{Z}_{60}$ generated by $\psi : 1 \mapsto 7$ and $\phi : 1 \mapsto 29$. Using (3) and (4) we obtain

$$\begin{aligned} 1 \ \ge\ & 2^{8(k-1)} \\ & \cdot(t_1t_7t_{11}t_{13}t_{17}t_{19}t_{23}t_{29})^{k_1+k_7+k_{11}+k_{13}+k_{17}+k_{19}+k_{23}+k_{29}} \\ & \cdot(t_2t_{14}t_{22}t_{26})^{2(k_2+k_{14}+k_{22}+k_{26})} \cdot (t_3t_{21}t_{27}t_9)^{2(k_3+k_{21}+k_{27}+k_9)} \\ & \cdot(t_4t_{28}t_{16}t_8)^{2(k_4+k_{28}+k_{16}+k_8)} \cdot (t_5t_{25})^{4(k_5+k_{25})} \cdot (t_6t_{18})^{4(k_6+k_{18})} \cdot (t_{12}t_{24})^{4(k_{12}+k_{24})} \\ & \cdot(t_{10})^{8k_{10}} \cdot (t_{15})^{8k_{15}} \cdot (t_{20})^{8k_{20}} \cdot (t_{30})^{8k_{30}} \end{aligned}$$

which (using Proposition 8) simplifies to

$$1 \ge 2^{4k_{15}+8k_{30}-8} \cdot 5^{2(k_{12}+k_{24})} \cdot 3^{4k_{20}}$$

In particular one of k_{20}, k_{30} is zero so either all β_i's are non-zero or all γ_i's are non-zero. Hence, by the above, one of H or L (and hence G) contains a non-abelian free subgroup. □

A Appendix: The case $m = 6$

This appendix gives a sketch proof of Theorem 1. We begin by giving a complete calculation of *all* the coefficients of the trace polynomial.

Let $\mathcal{A}(k)$ denote the set of subsets $S \subset \{1, \dots, k\}$ such that $s_1 - s_2 \neq 1$ for $s_1, s_2 \in S$. The maximum cardinality of $S \in \mathcal{A}(k)$ is the integer part $\lfloor k/2 \rfloor$ of $k/2$. For $0 \leq j \leq \lfloor k/2 \rfloor$, let $\mathcal{A}(k, j)$ denote the set of sets $S \in \mathcal{A}(k)$ of cardinality j.

Lemma 9 *Let $X, Y \in SL(2, \mathbb{C})$ be matrices with $\mathrm{tr}(X) = 0$, $\mathrm{tr}(Y) = 2\cos(\pi/m)$, $\mathrm{tr}(XY) = \lambda$, for some integer $m \geq 2$. Let $W = XY^{\alpha_1} \dots XY^{\alpha_k}$, where $1 \leq \alpha_i < m$ for each $1 \leq i \leq k$. Then the trace of W is given by the polynomial*

$$\mathrm{tr}(W) = c \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j B_j \lambda^{k-2j},$$

where

$$c = \prod_{j=1}^{k} \frac{\sin(\alpha_j \pi/m)}{\sin(\pi/m)},$$

$$B_j = \sum_{\{t_1,\dots,t_j\} \in \mathcal{A}(k,j)} \left(\prod_{s=1}^{j} b(t_s) \right),$$

$$b(j) = \frac{\sin^2(\pi/m) e^{i\pi(\alpha_{j+1} - \alpha_j)/m}}{\sin(\alpha_j \pi/m) \sin(\alpha_{j+1}\pi/m)}.$$

Proof

By standard trace identities, the trace of $W(X, Y)$ is determined by the traces of X, Y and XY, so it is sufficient to work with fixed matrices with the given traces. We define

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} e^{i\pi/m} & \lambda \\ 0 & e^{-i\pi/m} \end{pmatrix},$$

