

ONE-RELATOR PRODUCTS INDUCED FROM GENERALIZED TRIANGLE GROUPS

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ABSTRACT. We study the quotient of a free product of groups by the normal closure of a word that is contained in a certain type of 2-generator subgroup, and hence can be expressed in the form of a pushout involving a generalized triangle group. Using the theory of generalized triangle groups, we extend the range of conditions under which a number of results for one-relator products of groups, such as the Freiheitssatz and the solubility of the word problem, are known to hold.

1. INTRODUCTION

A *one-relator product* of two groups G_1, G_2 is the quotient group of the free product $G_1 * G_2$ by the normal closure of a single element, or *relator*, that is assumed not to be conjugate to an element of G_1 or of G_2 . Under suitable conditions on G_1 and G_2 and/or on the relator, one can prove analogues of classical theorems from one-relator group theory such as the Freiheitssatz [16] and Lyndon's Identity Theorem [14]. See [3] for a survey of results in this direction. One example of such a condition is that the relator be a fourth or higher power [11, 12, 13]. Another is that the relator be a third power that does not involve a letter of order 2 [2]. The general case of a third-power relator remains an open problem. In this paper we begin an attack on the problem by considering a special class of relator that is contained in a subgroup of $G_1 * G_2$ of the form $A * B$, where each of A, B is a cyclic subgroup of a conjugate of G_1 or G_2 . (For technical reasons we will restrict our attention to the case where either $A \cap B = \{1\}$ in $G_1 * G_2$, or A and B are subgroups of another cyclic subgroup C of $G_1 * G_2$.)

In another direction, recent work of Edjvet, Juhász and Shwartz [4, 7, 8, 17, 18, 19] shows that the Freiheitssatz often holds for short relations.

Now a one-relator product of (finite) cyclic groups, in which the relator is a power, is called a *generalized triangle group*. This terminology was introduced by Baumslag, Morgan and Shalen [1], and there is now a large body of knowledge about such groups (see for example [9] for a recent survey).

In our situation, we say that our one-relator product is *induced* from a generalized triangle group. It can be realised as a pushout of groups

$$(1.1) \quad \begin{array}{ccc} C_p * C_q & \longrightarrow & H \\ \downarrow & & \downarrow \\ G_1 * G_2 & \longrightarrow & G \end{array}$$

2000 *Mathematics Subject Classification*. Primary 20F05; Secondary 20F10; 20F65, 57M07.
The second named author was supported by Marie Curie Fellowship HPMF-CT-1999-00172.

where H denotes the corresponding generalized triangle group. (Note that, for the purposes of this paper, we relax the definition of generalized triangle group somewhat by allowing one of the cyclic factors to be infinite.)

One-relator products induced from generalized triangle groups have already cropped up (albeit not under that name) as critical cases requiring special consideration when proving general results. For example, one-relator groups referred to as ‘type $E(p, q, r)$ ’ in [11, 12, 13] (see also [3]) are induced from the triangle groups $T(p, q, r)$. The arguments in [12] involve detailed analysis of one-relator products ‘of type $E(2, 3, 4)^+$ ’, which are induced from generalized triangle groups of the form $\langle x, y \mid x^2 = y^3 = ((xyxy^2)^t xy)^4 \rangle$, $t \geq 1$. The ‘type E ’ condition also occurs in [10] as an obstruction to the straightforward calculation of Euler characteristics of generalized triangle groups.

The general philosophy is that we aim to reduce general problems about one-relator products induced from generalized triangle groups to those for the generalized triangle groups themselves, which may be more tractable since more is known about generalized triangle groups, and a greater array of techniques is available for their study.

More specifically, we will show that, for a one-relator product induced from a generalized triangle group in which the relator is at least a third power, the pushout (1.1) is *geometrically Mayer-Vietoris*, in the sense of [5]. This means that it provides a blueprint for the construction of an Eilenberg-MacLane space of type $K(G, 1)$, which in turn gives rise to Mayer-Vietoris sequences in homology and cohomology.

Our main results are as follows. Define a pair (a, b) of elements of $G_1 \cup G_2$ to be *admissible* if either they have a common root in G_1 or G_2 , or they do not have a common power in $G_1 * G_2$ other than the identity.

Theorem 1. *Let G be a one-relator product of the form $(G_1 * G_2)/N(R^r)$, where $r \geq 3$ and R is a word $W(a, U^{-1}bU)$ for some words U, W and admissible pair (a, b) of letters. Then*

- (1) *The natural maps $H \rightarrow G, G_1 \rightarrow G, G_2 \rightarrow G$ are all injective.*
- (2) *The pushout diagram (1.1) is geometrically Mayer-Vietoris.*

Theorem 2. *Let G be as in Theorem 1, and let H denote the generalized triangle group $\langle x, y \mid x^p = y^q = W(x, y)^r = 1 \rangle$, where p and q are the orders of a and b respectively. If the word problem is soluble for H, G_1 and G_2 then it is also soluble for G .*

Theorem 3. *Let G be a one-relator product of the form $(G_1 * G_2)/N(R^r)$, where $r \geq 3$ and R is a word $W(a, U^{-1}bU)$ for some words U, W and letters a, b such that $a^2 = 1$. Suppose that some cyclic permutation of R^r has the form $V_1 V_2$, where $0 < \ell(V_1) < \ell(R^r)$. Then $V_1 \neq 1$ in G .*

The remainder of the paper is organised as follows. In Section 2 below we recall some definitions and properties about words in free monoids, which we can apply to pictures over one-relator products. In Section 3 we discuss how certain one-relator products can be expressed as pushouts, and the implications of such expressions. The key facts about pictures are recalled in Section 4, and in Section 5 we introduce the notion of cliques in a picture, and of a clique-picture. In Section 6 we classify all cliques with short labels, using a spelling theorem for generalized triangle groups from [6]. In Section 7 we pull all the preliminary results together to prove our principal result, namely that clique-pictures over one-relator products

induced from generalized triangle groups with relator at least a third power satisfy the small-cancellation condition $C(6)$. Finally, we use this in Section 8 to prove the Freiheitssatz and other results for such one-relator products.

2. WORDS IN FREE MONOIDS

In this section we collect together some basic definitions and results on words in free monoids, which we use frequently in what follows.

Let A be an alphabet, and A^* the free monoid on A . We will often assume that A comes equipped with an involution, $a \mapsto \bar{a}$, possibly with fixed points. For the applications later in this paper, A will usually be the union of groups $G_1 \cup G_2$, and the involution will be group inverse.

We will use the notation \equiv to denote equality of words in A^* , as distinct from words that represent the same element of some quotient group.

The *length* of a word $W = a_1 \cdots a_n \in A^*$ is the integer $n \geq 0$, and is denoted $\ell(W)$. The *involute* of W is $\bar{W} = \bar{a}_n \cdots \bar{a}_1$. The word W has *period* λ if $a_i = a_{i+\lambda}$ whenever $1 \leq i \leq \ell(W) - \lambda$.

A *subword*, or *segment* of a word W is a word consisting of a collection of consecutive letters from W : $a_i a_{i+1} \cdots a_j$ for some $1 \leq i \leq j \leq n$. (Note that we do not allow the empty word as a subword.) Subwords of particular interest are the initial and terminal segments: the *initial segment* of W of length $m \leq n$ is $a_1 \cdots a_m$; the *terminal segment* of W of length m is $a_{n-m+1} \cdots a_n$.

The following is an easy observation and we omit the proof.

Lemma 2.1. *Suppose that x, y are letters and V a word such that $xV \equiv \bar{V}y$. Then either*

- (1) V is empty and $x = y$; or
- (2) $y = \bar{x}$ and $\bar{V} \equiv V$.

