Some results on one-relator surface groups

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To Fico González Acuña on his 60'th birthday

1 Introduction

This short note was inspired by a question from Fico González Acuña:

Question 1 *If α and β are two closed curves (nonsimple, in general) on an orientable surface S, such that the normal closures of α and β in $\pi_1(S)$ coincide, is β freely homopotic to $\alpha^{\pm 1}$?*

If S is noncompact, or has nonempty boundary, then $\pi_1(S)$ is free, and the answer to Question 1 is yes, by an old result of Magnus [7] on one-relator groups. (Essentially, the defining relator in a one-relator group on a given generating set is unique up to conjugacy and inversion.)

We will show (see Theorem 3.4 below) that Question 1 also has an affirmative answer in the case of a closed surface S. In this case Question 1 can be interpreted in terms of *one-relator surface groups*, as introduced by Hempel [3]. Among other results, Hempel proved analogues for one-relator surface groups of two theorems from one-relator group theory: (i) a one-relator surface group is locally indicable if and only if the relator is not a proper power in $\pi_1(S)$; (ii) a closed curve α in S lifts (up to homotopy) to a simple closed curve in the covering space corresponding to the normal closure of α in $\pi_1(S)$. These are analogues of results of Brodskiĭ[1] and Weinbaum [15] respectively. (In the latter case, the original form states that proper subwords of the defining relator represent nontrivial elements in a one-relator group.) Hempel [3] also proved (iii) that a power β^n of a simple closed curve β can belong to the normal closure in $\pi_1(S)$ of a curve α only in the obvious cases: either α is isotopic in S to β^m with $m|n$; or $n=1$ and α is a nonseparating curve in a punctured torus in S bounded by β.

The purpose of this note is to show that many other results from one-relator group theory have natural analogues for one-relator surface groups. In most (but not all) cases, the proofs can be obtained by using a trick from [3] to reduce us to the classical one-relator case.

Interest in one-relator surface groups first appeared in the work of Papkyriakopoulos [12], who reduced the Poincaré conjecture to two conjectures which can be expressed in terms of certain one-relator surface groups. (See [10, 13, 14] for more on these conjectures.)

2 One-relator surface groups

By a *one-relator surface group* we will mean, following Hempel [3], the quotient of the fundamental group $\pi_1(S)$ of a connected, orientable surface S by the normal closure of a single element α. We will denote this group by $\pi_1(S)/\alpha$. In particular, any (countable) one-relator group can be regarded as a one-relator surface group by choosing S to be noncompact (or ∂S to be nonempty). We will consistently abuse notation to regard α as an immersed closed curve in S.

Since one-relator quotients of the torus group are well understood, we may in practice restrict attention to the case where S is a closed orientable surface of genus at least 2. The following basic trick is employed by Hempel in [3] to reduce his analogue of Brodskiĭ's Theorem to the classical case. We follow [3] in using $\langle -, - \rangle$ to denote the integer-valued algebraic intersection pairing on $H_1(S)$ or $\pi_1(S)$ as appropriate.

Proposition 2.1 *Let S be a closed, connected, oriented surface of genus at least 2, and let α be a closed curve in S. Then*

1. *There is a non-separating simple closed curve β in S such that $\langle \alpha, \beta \rangle = 0$.*
2. *For any such β, there are connected surfaces F, F_0, F_1 and a closed curve α' in F, such that*
   - *(a) $F_0 \cong F_1$, $F_0 \subset F$ and $F_1 \subset F$;*
   - *(b) $\pi_1(F_0) \to \pi_1(F)/\alpha'$ and $\pi_1(F_1) \to \pi_1(F)/\alpha'$ are injective;*
   - *(c) $\pi_1(S)$ (resp. $\pi_1(S)/\alpha$) is an HNN-extension of $\pi_1(F)$ (resp. $\pi_1(F)/\alpha'$) with associated subgroups $\pi_1(F_0)$ and $\pi_1(F_1)$;*
   - *(d) Each of ∂F, ∂F_0 and ∂F_1 consists of two circles, each of which represents (a conjugate of) $\beta \in \pi_1(S)$.*

Proof. The first part is Lemma 2.1 of [3]. The second is implicit in the proof of Theorem 2.2 of [3]. For completeness we repeat the argument here. Let S_0 denote the surface obtained from S by cutting along β, let S_n be an isomorphic copy of S_0 for each integer n, and form a covering $\tilde{S}$ of S from $\bigcup_{n\in\mathbb{Z}} S_n$ by joining one of the two boundary components of S_n to the other boundary component of S_{n+1}, for all n. Note that $\tilde{S}$ is the infinite cyclic covering of S corresponding to the kernel of $\langle -, \beta \rangle : \pi_1(S) \to \mathbb{Z}$.