Then, for $1 \leq \alpha \leq m - 1$,

$$XY^\alpha = \begin{pmatrix} 0 & -e^{-i\alpha\pi/m} \\ e^{i\alpha\pi/m} & p(\alpha)\lambda \end{pmatrix}$$

with $p(\alpha) = \sin(\alpha\pi/m)/\sin(\pi/m)$. Now each entry in $W(X, Y)$ is a sum of terms, each of which is a product of an entry from each of XY^{α_j} $(1 \leq j \leq k)$. The leading monomial of $\mathrm{tr}(W(X, Y))$ necessarily consists of the product of the lower right entries of the XY^{α_j}, so is $c\lambda^k = \prod_{j=1}^{k} p(\alpha_j)\lambda^k$, as claimed. Each term contributing to the λ^{k-2j} monomial can be obtained from c by replacing each of j (non-overlapping) pairs of (cyclically) consecutive lower right entries by the upper right entry of the first member of the pair, followed by the lower left entry of the second member. Such a term is thus equal to $cb(s_1) \cdots b(s_j)$ for some $\{s_1, \dots, s_j\} \in \mathcal{A}(k, j)$, and the result follows. $\square$

Sketch proof of Theorem 1

Let

$$G = \langle\ x, y \mid x^2 = y^6 = w(x,y)^2 = 1\ \rangle,$$
$$\overline{G} = \langle\ x, y \mid x^2 = y^3 = \overline{w}(x,y)^2 = 1\ \rangle,$$

where $w(x,y) = xy^{\alpha_1} \ldots xy^{\alpha_k}$, $\overline{w}(x,y) = xy^{\beta_1} \ldots xy^{\beta_k}$ where for $1 \le i \le k$, $\beta_i = \alpha_i \bmod 3$, and $k > 3$. Let $\tau(\lambda), \sigma(\mu)$ denote the trace polynomials of $G, \overline{G}$ respectively. The only finite subgroups of $PSL(2,\mathbb{C})$ generated by an element of order 2 and an element of order 6 are cyclic and dihedral groups. Thus if G contains no non-abelian free subgroup we may assume that all the roots of its trace polynomial correspond to cyclic or dihedral essential representations.

Suppose first that G admits an essential cyclic representation, with kernel K. Then ± 1 are roots of $\tau(\lambda)$. By [9, Theorem 4.8] if 1 or -1 is a repeated root of $\tau(\lambda)$ then G has a non-abelian free subgroup. Thus we may assume that $\tau(\lambda) = c\lambda^{k-2}(\lambda^2 - 1)$ and in particular that G has an essential representation ρ onto the dihedral group of order 12, D_{12}. Now K has a deficiency 0 presentation, its abelianization K/K' is free abelian of rank 3, and conjugation by x induces the antipodal automorphism on K/K'. Moreover, a calculation shows that $\rho(K')$ is a non-trivial abelian subgroup of D_{12}, so K'/K'' is non-trivial. By [9, Corollary 3.2], K' (and hence G) contains a non-abelian free subgroup.

Hence we may assume that G has no essential cyclic representations, and thus $\tau(\lambda) = c\lambda^k$. Then as in the proof of Theorem 3 equations (3), (4) yield $(k_2, k_3) = (0,0), (1,0), (0,1)$ and thus $c = 1, \sqrt{3}, 2$, respectively. When k is even the existence of an essential dihedral representation implies that the alternating sum $\sum_{i=1}^{k}(-1)^i\alpha_i$ is congruent to 3 modulo 6 and thus $k_2 = 1$, $c = \sqrt{3}$.