A word W of period λ has the form P^n with $\ell(P) = \lambda$ ($< \ell(W)$) if $\ell(W) = n\lambda$. In this case we say that W is a *proper power*. Otherwise, W has the form $(PQ)^n P$ for some n , where $\ell(PQ) = \lambda$. If W is self-involutive ($W = \bar{W}$), then so are P and Q in the above.

Let $W = a_1 a_2 \cdots a_n \in A^*$. A *cyclic permutation*, or *cyclic conjugate* of W is a word $a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$, where σ is some power of the permutation $(1, 2, \dots, n)$. Thus W has $n = \ell(W)$ cyclic permutations, including W itself. A cyclic permutation arising from a non-identity permutation σ is called *proper*. The word W is equal to a proper cyclic permutation of itself if and only if W is a proper power.

We will also require the following observation.

Lemma 2.2. *Let A, B, C be words, and x, y letters, such that $\bar{A} \equiv A$, $\bar{B} \equiv B$ and $AB \equiv \bar{C}xCy$. Then there is a word W such that $\bar{W} \equiv W$ and each of A, B can be expressed as a word in $\{W, y, \bar{y}\}$.*

Proof. By hypothesis, $AB \equiv \bar{C}xCy$ has length $2\ell(C) + 2 = 2n$, say. Write

$$AB \equiv z_1 z_2 \cdots z_{2n}$$

and suppose that $\ell(A) = k$. Then $\bar{C} \equiv z_1 z_2 \cdots z_{n-1}$ and $C \equiv z_{n+1} \cdots z_{2n}$, so $\bar{z}_i = z_{2n-i}$ for $1 \leq i \leq 2n-1$, $i \neq n$. Also, since $\bar{A} \equiv A$ and $\bar{B} \equiv B$, we have $\bar{z}_i = z_{k+1-i}$ for all i (where we interpret indices modulo $2n$, for example $z_{-1} = z_{2n-1}$).

If $k = 0$, $k = 2n - 1$ or $k = 2n$, then the conclusions of the Lemma hold with $W \equiv B$, $W \equiv A$ or $W \equiv A$ respectively. Hence we may assume that $1 \leq k \leq 2n - 2$. Comparing the two expressions for \bar{z}_i above, we see that $z_j = z_{j-k-1}$ (indices interpreted modulo $2n$) for all j other than n and $2n$. Let $\mu = \text{hcf}(k+1, 2n)$ and define $W \equiv z_1 z_2 \cdots z_{\mu-1}$, the initial segment of AB of length $\mu - 1$. By the euclidean algorithm, $\mu = \alpha(k+1) + \beta(2n)$ for some integers α and β , and it follows that $z_j = z_{j+\mu}$ for all $j \not\equiv 0 \pmod{\text{hcf}(n, \mu)}$.

There are two possibilities to consider. If $2n/\mu$ is even, then $\mu \mid n$ and $z_j = z_{j+\mu}$ for all $j \not\equiv 0 \pmod{\mu}$. In particular,

$$W \equiv z_{2n-\mu+1} z_{2n-\mu+2} \cdots z_{2n-1} \equiv \bar{W}.$$

Moreover, n is congruent, modulo $2n$, to a multiple of $k+1$, so $x = z_n = z_{n+k+1} = \cdots = z_{2n-k-1} = \bar{z}_{2n} = \bar{y}$. Hence $z_{t\mu} \in \{y, \bar{y}\}$ for all t . Finally,

$$A \equiv W z_\mu W \cdots z_{k+1-\mu} W, \text{ and } B \equiv z_{k+1} W \cdots W z_{2n}.$$

Secondly, if $2n/\mu$ is odd, then μ is even, and n is congruent, modulo $2n$, to an odd multiple of $(k+1)/2$. Hence

$$x = z_n = z_{n+k+1} = \cdots = z_{2n-(k+1)/2} = z_{(k+1)/2} = \bar{z}_{2n-(k+1)/2} = \bar{x}.$$

By an exponent-sum count, it follows also that $\bar{y} = y$. Hence $z_{t\mu} = y = \bar{y}$ for all t , while $z_{t\mu/2} = x = \bar{x}$ for all odd t . Also, $\bar{z}_i = z_{2n-i} = z_{\mu-i}$ for $i = 1, \dots, \mu - 1$, $i \neq \mu/2$, and $\bar{z}_{\mu/2} = x = z_{\mu/2}$, so $\bar{W} = W$, $A \equiv (Wy)^{\frac{k+1}{\mu}-1} W$ and $B \equiv y(Wy)^{\frac{2n-k-1}{\mu}-1}$. \square

A *cyclic subword* of W is any initial segment of a cyclic permutation of W . A cyclic subword of W is *uniquely positioned* (cf. [20]) if it is equal to an initial segment of precisely one of the $\ell(W)$ cyclic permutations of W . (In particular, this is possible only if W is not a proper power.)

Theorem 2.3. (Weinbaum [20].) *If W is any word with $\ell(W) > 1$ that is not a proper power, then some cyclic permutation of W has the form UV with U and V uniquely positioned subwords of W .*

Suppose that $W \equiv R^r$ with R not a proper power and $\ell(R) > 1$. A *piece* of W is a cyclic subword of R that is either not uniquely positioned, or that is also equal to a cyclic subword of \bar{R} when \bar{R} is not a cyclic permutation of R . (When R and \bar{R} are cyclic permutations of each other, then a piece is just a non-uniquely positioned cyclic subword of R .) A *non-piece* is a cyclic subword of R that is not a piece, in other words a uniquely positioned cyclic subword that is not a cyclic subword of \bar{R} (unless \bar{R} is a cyclic permutation of R).

If the word R contains no fixed points of the involution $a \mapsto \bar{a}$, then the above result of Weinbaum can be strengthened ([2]): U and V can be chosen to be non-pieces.

If W is a word and a a letter, we denote the number of occurrences of a in W by $\nu(a, W)$, and define the *exponent-sum* of a in W by $\epsilon(a, W) = \nu(a, W) - \nu(\bar{a}, W)$. Both $\nu(a, W)$ and $\epsilon(a, W)$ are invariant under taking cyclic permutations. We will often use occurrence counts and exponent-sum counts without further explanation. For example, if we know that words Ux and Uy are cyclic permutations of one another (where x and y are letters, then an occurrence count immediately shows that $x = y$).

3. PUSHOUTS

Consider a group G given by a one-relator product description

$$G = (G_1 * G_2) / N(R^r),$$

where R is a cyclically reduced free product word in $G_1 * G_2$ of length at least 2, $r \geq 2$ and $N(R^r)$ denotes the normal closure of R^r in $G_1 * G_2$.

If the word R is contained in a subgroup $A * B$ of the free product $G_1 * G_2$, where each of A, B is a cyclic subgroup of a conjugate of G_1 or G_2 , then we say that G is *induced from a generalized triangle group*. Taking S, T to be generators of these cyclic groups, and p, q the orders of S, T respectively in $G_1 * G_2$, we construct the generalized triangle group

$$H = \langle x, y \mid x^p = y^q = R'(x, y)^r \rangle,$$

where the word R' satisfies $R'(S, T) \equiv R$ in $G_1 * G_2$. (Note that we allow $p = \infty$ and/or $q = \infty$ here.)

The group G is then defined by the pushout diagram (1.1).

We also refer to the pushout (1.1), or the choice of U, V , as a *generalized triangle group description* of G .