There is a minimum $n \geq 0$ such that $S_0 \cup S_1 \cup \cdots \cup S_n$ contains a closed curve α' homotopic to a lift of α. Define $F = S_0 \cup S_1 \cup \cdots \cup S_n$, $F_0 = S_0 \cup \cdots \cup S_{n-1}$ and $F_1 = S_1 \cup \cdots \cup S_n$ (provided $n > 0$). Then property 2b follows from the classical Freiheitssatz of Magnus for one-relator groups, using the fact that α' cannot be homotoped into F_0 or F_1. The remaining properties are clear from the construction.

For the case $n = 0$ we adapt the construction slightly as follows: F_0 and F_1 are annuli which are regular neighbourhoods in $\tilde{S}$ of the two boundary compnents of S_0 (with $F_0 \cong F_1$ via a covering transformation), and $F = F_0 \cup S_0 \cup F_1$. □

3 Results using Hempel's trick

In this section we list some results which follow easily from Hempel's trick. The first two were proved by Hempel in [3].

Theorem 3.1 [3, Theorem 2.2] *Let S be an oriented surface and α an essential closed curve in S. Then the following are equivalent:*

1. *α is not homotopic to β^m for any curve β and any integer $m > 1$;*
2. *$\pi_1(S)/\alpha$ is locally indicable;*
3. *$\pi_1(S)/\alpha$ is torsion-free.*

Theorem 3.2 [3, Corollary 2.4] *Let S be an oriented surface and α a closed curve in S. Then each lift of α to the regular covering corresponding to the normal closure of α in $\pi_1(S)$ is (homotopic to) a simple closed curve.*

Corollary 3.3 *If α is homotopic to β^m in $\pi_1(S)$ for some curve β and integer $m \geq 1$, then β has order m in $\pi_1(S)/\alpha$.*

Proof. Clearly $\beta^m = 1$ in $\pi_1(S)/\alpha$. On the other hand, β^m lifts to a simple closed curve (up to homotopy) in the covering correpsonding to the normal closure N os α in $\pi_1(S)$, so for $0 < k < m$, β^k does not lift to a closed curve. In other words, $\beta^k \notin N$. □

The next result answers Question 1, and generalises the result of Magnus that was mentioned in the introduction.

Theorem 3.4 *Let S be an oriented surface and α, β two closed curves in S whose normal closures in $\pi_1(S)$ coincide. Then α is freely homotopic to $\beta^{\pm 1}$.*

Proof. For this we use the proof, as well as the statement, of Proposition 2.1. If either of α, β is nullhomotopic, then clearly so is the other, so we may assume that both α and β are essential. Let γ be a simple closed curve in S such that $\langle \alpha, \gamma \rangle = 0$ (and hence also $\langle \beta, \gamma \rangle = 0$, since the normal closures of α and β in $\pi_1(S)$ coincide). In the notation of Proposition 2.1, suppose that $F = S_0 \cup \cdots \cup S_n$ contains a closed curve α' homotopic to a lift of α, and that n is minimal with this property.

Similarly, suppose that $F' = S_0 \cup \cdots S_{n'}$ contains a closed curve β' homotopic to a lift of β, and that n' is minimal with respect to this property. Suppose that $n' < n$. Then $F' \subset F_0$, and $\pi_1(F_0)$ embeds into $\pi_1(S)/\alpha$. Hence β', and hence also β, must be nullhomotopic, contrary to assumption. Hence $n' \geq n$. By a symmetric argument $n \geq n'$, so $n = n'$ and $F' = F$.

Moreover, $\alpha' = 1$ in $\pi_1(F)/\beta'$, since $\alpha = 1$ in $\pi_1(S)/\beta$ whch is an HNN extension of $\pi_1(F)/\beta'$. Similarly, $\beta' = 1$ in $\pi_1(F)/\alpha'$. Using Magnus' original theorem for one-relator groups, [7], we see that α' is conjugate in the free group $\pi_1(F)$ to β' or its inverse. Hence α is conjugate in $\pi_1(S)$ to β or its inverse, as claimed. □

The next generalises a result of Dyer and Vasquez [2] for ordinary one-relator groups, and of Papkyriakopoulos [13] for certain one-relator surface groups.