We proceed by calculating the coefficients in $\tau(\lambda), \sigma(\mu)$ and split the proof into three cases, depending on the value of c. Consider first the form of $\sigma(\mu)$ in the cases $c = 1, \sqrt{3}$. The only finite subgroups of $PSL(2,\mathbb{C})$ generated by projective matrices X, Y of orders 2, 3 respectively are $\mathbb{Z}_6, D_6, A_4, S_4, A_5$, so the possible roots of σ are $\pm\sqrt{3}, 0, \pm 1, \pm\sqrt{2}, (\pm 1 \pm \sqrt{5})/2$. If ± 1 or $\pm\sqrt{3}$ occurs as a root of σ then G admits an essential representation to A_4 or $\mathbb{Z}_6$. In either case $\sum_{i=1}^{k}\beta_i = 0 \bmod 3$, and we can define a representation $\rho : G \to \mathbb{Z}_6$ by $\rho(x) = 3 \bmod 6$ and $\rho(y) = 1 \bmod 6$. By assumption, ρ is not essential, so $\rho(w) = 0 \bmod 6$ and $c = \tau(1) = \pm 2$, a contradiction. Thus

$$\sigma(\mu) = \mu^r(\mu^2 - 2)^s(\mu^4 - 3\mu^2 + 1)^t \tag{8}$$

where $r, s, t \ge 0$ satisfy $r + 2s + 4t = k$. Since $\sigma(\sqrt{3}) \in \{\pm 1, \pm\sqrt{3}, \pm 2\}$ we have $r = 0, 1$. If k is even then $r = 0$, and (since $\sum_{i=1}^{k}(-1)^i\alpha_i$ is congruent to 0 modulo 3) we also have $\sigma(0) = \pm 2$ so $s = 1$.

Case 1: $c = 1$.

In this case k is odd and $\alpha_i \in \{1, 5\}$ for each $1 \le i \le k$. By Lemma 9, the coefficient $-B_1$ of λ^{k-2} in $\tau(\lambda)$ is given by $B_1 = \sum_{i=1}^{k} b(i)$, where for each $1 \le i \le k$

$$b(i) := \begin{cases} 1 & \text{if } \alpha_i = \alpha_{i+1} \\ \frac{-1+\sqrt{-3}}{2} & \text{if } \alpha_i = 1,\ \alpha_{i+1} = 5 \\ \frac{-1-\sqrt{-3}}{2} & \text{if } \alpha_i = 5,\ \alpha_{i+1} = 1 \end{cases}$$

(where α_{k+1} is defined equal to α_1). A similar analysis for $\sigma(\mu)$ shows that the coefficient $-B'_1$ of μ^{k-2} is given by $B'_1 = \sum_{i=1}^{k} b'(i)$ where

$$b'(i) := \begin{cases} 1 & \text{if } \beta_i = \beta_{i+1} \\ \frac{1+\sqrt{-3}}{2} & \text{if } \beta_i = 1,\ \beta_{i+1} = 2 \\ \frac{1-\sqrt{-3}}{2} & \text{if } \beta_i = 2,\ \beta_{i+1} = 1 \end{cases}$$

Since the coefficient of λ^{k-2} in $\tau(\lambda)$ is zero, we have that k is a multiple of 3 – say $k = 3\ell$ where $\ell > 1$ – and each possible value of $b(i)$ occurs precisely ℓ times. It follows that $B'_1 = 2\ell$. On the other hand we can compute the coefficient of μ^{k-2} in $\sigma(\mu) = \mu(\mu^2 - 2)^s(\mu^4 - 3\mu^2 + 1)^t$ as $-2s - 3t$. We thus obtain the simultaneous diophantine equations

$$1 + 2s + 4t = 3\ell, \quad 2s + 3t = 2\ell, \quad s, t, \ge 0, \ell > 1$$

with the unique solution $s = 0, t = 2, \ell = 3$, and so $k = 9$.

Now consider the coefficient B_2 of λ^5 in $\tau(\lambda)$ and the coefficient B'_2 of μ^5 in $\sigma(\mu)$. Using Lemma 9 we can deduce

$$2B_2 = B_1^2 - \sum_{i=1}^{9} b(i)^2 - 2\sum_{i=1}^{9} b(i)b(i+1)$$

where $b(10)$ is defined equal to $b(1)$. Since $B_1 = B_2 = 0$ and the $b(i)$'s are equally distributed amongst the three possible values it follows that $\sum_{i=1}^{9} b(i)b(i+1) = 0$.