The 2-generator subgroup $A * B$ need not be unique. It can happen that two or more such two-generator subgroups of $G_1 * G_2$ contain the word W . If $A * B, A' * B'$ are two such subgroups, with $A * B \subseteq A' * B'$, and $\{S, T\}, \{S', T'\}$ are suitable generating sets for $A * B, A' * B'$ respectively, then S, T can be expressed as words in S', T' , say $S = U(S', T'), T = V(S', T')$. If $R = R'(S, T) = R''(S', T')$, then R'' is obtained from R' by replacing the letters S, T by the words U, V respectively. (Hence R'' is a longer word than R' . Equivalently, S' and T' can be chosen with $\ell(S') + \ell(T') < \ell(S) + \ell(T)$.) In this case we say that the generalized triangle group description of G given by R'' is a *refinement* of that given by R' .

Let p', q' be the orders of S', T' respectively, and H' be the generalized triangle group $\langle x, y \mid x^{p'} = y^{q'} = R''(x, y)^r = 1 \rangle$. Then this refinement can be expressed as the existence of a commutative diagram

$$(3.1) \quad \begin{array}{ccc} C_{p'} * C_{q'} & \longrightarrow & H' \\ \downarrow & & \downarrow \\ C_p * C_q & \longrightarrow & H \\ \downarrow & & \downarrow \\ G_1 * G_2 & \longrightarrow & G \end{array}$$

in which both squares are pushouts.

If a generalized triangle group description of G has no refinement, then we say it is *maximal*. This corresponds to $A * B$ being maximal among all subgroups of $G_1 * G_2$ that are generated by 2 elements of conjugates of G_1 or G_2 and contain R . Since refinement reduces the total length of the generating set, any generalized triangle group description for G has a maximal refinement.

We will be particularly interested in generalized triangle group descriptions for which one of p, q is equal to 2. If, in the refinement (3.1), we have $p' = 2$ and/or $q' = 2$, then we say that (3.1) is a 2-refinement. A generalized triangle group description (1.1) is called 2-maximal if it has no 2-refinement. As above, any

generalized triangle group description with $p = 2$ and/or $q = 2$ has a 2-maximal 2-refinement.

We will show, among other things, that when $r \geq 3$ the pushout (1.1) is *geometrically Mayer-Vietoris*, as defined in [5]. This means that we can construct an Eilenberg-MacLane space $K(G, 1)$ as a double-mapping cylinder of maps $K(C_p * C_q, 1) \rightarrow K(G_1 * G_2, 1)$ and $K(C_p * C_q, 1) \rightarrow K(H, 1)$ inducing the given maps on fundamental groups. The term Mayer-Vietoris is used because the topological set-up then provides Mayer-Vietoris sequences linking the (co-)homology of G with that of $C_p * C_q$, $G_1 * G_2$ and H . (See [5] for details.)

4. PICTURES

In this section we recall the basics about pictures over a one-relator product.

A *picture* Γ over G on an oriented surface Σ (usually D^2) consists of the following: a collection of disjoint closed discs in the interior of Σ called *vertices*; a finite number of disjoint arcs, each of which is either: (i) a simple closed curve in the interior of Σ that meets no vertex, (ii) an arc joining two vertices (or one vertex to itself), (iii) an arc joining a vertex to the boundary $\partial\Sigma$ of Σ , or (iv) an arc joining $\partial\Sigma$ to $\partial\Sigma$; a collection of *labels*, one at each corner of each *region* of Γ (i.e. connected component of the complement in Σ of the arcs and vertices) at a vertex, and one along each component of the intersection of the region with $\partial\Sigma$.

Each label of Γ is an element of $G_1 \cup G_2$.

Reading the labels round a vertex in the *clockwise* direction yields $R^{\pm r}$ (up to cyclic permutation), as a cyclically reduced word in $G_1 * G_2$.

A region is a *boundary* region if it meets $\partial\Sigma$, and an *interior* region otherwise.

If $\Sigma = S^2$ or if $\Sigma = D^2$ and no arcs of Γ meet ∂D^2 then Γ is called *spherical*. In the latter case ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that Γ contains at least one vertex or arc), which is called the *distinguished* region. All other regions are *interior*.

The labels of any region Δ of Γ are required all to belong to either G_1 or G_2 . We talk of G_1 -regions and G_2 -regions accordingly. Each arc is required to separate a G_1 -region from a G_2 -region. Observe that this is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from G_1 and G_2 . A region bounded by arcs that are closed curves will have no labels; nevertheless the above convention requires that it be designated a G_1 - or G_2 -region. An important rule for pictures is that the labels within any G_1 -region (respectively G_2 -region) allow the solution of a quadratic equation in G_1 (respectively G_2). The labels around any given boundary component of the region are formed into a single word reading anti-clockwise. The resulting collection of elements of G_1 or G_2 is required to have genus no greater than that of the region (in the sense of [3]). This technical general requirement is much simpler in the commonest case of a simply connected region - it means merely that the resulting word represents the identity element in G_1 or G_2 .

The *boundary label* of a picture Γ on D^2 is the word obtained by reading the labels on ∂D^2 in an *anticlockwise* direction. This word represents the identity element of G . It may be assumed to be cyclically reduced as a word in $G_1 * G_2$. If Γ is spherical, then the boundary label is an element of $G_1 \cup G_2$ determined by the other labels in the distinguished region.

Two distinct vertices of a picture are said to *cancel* along an arc e if they are joined by e and if their labels, read from the endpoints of e , are mutually inverse words in $G_1 * G_2$.

The point is that such vertices can be removed from a picture via a sequence of so-called *bridge moves*, followed by deletion of a *dipole* (see below) without changing the boundary label. This gives an alternative picture with the same boundary label and two fewer vertices. We refer the reader to [11] for details on bridge moves, but the basic idea is as follows. Let γ denote an arc in the surface Σ , that meets the picture Γ only in its endpoints, which are interior points of arcs of Γ . A bridge move is the result of altering Γ by surgery along γ . It is allowed provided that the resulting picture satisfies the above rules concerning labels within a region. The most common example of this in practice is that γ divides a simply-connected region (say a G_1 -region) into two parts. The requirement for a bridge move is then that, in the resulting subdivision of the region-label into two subwords, each of the two subwords represent the identity in G_1 .

We say that a picture is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of G occurs as the boundary label of some reduced picture on D^2 . A picture is *connected* if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed curve or joins two points of $\partial \Sigma$, unless the picture consists only of that arc. A *dipole* is a connected spherical picture containing precisely two vertices. It is easy to check that the vertices of a dipole cancel. A connected component Γ_0 of a picture Γ is a *dipole* if it contains precisely two vertices, does not meet $\partial \Sigma$, and none of its interior regions contain other components of Γ .

If the Freiheitssatz holds for G (in other words, the natural maps $G_1 \rightarrow G$ and $G_2 \rightarrow G$ are injective), then every spherical picture has boundary label the identity element of G_1 or of G_2 . Any such spherical picture represents an element of the second homotopy group $\pi_2(X)$, where X is a space formed from an Eilenberg-MacLane space $K(G_1 * G_2, 1) = K(G_1, 1) \vee K(G_2, 1)$ by attaching a 2-cell along a path representing R^r . Conversely, any element of $\pi_2(X)$ can be represented by a spherical picture with trivial boundary label.