Theorem 3.5 *Let S be an oriented surface and α an essential closed curve in S. Suppose that $\alpha = \beta^m$ in $\pi_1(S)$, with m maximal. Then the space formed by attaching a $K(\mathbb{Z}_m, 1)$-space X_m to S by identifying β with a curve in X_m that generates $\pi_1(X_m)$ is a $K(\pi_1(S)/\alpha, 1)$-space.*

(Note that, in the case $m = 1$ of the theorem, we may take X_m to be a disc, whose boundary is the curve to be identified with $\beta = \alpha$. In other words, the theorem says that the space formed by attaching a 2-cell to S along a non-power essential curve α is aspherical.)

Proof. By Proposition 2.1, there is a surface F with homeomorphic subsurfaces F_0 and F_1, such that S is homotopy equivalent to the double mapping cylinder Y formed from F and $F_0 \times [0, 1]$ by identifying $F_0 \times \{0\}$ with $F_0 \subset F$ and $F_0 \times \{1\}$ with $F_1 \subset F$. By the theorem of Dyer and Vasquez [2], $Z := F \cup_\beta X_m$ is aspherical. Since F_0, F_1 and $F_0 \times [0, 1]$ are aspherical, and the inclusion-induced maps $F_0 \to Z$, $F_1 \to Z$ are π_1-injective, it follows from a theorem of Whitehead [16] that Y is aspherical, as claimed. □

Arguing as in [2], we deduce from this an analogue of Lyndon's Identity Theorem [5], and the resulting structure of the (co-) homology of $\pi_1(S)/\alpha$.

Corollary 3.6 *Let S, α, β and m be as in the Theorem. Let $G = \pi_1(S)/\alpha$, let N be the normal closure of α in $\pi_1(S)$, and C the cyclic subgroup of G generated by β (which has order precisely m, by Corollary 3.3). Then $N/[N, N] \cong \mathbb{Z}G \otimes_{\mathbb{Z}C} \mathbb{Z} \cong \mathbb{Z}(G/C)$ as a (left) $\mathbb{Z}G$-module.*

Proof. Let K denote the $K(G, 1)$-space constructed in the theorem. Then its universal cover $\tilde{K}$ is constructed from the regular cover S_N of S corresponding to N by attaching copies of the universal cover of X_m, one for each left coset of C in G. There is a long exact sequence

$$\cdots \to H_k(S_N) \to H_k(\tilde{K}) \to H_k(\tilde{K}, S_N) \to H_{k-1}(S_N) \to \cdots$$

in which $H_k(\tilde{K}) = 0$ for $k \geq 1$ by the Theorem, and

$$H_k(\tilde{K}, S_N) \cong \mathbb{Z}\,(G/C) \otimes_{\mathbb{Z}} H_k(\tilde{X}_m, S^1) \cong \mathbb{Z}\,(G/C) \otimes_{\mathbb{Z}} H_{k-1}(S^1)$$

for $k \geq 2$ since X_m is aspherical. Hence

$$N/[N,N] \cong H_1(S_N) \cong H_2(\tilde{K}, S_N) \cong \mathbb{Z}\,(G/C) \otimes H_1(S^1) \cong \mathbb{Z}\,(G/C)$$

as claimed. □

Corollary 3.7 *Let G and C be as in the previous corollary, and M a left $\mathbb{Z}\,G$-module. Then for each $q > 2$ there are isomorphisms $H_q(G, M) \cong H_q(C, M)$ and $H^q(G, M) \cong H^q(C, M)$.*

Combining the above corollary with a theorem of Serre [4] yields further consequences:

Corollary 3.8 *Let G and C be as in the previous corollary, and let H be a finite subgroup of G. Then there is a unique double coset HgC such that $H \subseteq gCg^{-1}$.*

Corollary 3.9 *Let G and C be as in the previous corollary. Then every element of finite order in G belongs to a conjugate of C.*

This last result generalises a theorem of Magnus, Karrass and Solitar [9] for one-relator groups.

We have not yet addressed the oldest results of one-relator group theory, Magnus' Freiheitssatz [6] and his solution of the word problem [8]. The Freiheitssatz for a one-relator group says that any proper subset of the generators, omitting a letter which essentially occurs in the relator, freely generates a free subgroup. Such subgroups are now known as *Magnus subgroups.*

The word problem is the algorithmic problem of deciding whether any given word in the generators represents the identity element of the group. For one-relator groups a stronger proerty is true: one can algorithmically decide whether any given word represents an element of the Maguns subgroup generated by any given recursive subset of the generators. This is called the *generalized word problem* for Magnus subgroups. (In the case of a finite presentation, all subsets of the generators are recursive.)

We will prove the analogues of both these results for one-relator surface groups. In general, this will require some more effort than just applying Proposition 2.1. However, there are special cases of both results which can be immediately deduced from Proposition 2.1.