A similar analysis shows that $\sum_{i=1}^{9} b'(i)^2 = 0$, $\sum_{i=1}^{9} b'(i)b'(i+1) = 6$, from which we can deduce $B'_2 = 12$. But the coefficient of μ^5 in $\sigma(\mu) = \mu(\mu^4 - 3\mu^2 + 1)^2$ is 11. This contradiction completes Case 1.

Case 2: $c = \sqrt{3}$.

Then $\alpha_i \in \{1, 5\}$ for all but one value of i, for which $\alpha_i \in \{2, 4\}$. Without loss of generality we may assume that $\alpha_k = 2$ and $\alpha_i \in \{1, 5\}$ for $1 \le i < k$. As in Case 1, consideration of the coefficient of λ^{k-2} in $\tau(\lambda)$ and of μ^{k-2} in $\sigma(\mu)$ yield diophantine equations in s, t, k. We find that the only solutions with $k > 3$ are (i) $s = 2$, $t = 0$, $k = 5$; (ii) $s = 0$, $t = 2$, $k = 9$; (iii) $s = 1$, $t = 2$, $k = 11$; (iv) $s = 0$, $t = 4$, $k = 17$; (v) $s = 0$, $t = 2$, $k = 8$. We can rule out solution (v) since k is even and $s \ne 1$.

For the remaining solutions, consideration of the coefficient of λ^{k-4} in $\tau(\lambda)$ and the coefficient of μ^{k-4} in $\sigma(\mu)$ yield additional diophantine equations which reduce us to solution (i). A computer search reveals that the only word $w(x,y)$ (up to cyclic permutation, inversion, and automorphisms of $\langle\, y \mid y^6 = 1 \,\rangle$) such that $\tau(\lambda), \sigma(\mu)$ are of the required form is $w(x,y) = xy^5xyxyxy^5xy^2$. A calculation in GAP [7] shows that in this case G has a subgroup of index 6 admitting a free homomorphic image of rank 2.

Case 3: $c = 2$**.**
In this case k is odd, the α_i are all odd, and $\alpha_i = 3$ for precisely one value of i. Without loss of generality we may assume that $\alpha_k = 3$ and $\alpha_i \in \{1,5\}$ for $1 \le i < k$. Again, the coefficient $-B_1$ of λ^{k-2} is given by $B_1 = \sum_{i=1}^{k} b(i)$ where $b(i)$ is as in Case 1 for $i < k-1$,

$$b(k-1) := \begin{cases} \frac{1+\sqrt{-3}}{4} & \text{if } \alpha_{k-1} = 1 \\ \frac{1-\sqrt{-3}}{4} & \text{if } \alpha_{k-1} = 5 \end{cases}$$

and

$$b(k) := \begin{cases} \frac{1-\sqrt{-3}}{4} & \text{if } \alpha_1 = 1 \\ \frac{1+\sqrt{-3}}{4} & \text{if } \alpha_1 = 5 \end{cases}$$

Note that $b(1), \ldots, b(k-2)$ are algebraic integers. From the equation $B_1 = 0$ it follows that $b(k-1) + b(k)$ is also an algebraic integer, and this can only happen if $\alpha_1 + \alpha_{k-1} = 6$. Assume inductively that $\alpha_t + \alpha_{k-t} = 6$ (and hence $b(k-t) = b(t-1)$, where $b(0)$ is defined equal to $b(k)$) for $1 \le t < u$, for some $u \le (k-1)/2$. Then from the equation $B_u = 0$ it turns out that $b(k-u) + b(u-1)$ is an algebraic integer, and this can only happen if $\alpha_u + \alpha_{k-u} = 6$.

Thus $\alpha_t + \alpha_{k-t} = 6$ for all $1 \le t \le (k-1)/2$, so the third relator of G has the form $(U(x,y)xU(x,y)^{-1}y^3)^2$ for some word U. In passing to $\overline{G}$, we kill y^3, so the relator collapses to x^2, and $\overline{G} \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Hence $\overline{G}$, and so also G, contains a non-abelian free subgroup, as claimed. □

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