Two arcs of a picture Γ are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region Δ of Γ . We will also use the term *parallel* to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a *class of parallel arcs*. Given a parallel class of $n > 1$ arcs joining vertices u and v of a picture, consider the $n - 1$ two-sided regions separating these arcs. Each such region has a corner label x_u at u and a corner label x_v at v , and the picture axioms imply that $x_u x_v = 1$ in G_1 or G_2 . The $n - 1$ corner labels at u spell a cyclic subword P of length $n - 1$ of the

relator or its inverse. Similarly the corner labels at v spell out a cyclic subword P' of length $n - 1$. Moreover, $P' \equiv P^{-1}$. If we assume that the picture is reduced, then u and v do not cancel, so the cyclic permutations of $R^{\pm r}$ of which P and P' are initial segments are not equal. Hence P and P' are pieces.

We say that a vertex u of a picture satisfies the local $C(n)$ condition if it is joined to at least n neighbours in the picture.

We say that a picture satisfies $C(n)$ if every interior vertex satisfies local $C(n)$.

Note that, if some interior vertex fails to satisfy $C(n)$, then some cyclic permutation of the relator can be written in the form

$$P_1 x_1 \cdots P_k x_k$$

with $k < n$, where the P_i are pieces and the x_i are letters.

5. CLIQUES IN PICTURES

Suppose that $G = (G_1 * G_2)/N(W^r)$ is a one-relator group induced from a generalized triangle group $H = \langle x, y \mid x^p = y^q = W'(x, y)^r \rangle$, via words $S, T \in G_1 * G_2$ of orders p, q respectively, conjugate to elements of $G_1 \cup G_2$.

Since $W = W'(S, T)$ is cyclically reduced, we can without loss of generality assume that there are letters $a, b \in G_1 \cup G_2$ and a word $U \in G_1 * G_2$, such that $S \equiv UaU^{-1}$, $T = b$ and $ST \equiv UaU^{-1}b$ is reduced.

Lemma 5.1. *Suppose that the above generalized triangle group description for G is maximal (or 2-maximal, if $p = 2$ or $q = 2$), and the (a, b) is an admissible pair. Then for $s, u \not\equiv 0 \pmod p$ and $t, v \not\equiv 0 \pmod q$, the word $Ua^sU^{-1}b^t$ cannot be a proper cyclic conjugate of $Ua^uU^{-1}b^v$ in $G_1 * G_2$.*

Proof. If $Ua^sU^{-1}b^t$ is a proper cyclic conjugate of $Ua^uU^{-1}b^v$, then $\{a^s, b^t\} = \{a^u, b^v\}$ by an occurrence count. If $a^s = a^u$, then $b^t = b^v$, and $Ua^sU^{-1}b^t$ is a proper cyclic conjugate of itself, hence a proper power. Say $Ua^sU^{-1}b^t \equiv (Vx)^n$ for some word V and letter x . But then Vx is an initial segment of Ua^s and xV is a terminal segment of a^sU^{-1} , so $V \equiv V^{-1}$ and (if $n > 2$) $x = x^{-1}$, or equivalently $V^2 = x^2 = 1$. Moreover, $x = b^t$, so $b^{2t} = 1$. $W'(S, T)$ is conjugate to a word in V and b , leading to a 2-refinement of the generalized triangle group description of G , contrary to the assumption of (2-) maximality.

If $n = 2$ then $x^2 \neq 1$ in general, but $x = a^s = b^t$, so by admissibility there is a letter c such that a and b are powers of c . Hence $W \equiv W'(V, c)$ for some word W' . Since $V^2 = 1$, this is a 2-refinement, contrary to hypothesis.

On the other hand, if $a^s \neq a^u$, then $a^s = b^v$, and $b^t = a^u$. Moreover, $Ua^sU^{-1}a^u$ is a proper cyclic conjugate of $Ua^uU^{-1}a^s$. By an exponent-sum count this is possible only if $s + u \equiv 0 \pmod p$, so $Ua^sU^{-1}b^t$ is a cyclic conjugate of its own inverse, and thus has the form AB where $A^2 = B^2 = 1$. By Lemma 2.2 it follows that

$$Ua^sU^{-1}b^t \equiv (Wa^{\varepsilon(1)s}) \cdots (Wa^{\varepsilon(k)s})$$

for some word W with $W^2 = 1$, some natural number k and some $\varepsilon(i) = \pm 1$. Moreover, if k is odd, then a^s is the middle letter of W so has order 2, a contradiction, since $a^{-s} = a^t \neq a^s$. Hence U is a word in W and a^s . Since (a, b) is an admissible pair, there exists a letter c such that a and b are both powers of c . Hence U is a word in W and c , giving a 2-refinement of the relator $(UaU^{-1}b)^r$, again a contradiction. \square

If u, v are two vertices in a picture over G that are joined by an arc e , then we may use the endpoints of e as the starting points for reading the labels L_u and L_v of u and v respectively. In each case the label is a cyclic permutation of $W'(S, T)^{\pm r}$. We may assume, without loss of generality, that the word $W'(x, y)$ begins with the letter x . Choose a cyclic permutation $W''(x, y)$ of $W(x, y)^{-1}$ that also starts with x .

Now each of L_u and L_v^{-1} is a cyclic permutation of $W'(S, T)^r$ or of $W''(S, T)^r$, say

$$L_u = YZ, \text{ where } ZY = W'(S, T)^r \text{ or } ZY = W''(S, T)^r,$$

$$L_v^{-1} = Y'Z', \text{ where } Z'Y' = W'(S, T)^r \text{ or } Z'Y' = W''(S, T)^r.$$

We define $u \sim v$ if and only if $\ell(Y') \equiv \ell(Y) \pmod{\ell} = \ell(UaU^{-1}b)$. Although the subwords Y and Y' are not uniquely defined, it follows from Lemma 5.1 that their lengths are unique modulo ℓ , and so the relation \sim is well-defined.

The point of the relation \sim is that, when $u \sim v$, then the 2-vertex subpicture consisting of u and v , joined by e and any arcs parallel to e , has boundary label a word in $\{S, T\}$, after cyclic reduction and cyclic permutation. (Indeed, the cyclic reduction of the label can be achieved by performing bridge moves to make the number of edges parallel to e be a multiple of $\ell/2$.)

Now let \equiv denote the transitive, reflexive closure of \sim . Then \equiv is an equivalence relation on vertices. After a sequence of bridge moves, we may assume that arcs joining equivalent vertices do so in parallel classes each containing a multiple of $\ell/2$ arcs. Define a *clique* to be the subpicture consisting of any \equiv -equivalence class of vertices, together with all arcs between vertices in that \equiv -class (assumed to occur in parallel classes of multiples of $\ell/2$ arcs), and all regions that are enclosed entirely by such arcs.

Our idea is to form a *quotient picture* by contracting each clique to a point, which we regard as a vertex in the quotient. For this to make sense, we require each clique to be *simply connected*, in the sense that it is a picture on a simply-connected surface - ie a topological disc (and hence, in particular, has a single boundary component).

Definition Let G be a one-relator product induced from a generalized triangle group as above, and let Γ be a picture on a surface Σ , such that every clique of Γ is simply-connected. Then the *clique-quotient* of Γ is the picture $\bar{\Gamma}$ formed from Γ by contracting each clique to a point, and regarding it as a vertex.

A *clique-picture* over G is the clique-quotient of some (reduced) picture over G . The label of a vertex in a clique-picture is called a *clique-label*.

As in ordinary pictures, two arcs in a clique-picture are termed *parallel* if they bound a two-sided, simply-connected region. A pair of parallel arcs thus gives rise to an equation $xy = 1$ between letters x and y occurring in two clique-labels. As before, we extend the relation of parallelism to its reflexive, transitive closure, and refer to the corresponding equivalence classes as *parallel classes* of arcs. A parallel class of n arcs thus identifies cyclic subwords of length $n - 1$ of two clique-labels. Any such subword will be called a *clique-piece*, or (where no confusion is caused) simply a *piece*.