Proposition 3.10 *Let S be a closed oriented surface, α a closed curve in S, and β a simple closed curve in S such that α is not homotopic to a curve disjoint from β, and that $\langle\alpha,\beta\rangle = 0$. Then $\pi_1(S \setminus \beta) \to \pi_1(S)/\alpha$ is injective.*

Proof. This is immediate from the proof of Proposition 2.1, since $S \setminus \beta \cong \mathrm{Int}(S_0) \subseteq F$ (in the notation of 2.1), and the natural maps $\pi_1(S_0) \to \pi_1(F_0) \to \pi_1(S)/\alpha$ are injective. □

Proposition 3.11 *Let $G = \langle u_1, \ldots, u_{2g} \mid [u_1, u_2] \cdots [u_{2g-1}, u_{2g}]\rangle$, let W be a word in the generators of G such that u_1 appears in W with exponent-sum zero, let N be the normal closure of W in G and let H be the subgroup of G generated by $\{u_2, \ldots, u_{2g}\}$. Then there exists an algorithm which, given a word U in the generators of G, will determine whether or not $U \in NH$; and if so will find the (unique) word V in $\{u_2, \ldots, u_{2g}\}$ such that $U^{-1}V \in N$.*

Again this follows more or less immediately from Proposition 2.1, where β is the closed curve representing u_2. We omit the details, since a stronger result will be proved in the next section.

Combining this last result with the observation that we may algorithmically find an automorphism of G sending W to a word in which u_1 appears with exponent-sum zero, we immediately obtain a solution to the absolute word problem for G/N.

Corollary 3.12 *Let G, and N be as in Proposition 3.11. Then the word problem for G/N is soluble.*

4 Further results

In this section we complete the proofs of the Freiheitssatz and the solution of the generalized word problem for one-relator surface groups. First we prove the Freihietssatz.

Theorem 4.1 *Let S be a closed oriented surface, α a closed curve in S, and β a simple closed curve in S such that α is not homotopic to a curve disjoint from β. Then $\pi_1(S \setminus \beta) \to \pi_1(S)/\alpha$ is injective.*

Proof. The result is trivial if S is a torus, so we may assume that S has genus $g \geq 2$. We may also assume that α is not a proper power in $\pi_1(S)$, since if $\pi_1(S\setminus\beta) \to \pi_1(S)/\alpha$ is injective then so is $\pi_1(S \setminus \beta) \to \pi_1(S)/\alpha^m$ for all m.

We first strengthen the first part of Proposition 2.1 to obtain a simple closed curve γ, disjoint from β, with $\langle\alpha,\gamma\rangle = 0$. Choose a simple closed curve β' that meets β tranversely in a single point. Then a regular neighbourhood N of $\beta\cup\beta'$ is a punctured torus, so $S \setminus N$ is a punctured surface of genus $g - 1 \geq 1$. Take γ to be a simple closed curve in the kernel of the restriction of $\langle\alpha, -\rangle$ to $\pi_1(S \setminus N)$.

Now consider the cover S_K of S corresponding to the kernel K of $\langle -, \gamma\rangle : \pi_1(S) \to \mathbb{Z}$. Let A be a small regular neighbourhood of β in S, such that each component of $A\cap\alpha$ is an embedded arc joining the two components of ∂A. Then A is an annulus. Moreover, $\langle\beta,\gamma\rangle = 0$, so A lifts to an infinite collection of annuli A_n ($n \in \mathbb{Z}$) in S_K, such that $A_{n+1} = \tau(A_n)$, where τ is a generator of the covering transformation group. Let T denote the preimage in S_K of $S \setminus A$, so that $T = S_K \setminus (\bigcup_{n\in Z} A_n)$.

Since $\langle\alpha,\gamma\rangle = 0$, α lifts to an infinite collection $\{\alpha_n, n \in \mathbb{Z}\}$ of closed curves in S_K, where $\alpha_{n+1} = \tau(\alpha_n)$. Now α_0 intersects a nonzero finite number of the A_n. Let λ, μ denote the least and greatest indices n such that $\alpha_0 \cap A_n \neq \emptyset$, and assume that α has been isotoped to minimise $\mu - \lambda$.