We will say that an (interior) vertex of a clique-picture over G satisfies the local $C(n)$ property if it has at least n neighbours. A clique-picture in which every interior vertex satisfies the local $C(n)$ property is said to satisfy $C(n)$.

Note that the vertices of $\bar{\Gamma}$ are labelled, in general by words in $G_1 * G_2$ that are the images under the natural map $\mathbb{Z}_p * \mathbb{Z}_q \rightarrow G_1 * G_2$ of elements of the kernel of $\mathbb{Z}_p * \mathbb{Z}_q \rightarrow H$. Note also that the definition of $\bar{\Gamma}$ depends on the precise generalized triangle group description of G . In practice, where there is ambiguity, we will assume that we are working with a maximal such description that is 2-maximal if either $p = 2$ or $q = 2$, and maximal otherwise.

6. CLIQUES WITH SHORT LABELS

A typical clique-label in a clique-picture over $G = (G_1 * G_2)/N((UaU^{-1}b)^r)$ will have the form (up to cyclic permutation)

$$Ua^{\alpha(1)}U^{-1}b^{\beta(1)}Ua^{\alpha(2)}U^{-1}b^{\beta(2)} \dots Ua^{\alpha(k)}U^{-1}b^{\beta(k)}$$

for some k . It follows from Lemma 5.1 that a vertex with this label satisfies local $C(k)$. We are interested in proving that clique-pictures over G satisfy $C(6)$ whenever $r \geq 3$, so as a first step we need to classify all possible clique-labels of the above type with $k \leq 5$. We do this by means of the spelling theorem for generalized triangle groups from [6].

Theorem 6.1. [6] *Let $H = \langle x, y \mid x^p = y^q = W(x, y)^r \rangle$ be a generalized triangle group, with*

$$W = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_k}y^{\beta_k}, \quad (k \geq 1, 0 < \alpha_i < p, 0 < \beta_i < q),$$

and let

$$V(x, y) = x^{\gamma_1}y^{\delta_1} \dots x^{\gamma_\ell}y^{\delta_\ell}, \quad (\ell \geq 1, 0 < \gamma_i < p, 0 < \delta_i < q)$$

be a word that is equal to 1 in H . Then

$$\ell \geq k(r-1) + 1.$$

Corollary 6.2. *If $r \geq 3$ and $\ell \leq 5$ in the Theorem, then $k \leq 2$ and $r \leq 5$. If in addition $p = 2$ or $q = 2$, then $k = 1$.*

Proof. The first statement follows immediately from the inequality $\ell \geq k(r-1) + 1$, as does the second statement, except possibly in the case where $\ell = 5$, $r = 3$ and $k = 2$.

If $p = 2$ or $q = 2$ then ℓ must be even if k is even. (Compare the exponent sums of x or y in W^r and V .) This contradicts $\ell = 5$. \square

Now consider the one-relator product $G = (G_1 * G_2)/N(UaU^{-1}b)^3$, where $a^p = 1 = b^q$. In particular, G is induced from the triangle group $T(p, q, 3) = \langle x, y \mid x^p = y^q = (xy)^3 = 1 \rangle$. (Note that we include the case $p = \infty$ and/or $q = \infty$, where for example $T(p, \infty, 3) \cong C_p * C_3$.)

This is precisely the situation where Corollary 6.2 does not immediately imply that clique-pictures satisfy the $C(6)$ property. In order to prove $C(6)$, we will need to consider all possible cliques with labels of length up to $5\ell(UaU^{-1}b)$. Any such clique arises from a relation in $T(p, q, 3)$ that can be expressed as a word of free-product length at most 10 in $C_p * C_q = \langle x, y \mid x^p = y^q = 1 \rangle$. The following can be readily verified by direct calculation in $T(p, q, 3)$ for all q (for example using the

action of $T(p, q, 3)$ on the spherical, euclidean or hyperbolic plane). The details are left to the reader.

Proposition 6.3. *Suppose that $V \equiv x^{\alpha(1)}y^{\beta(1)} \dots x^{\alpha(n)}y^{\beta(n)}$ is equal to the identity in $T(p, q, 3)$, where $0 < \alpha(i) < p$ and $0 < \beta(i) < q$ for all i , $2 \leq p \leq q$, and $1 \leq n \leq 5$. Then, up to cyclic permutation and inversion in $C_p * C_q$, V is one of:*

- (1) $(xy)^3$;
- (2) $(xy)^2x^2(yx)^2y^2$ (if $p > 2$);
- (3) $(xyxy^2)^2$ (if $q > 2 = p$);
- (4) $(xy^2)^q$ (if $p = 2$ and $q \in \{4, 5\}$);
- (5) $xy^2xyxy^3xyxy^2$ (if $p = 2$ and $q > 3$).

As an immediate consequence, we have:

Corollary 6.4. *Let G be as above. Then, up to cyclic permutation and inversion in $G_1 * G_2$, any clique-label of length less than $6\ell(UaU^{-1}b)$ in a clique-picture over G has one of the forms:*

- (1) $(UaU^{-1}b)^3$,
- (2) $(UaU^{-1}b)^2Ua^2U^{-1}(bUaU^{-1})^2b^2$ (if $p > 2$),
- (3) $(UaU^{-1}bUaU^{-1}b^2)^2$ (if $q > 2 = p$),
- (4) $(UaU^{-1}b^2)^q$ (if $p = 2$ and $q \in \{4, 5\}$),
- (5) $UaU^{-1}b^2UaU^{-1}bUaU^{-1}b^3UaU^{-1}bUaU^{-1}b^2$ (if $p = 2$ and $q > 3$).

7. MAIN THEOREM

In this section we prove our principal result about clique-pictures,

Theorem 7.1. *If $G = (G_1 * G_2)/N(W(UaU^{-1}, b)^3)$ is a maximal generalized triangle group description of a one-relator product G , and (a, b) an admissible pair, then every clique-picture over G satisfies $C(6)$.*

This result follows immediately from Corollary 6.2 and Lemma 5.1 except possibly in the case where $k = \ell(W) \leq 2$. (Indeed, $k = 1$ in the case where $p = 2$ or $q = 2$.) If $k = 1$ then $R \sim Ua^\alpha U^{-1}b^\beta$ for some α, β , while if $k = 2$ then $R \sim Ua^\alpha U^{-1}b^\beta Ua^\gamma U^{-1}b^\delta$ for some $\alpha, \beta, \gamma, \delta$, where $a^2 \neq 1 \neq b^2$.

The simplest instance of the result is when $a^2 = 1 = b^2$.

Proposition 7.2. *If $R \equiv (UaU^{-1}b)$ where $a^2 = b^2 = 1$, then every clique in a reduced picture over $G = (G_1 * G_2)/N(R^r)$ consists of a single vertex, and pictures (and so also clique-pictures) over G satisfy $C(2r)$.*

Proof. If u and v are two vertices in a picture such that $u \sim v$, then after bridge moves if necessary, they are connected by a parallel class of arcs identifying subwords $U^{\pm 1}$ of the vertex labels of u and v . Hence the subpicture consisting of u, v and this class of arcs has boundary label $[aU^{-1}b(UaU^{-1}b)^{r-1} \cdot b(UaU^{-1}b)^{r-1}Ua]^{\pm 1} = 1$ in $G_1 * G_2$. Hence u and v cancel. Hence every clique in a reduced picture consists of a single vertex, as claimed. It follows that there is no difference between reduced clique-pictures and reduced pictures.

Now R is a cyclic permutation of R^{-1} . Hence any piece in a reduced (clique-)picture is a non-uniquely positioned cyclic subword of R . But some cyclic permutation of R is a product of two uniquely positioned subwords, by Weinbaum's Theorem 2.3. Hence a cyclic permutation of R^r is a product of $2r$ non-pieces, and $C(2r)$ is satisfied. \square

If $a^2 = 1 \neq b^2$, then Theorem 7.1 reduces to the following.

Theorem 7.3. *If $a^2 = 1 \neq b^2$ and $G = (G_1 * G_2)/N((UaU^{-1}b)^3)$ is a 2-maximal generalized triangle group description, then every clique-picture over G satisfies $C(6)$.*

When $a^2 \neq 1 \neq b^2$, we will use Theorem 7.3 to prove the following result, hence eliminating the second possibility for R .

Proposition 7.4. *Let G be a generalized triangle group of the form $\langle x, y \mid x^p = y^q = (x^a y^b x^c y^d)^3 = 1 \rangle$, where $p, q \geq 3$. Then for any $\alpha_i \in \{1, \dots, p-1\}$, $\beta_i \in \{1, \dots, q-1\}$ ($i = 1, \dots, k$, $1 \leq k \leq 5$), the word $x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_k} y^{\beta_k}$ is nontrivial in G .*

Proof. The result is immediate from small cancellation theory if pictures over G satisfy $C(6)$. Assume therefore that $C(6)$ fails. Then some subword of the relator of the form $(x^\alpha y^\beta)^{\pm 1}$ is a piece. Without loss of generality $x^\alpha y^\beta$ is a piece. If $x^\alpha y^\beta \equiv x^c y^d$ then the relator is a 6-th power $(x^\alpha y^\beta)^6$, and the result follows from Corollary 6.2.

Hence we may assume that $x^\alpha y^\beta \equiv (y^b x^c)^{-1}$. In other words $y^{2b} = 1 = x^{a+c}$. It follows that y^b has order 2, and y^d has order q' , for some $q' \mid q$. Hence G has a triangle group description obtained by embedding $C_2 * C_{q'}$ into $C_p * C_q$ as $\langle y^b \rangle * \langle x^a y^d x^{-a} \rangle$.

Moreover, if x^a does not have order 2, this description is 2-maximal. We may thus apply Theorem 7.3 to deduce that clique-pictures satisfy $C(6)$. Standard small-cancellation arguments now show that no clique-picture has shorter boundary label than the shortest possible clique-label, which by Corollary 6.4 is just the relator itself. The result follows.

Suppose then that $z = x^a$ has order 2. Then G is a free product of $\langle x \mid x^p = 1 \rangle$ and $H = \langle z, y \mid z^2 = y^q = (zy^bzy^d)^3 = 1 \rangle$, amalgamating the order 2 subgroups generated by x^a and z . If $W = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_k} y^{\beta_k} = 1$ in G , then at least two of the α_i are not divisible by a , for otherwise they are all divisible by a , and $W = 1$ in H , contradicting Corollary 6.2. Since W is not reduced in the amalgamated free product decomposition, we may assume, up to cyclic permutation, that $\alpha_1 \neq a \neq \alpha_{i+1}$ for some $i \geq 1$, $\alpha_j = a$ for $2 \leq j \leq i$, and $y^{\beta_1} x^a \dots x^a y^{\beta_i} = x^a$ in H . But $\leq i < k \leq 5$, so again this contradicts Corollary 6.2. This completes the proof. \square

When combined with Lemma 5.1, this result proves the $C(6)$ property for clique-pictures over a one-relator product of the form $(G_1 * G_2)/N((Ua^\alpha U^{-1} b^\beta U a^\gamma U^{-1} b^\delta)^3)$. To complete the proof of Theorem 7.1 we prove the following.

Theorem 7.5. *If $a^2 \neq 1 \neq b^2$ and $G = (G_1 * G_2)/N((UaU^{-1}b)^3)$ is a maximal generalized triangle group description, then every clique-picture over G satisfies $C(6)$.*

The proofs of Theorems 7.3 and 7.5 can be carried out in parallel. In each case, we write U as $u_1 u_2 \dots u_N$, where $N = \ell(U)$. The u_i and their inverses are elements of $G_1 \cup G_2$, some of which coincide. Let p, q be the orders of a and b respectively. Let Ω be a set

$$\Omega = \{A_1, \dots, A_{p-1} B_1, \dots, B_{q-1}, X_1, \bar{X}_1, \dots, X_N, \bar{X}_N\}$$

of cardinality $2N + p + q - 2$, with involution $A_i \leftrightarrow A_{p-i}$, $B_i \leftrightarrow B_{q-i}$, $X_i \leftrightarrow \bar{X}_i$, and let

$$\tilde{R} = A_1 X_1 \dots X_N B_1 \bar{X}_N \dots \bar{X}_1 \in \Omega^*.$$

Then R is the image of \tilde{R} under the involution-preserving homomorphism $\phi : \Omega^* \rightarrow G_1 * G_2$ induced by the map $A_i \mapsto a^i$, $B_i \mapsto b^i$, $X_i \mapsto u_i$, $\bar{X}_i \mapsto u_i^{-1}$.

Now $\ker(\phi)$ is a congruence on Ω^* , closed under involution, so $\ker(\phi) \cap \Omega$ is an equivalence relation that is also closed under involution. Call this equivalence relation \cong .

If clique-pictures over G do not satisfy $C(6)$, then we can construct a clique-picture with precisely one interior clique having at most 5 neighbours, \mathcal{P}_0 say. Replacing the vertex labels R^3 by \tilde{R}^3 , each set of parallel arcs emanating from the interior clique yields a pair of cyclic subwords P_i, Q_i of $\tilde{R}^{\pm 1}$ that are identified under \cong . Let \sim denote the smallest involution-closed equivalence relation on Ω such that $P_i \sim Q_i$ for each i . Then \sim is, in general, a smaller equivalence relation than \cong . Indeed, we shall make the following assumption.

Assumption Among all clique-pictures \mathcal{P}_0 as above, ours is chosen to minimise the resulting equivalence relation \sim .

A clique has label of the form

$$Ua^{\alpha(1)}U^{-1}b^{\beta(1)}Ua^{\alpha(2)}U^{-1}b^{\beta(2)} \dots Ua^{\alpha(k)}U^{-1}b^{\beta(k)}$$

for some integers $\alpha(1), \dots, \alpha(k) \in \{1, \dots, p-1\}$ and $\beta(1), \dots, \beta(k) \in \{1, \dots, q-1\}$, where p is the order of a and q is the order of b . Moreover, by Corollary 6.4 we know that $k \geq 3$. We introduce another involution-closed equivalence relation \approx on Ω , containing \sim , by taking \approx to be the smallest involution-closed equivalence class on Ω that contains \sim and satisfies $A_1 \approx A_2 \approx \dots \approx A_{p-1}$ and $B_1 \approx B_2 \approx \dots \approx B_{q-1}$.

Let A and B denote the \approx -classes containing A_1, \dots, A_{p-1} and B_1, \dots, B_{q-1} respectively. (Possibly $A \approx B$.) Then clique labels, interpreted as words in $(\Omega/\approx)^*$, all have the form $(UAU^{-1}B)^k$ for $k \geq 3$, and $A \approx \bar{A}$ and $B \approx \bar{B}$, and so clique-pictures satisfy $C(6)$ by Proposition 7.2. But, by construction, we have a clique-picture \mathcal{P}_0 (labelled in Ω^*) that fails the $C(6)$ condition. Hence when we reinterpret the labels of \mathcal{P}_0 in $(\Omega/\approx)^*$ we must have cancellation. Thus $UAU^{-1}B$ is a proper cyclic permutation of itself (that is, a proper power) in $(\Omega/\approx)^*$. Say $UAU^{-1}B \approx (VB)^m$ where $m \geq 1$ and VB is not a proper power in $(\Omega/\approx)^*$.