Define $S_0 = T \cup A_\lambda \cup \cdots \cup A_\mu$, $S_1 = T \cup A_\lambda \cup \cdots \cup A_{\mu-1}$, and $S_2 = T \cup A_{\lambda+1} \cup \cdots \cup A_\mu$. Then $S_1 \cong S_2$ via τ, the inclusion-induced maps $\pi_1(S_i) \to \pi_1(S_0)/\alpha_0$ ($i = 1, 2$) are injective (by Magnus' Freiheitssatz [6]), and $\pi_1(S)/\alpha$ is an HNN-extension of $\pi_1(S_0)/\alpha_0$ with associated subgroups $\pi_1(S_1)$, $\pi_1(S_2)$ and isomporphism $\tau_* : \pi_1(S_1) \to \pi_1(S_2)$.

It follows that $\pi_1(T) \to \pi_1(S)/\alpha$ is injective. Since $\pi_1(T)$ is the kernel of $\langle -,\gamma\rangle : \pi_1(S \setminus \beta) \to \mathbb{Z}$ and since $\langle\alpha,\gamma\rangle = 0$, it also follows that the inclusion-induced map $\pi_1(S \setminus \beta) \to \pi_1(S)/\alpha$ is injective. □

In a similar manner, we can obtain the solution of the generalised word problem for Magnus subgroups in one-relator surface groups.

Theorem 4.2 *Let $G = \langle u_1, \ldots, u_{2g} \mid [u_1, u_2] \cdots [u_{2g-1}, u_{2g}]\rangle$, let W be a word in the generators of G, let N be the normal closure of W in G and let H be the subgroup of G generated by $\{u_2, \ldots, u_{2g}\}$. Then there exists an algorithm which, given a word U in the generators of G, will determine whether or not $U \in NH$; and if so will find the (unique) word V in $\{u_2, \ldots, u_{2g}\}$ such that $U^{-1}V \in N$.*

Proof. We follow the proof of Theorem 4.1, letting α be the closed curve represented by W (up to isotopy), and β the simple closed curve represented by u_2.

In order to find γ we replace the pair of generators $\{u_3, u_4\}$ by another basis $\{u_3', u_4'\}$ of $\langle u_3, u_4\rangle$, such that, on rewriting W in terms of the new generators

$$\{u_1, u_2, u_3', u_4', u_5, \ldots, u_{2g}\},$$

the generator u_3' appears with exponent-sum zero. Then we take γ to be the simple closed curve representing u_4'. Note that this process can be carried out algorithmically - the euclidean algorithm produces a product of elementary matrices which transforms $\{u_3, u_4\}$ to $\{u_3', u_4'\}$ modulo commutators. Realising each elementary matrix by a Nielsen transformation produces an automorphism of $\langle u_3, u_4\rangle$ carrying u_3 to u_3' and u_4 to u_4'. Note also that $[u_3', u_4']$ is conjugate to $[u_3, u_4]^{\pm 1}$ [11]. Indeed, by a further Nielsen transformation if necessary, followed by an inner automorphism of $\langle u_3', u_4'\rangle = \langle u_3, u_4\rangle$, we may assume that $[u_3', u_4'] = [u_3, u_4]$. Thus our automorphism extends to an automorphism of G fixing u_i for $i \neq 3, 4$.

Let us assume that the above algorithmic automorphism has been carried out, so that u_3 appears in W with exponent-sum zero, and we can choose γ to be a curve representing u_4. The homomorphism $\langle -,\gamma\rangle : G \to \mathbb{Z}$ can then be interpreted as the exponent-sum of u_3, and its kernel K is generated by conjugates of u_i ($i \neq 3$) by powers of u_3. Since $W \in K$, we may rewrite W as a word $\tilde{W}$ in these generators. Let λ, μ be the least and greatest indices n respectively such that $u_3^{-n} u_1 u_3^n$ occurs in $\tilde{W}$.

Let G_0 be the one-relator group with generators

$$\{u_3^{-n} u_1 u_3^n; \lambda \leq n \leq \mu\} \cup \{u_3^{-n} u_j u_3^n; n \in \mathbb{Z}, j = 2, 4, 5, \ldots, 2g\}$$

and relator $\tilde{W}$. The proof of Theorem 4.1 then expresses G/N as an HNN-extension of G_0, in which the associated subgroups are the Magnus subgroups obtained by omitting $u_3^{-\mu} u_1 u_3^{\mu}$ and $u_3^{-\lambda} u_1 u_3^{\lambda}$ respectively from the generating set. By the solutions of the generalised word problems for one-relator groups and for HNN-extensions, it is decidable whether or not the generator u_1 may be eliminated from U in G/N, as required.

[Note that, while the proof of Theorem 4.1 makes use of the assumption that W is not a proper power in G, the HNN-construction of G/N described there does not depend on that assumption. We may therefore use it in full generality for the purposes of the present proof.] $\square$

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