The label of the interior clique in our picture \mathcal{P}_0 has the form, up to cyclic permutation,

$$P_1x_1 \dots P_nx_n$$

for some $n \leq 5$, where each P_i is a piece, and each x_i is a letter. We say that P_i is a *good* piece if it does not represent a piece over Ω/\approx , and *bad* if it does. Since VB is not a proper power in $(\Omega/\approx)^*$, any piece of length at least $\ell(VB) - 1$ is good, by Lemma 5.1.

Lemma 7.6. $\ell(P_i) < (m-1)\ell(VB)$ for all $i = 1, \dots, n$.

Proof. Certainly, $\ell(P_i) < \ell(R) = m\ell(VB)$, by Lemma 5.1. Hence P_i is a cyclic subword of $Ua^\alpha U^{-1}b^\beta$ for some α, β . Similarly, Q_i is a cyclic subword of $Ua^\gamma U^{-1}b^\delta$ for some γ, δ . These subwords cannot be identically positioned, since the clique-picture \mathcal{P}_0 is reduced. If the final letters b^β and b^δ belong to P_i and Q_i respectively, then $P_i \equiv X_1^{-1}b^\beta X_2$ and $Q_i \equiv X_3^{-1}b^\delta X_4$, for some initial segments X_1, \dots, X_4 of U . Without loss of generality, $\ell(X_1) < \ell(X_3)$, so $b^\beta X \equiv X^{-1}b^\delta$ where X is the initial segment of U of length $\ell(X_3) - \ell(X_1)$. It then follows from Lemma 2.1 that, $\delta = -\beta$. If one of the subwords does not contain the final letter, then we may still assume that $\delta = -\beta$. (For example, if Q_i is a subword of $Ua^\gamma U^{-1}$ then it is a

subword of $Ua^\gamma U^{-1}b^{-\beta}$.) Similarly, we may assume that $\gamma = -\alpha$, and hence Q_i is a cyclic subword of $(Ua^\alpha U^{-1}b^\beta)^{-1}$.

Now assume that $\ell(P_i) \geq (m-1)\ell(VB)$. Since $\ell(P_i) = \ell(Q_i) > \ell(R)/2$, the cyclic subwords P_i and Q_i^{-1} of $Ua^\alpha U^{-1}b^\beta$ intersect (possibly at both ends). Thus there is a cyclic subword T of $(Ua^\alpha U^{-1}b^\beta)^2$ of length $2\ell(P_i) - \varepsilon$ ($\varepsilon > 0$), with initial segment P_i and terminal segment Q_i^{-1} . Let X_1 denote the initial segment of $P_i \equiv Q_i$ of length $\ell(P_i) - \varepsilon$, and X_2 the terminal segment of P_i of length ε . Then X_2 represents the intersection of the subwords P_i and Q_i^{-1} , so is also an initial segment of Q_i^{-1} . Hence X_2^{-1} is a terminal segment of $Q_i \equiv P_i$, so $X_2^{-1} \equiv X_2$. Moreover, the terminal segment of Q_i^{-1} of length $\ell(Q_i) - \varepsilon$ is equal to X_1^{-1} . Hence $T \equiv X_1 X_2 X_1^{-1} \equiv T^{-1}$. In particular, $\ell(T) \geq \ell(P_i) \geq (m-1)\ell(VB)$. Indeed $\ell(T)$ is odd, so $\ell(T) > (m-1)\ell(VB)$.

Now let T_0 denote the middle segment of T of length $1 + (m-1)\ell(VB)$. Considering T_0 as a cyclic subword of $(VB)^{2m}$ in $(\Omega/\approx)^*$, we see that $(VB)^m$ is either \approx -equivalent to VT_0 or to a cyclic conjugate of $V'T_0$ where V' is a conjugate of $b^{\beta(i)}$ for some i .

Write

$$Ua^\alpha U^{-1}b^\beta \sim V_1 \cdot b^{\beta(1)} \cdot V_2 \cdot b^{\beta(2)} \dots V_m \cdot b^{\beta(m)}.$$

In the first case (where $(VB)^m \approx VT_0$), all m letters $b^{\beta(i)}$ appearing explicitly in this expression belong to T_0 . Also all but one of the V_i - say V_j , appear in T_0 , and can be arranged into cancelling pairs - each V_i appearing in T_0 is identical to \bar{V}_k for some V_k appearing in T_0 (with $k \neq i$). Since also $V_i \equiv \bar{V}_{m+1-i}$ for all i , it follows that $V_j \equiv \bar{V}_j$. Hence T_0 is contained in, a cyclic subword $T_1 \equiv V_j T_0 V_j$ of $(Ua^\alpha U^{-1}b^\beta)^2$ such that $T_1 \equiv \bar{T}_1$. In particular, $Ua^\alpha U^{-1}b^\beta$ has a cyclic permutation of the form YZ with $Y^2 = Z^2 = 1$. This yields a refinement of R , by Lemma 2.2.

The argument in the second case (where $(VB)^m \approx V'T_0$) is similar. Here we have $Ua^\alpha U^{-1}b^\beta$ a cyclic permutation of $T_0 S b^{\beta(i)} S'$, for some i , where S is the terminal segment of V_{i-1} of length $(\ell(V) - 1)/2$, and S' is the initial segment of V_i of length $(\ell(V) - 1)/2$. Now the initial segment of V_j of length $(\ell(V) - 1)/2$ is inverse to the terminal segment of V_{m+1-j} of the same length for each j . Moreover, for each V_j appearing in T this initial segment is inverse to a terminal segment of some V_k appearing in T . By the pigeonhole principle, $S' \equiv \bar{S}$, so $Ua^\alpha U^{-1}b^\beta$ is a cyclic permutation of $T_2 b^{\beta(i)}$ with $T_2 \equiv S' T_0 S \equiv \bar{T}_2$. By Lemma 2.2 we again obtain a refinement of R .

To complete the proof, note that, if either of a or b has order 2, then the refinement obtained above is in fact a 2-refinement, so we contradict the hypotheses of Theorem 7.3 or 7.5. \square

Lemma 7.7. *If each P_i coincides with Q_i as a subword of \tilde{R}^r in $(\Omega/\approx)^*$, then R has a refinement, which is a 2-refinement if one of a , b has order 2.*

Proof. Recall that $\tilde{R} \approx (VB)^m$ in $(\Omega/\approx)^*$. If $x \in \Omega$ with $x \approx B$, then $x \sim b^\beta$ for some β , and since \sim is the equivalence relation imposed by matching P_i with Q_i , it follows that the position of x in \tilde{R} is congruent to 0 modulo $\ell(VB)$. Thus $x \not\approx B$ for any letter x of V . Similarly, VB has a cyclic permutation AV' , and $x \not\approx A$ for any letter x of V' . There are two possibilities. If $A \approx B$ then $V' \approx V \approx \bar{V}$; while if $A \not\approx B$ then $V \approx V_1 A \bar{V}_1$, $V' \approx \bar{V}_1 B V_1$. In the first case,

$$R \equiv V b^{\beta(1)} V b^{\beta(2)} \dots V b^{\beta(m)},$$

yielding a 2-refinement. In the second case,

$$R \equiv V_1 a^{\alpha(1)} \bar{V}_1 b^{\beta(1)} \dots V_1 a^{\alpha(m)} \bar{V}_1 b^{\beta(m)},$$

yielding a refinement. In particular, if either $a^2 = 1$ or $b^2 = 1$, this is a 2-refinement. \square

Corollary 7.8. *If all the pieces P_i are good, then R has a refinement, which is a 2-refinement if one of a, b has order 2.*

Proof. By definition, no P_i represents a piece of \tilde{R}^3 over $(\Omega/\approx)^*$. This means that P_i and Q_i coincide exactly as cyclic subwords of $(VB)^m$ in $(\Omega/\approx)^*$ (up to cyclic permutation of the copies of VB). The result follows from Lemma 7.7. \square

As a consequence, we can assume that at least one of the P_i is not good, and therefore has length strictly less than $\ell(VB) - 1$. But then we have

$$3m\ell(VB) \leq km\ell(VB) = \sum_i (\ell(P_i) + 1) \leq [(n-1)(m-1) + 1]\ell(VB) \leq (4m-3)\ell(VB).$$

It follows that $k = 3$ and $n = 5$. In particular, since $k = 3$, the clique label of the interior clique of \mathcal{P}_0 is precisely $(aUbU^{-1})^3$, by Corollary 6.4.

Moreover, precisely one of the P_i is not good, since otherwise we have a contradiction

$$3m\ell(aV) = \sum_i (\ell(P_i) + 1) \leq [3(m-1) + 2]\ell(aV).$$

Suppose without loss of generality that P_2, \dots, P_5 are good, but not P_1 .

Remark 7.9. Without loss of generality, we may assume that no cyclic subword of R of length $\ell(VB)$ that contains P_1 is a piece.

Proof. To see this, note that any such subword is necessarily a good piece, so there is a cyclic subword Q' of \tilde{R} , with $Q' \not\equiv P_1$ in Ω^* but $Q' \sim P_1$, such that, in $(\Omega/\approx)^*$, one obtains Q' from P_1 by cyclically permuting the copies of VB in $(VB)^m \approx \tilde{R}$. Let us form a new clique-picture \mathcal{P}_1 from \mathcal{P}_0 by re-labelling the boundary clique whose label contains Q_1 in such a way as to replace the subword Q_1 of the label by Q' . Since $Q_1 \sim P_1 \sim Q'$ this clique-picture is no different from the old one, over G . However, we have changed the Ω -labelling, and hence potentially the resulting equivalence relation \sim , to a new equivalence $\sim' \subset \sim$ (since \sim' is induced by the relations $P_1 \sim' Q', P_i \sim' Q_i, i = 2, \dots, 5$). But we have an underlying assumption of minimality of \sim , so the relations \sim' and \sim coincide. We may therefore apply Lemma 7.7 to obtain a (2-) refinement, contrary to hypothesis. \square

Now U is a piece, for otherwise all clique labels are products of 6 non-pieces. Hence P_1 is not a subword of U (or of U^{-1}). Thus P_1 is either a subword of $U_1 a U_1^{-1}$, or of $U_2^{-1} b U_2$, where U_1 and U_2 denote the terminal and initial segments respectively of U of length $\ell(V)$. Without loss of generality we assume the latter. It follows that $U_2^{-1} b U_2$ is not a piece (see above Remark).

Now write $U \equiv U_2 V_2 \equiv V_1 U_1$. If we can show that $V_2 a V_2^{-1}$ is also not a piece, then a cyclic permutation of the relator $(U a U^{-1} b)^3$ is a product of 6 non-pieces, and $C(6)$ follows. Hence the following result completes the proof of Theorems 7.3 and 7.5.

Lemma 7.10. *In the above notation, $V_2 a V_2^{-1}$ is not a piece.*

Proof. If $V_2aV_2^{-1}$ is a piece, then it is a good piece. We need to consider two possibilities.

1) $V_2aV_2^{-1}$ is identical to an initial or terminal segment of $Ua^\alpha U^{-1}$ for some α .

By Lemma 2.1 we must have $\alpha = -1$. Without loss of generality, let us assume that $V_2aV_2^{-1}$ is identical to an initial segment of $Ua^{-1}U^{-1}$. Then, depending on the parity of m , we deduce that:

$$UaU^{-1} \equiv (V_1a)^{m/2}(V_1a^{-1})^{(m-2)/2}V_1, \quad V_1 \equiv V_1^{-1}, \quad a \text{ a power of } b \text{ (} m \text{ even),}$$

giving a 2-refinement of the relator, or

$$UaU^{-1} \equiv (V_1b^\beta)^{(m+1)/2}(V_1^{-1}b^\beta)^{(m-1)/2}V_1^{-1}, \quad b^{2\beta} = 1, \quad V_1 \text{ conjugate to } a, \text{ (} m \text{ odd),}$$

giving a refinement of the relator, which is a 2-refinement if one or both of a, b has order 2.

2) $V_2aV_2^{-1}$ is identical to a nonextremal segment of $V_2a^\alpha U^{-1}b^\beta Ua^\gamma V_2^{-1}$ for some α, β, γ .

Without loss of generality, we can assume that $V_2aV_2^{-1}$ is identical to a noninitial segment of $V_2a^\alpha U^{-1}b^\beta U$. If this segment contains the $\ell(V_2a)$ -th letter a^α , then $\alpha = -1$ by Lemma 2.1. Otherwise we are at liberty to set $\alpha = -1$ without altering the situation. Hence there is an initial segment T of $V_2a^{-1}U^{-1}b^\beta U$ which has a terminal segment $V_2aV_2^{-1}$ and an initial segment $V_2a^{-1}V_2^{-1}$, and so $T^{-1} \equiv T$ (using the fact that $\ell(T) \leq \ell(V_2a^\alpha U^{-1}b^\beta U) < 2\ell(V_2aV_2^{-1})$).

But $\ell(T) \geq \ell(V_2a^\alpha U^{-1}b^\beta U) = \ell(UaU^{-1}b) - 1$, so T has a middle segment T_0 of length $\ell(UaU^{-1}b) - 1$ with $T_0^{-1} \equiv T_0$, and cT_0 is a cyclic permutation of $Ua^{-1}U^{-1}b^\beta$ for some letter c . Moreover, it must be a proper cyclic permutation, since the middle letter of T cannot be one of the letters a^{-1} or b^β appearing explicitly in $V_2a^{-1}U^{-1}b^\beta U$. Hence $Ua^{-1}U^{-1}b^\beta$ is a proper power, by Lemma 5.1, and it follows that we have a refinement of the relator. Again, if either $a^2 = 1$ or $b^2 = 1$, then this is a 2-refinement.

In both cases we have a (2-) refinement of the relator, contrary to hypothesis, and the proof is complete. \square

This completes the proofs of Theorems 7.3 and 7.5, and hence of Theorem 7.1.

8. APPLICATIONS

In this section, we apply the Main Theorem 7.1 to prove the results stated in the Introduction on the structure, cohomology, etc. of one-relator products induced from generalized triangle groups of type $(2, q, r)$ for $r \geq 3$.

Proof of Theorem 1. We may assume, without loss of generality, that the generalized triangle group description (1.1) is maximal.

Suppose first that one of the natural maps $H \rightarrow G, G_1 \rightarrow G, G_2 \rightarrow G$ is not injective. Then there is a picture over G on D^2 whose boundary label represents a nontrivial element of H, G_1 or G_2 . Among all such pictures, let Γ be one with the smallest possible number of cliques.

Then every clique of Γ is simply connected with nonempty boundary label. For suppose that some clique C is not simply-connected, and let γ be an inner boundary component of C . Then γ bounds a proper subpicture Δ of Γ whose boundary label is (up to conjugacy) an element of H . By minimality this boundary label is trivial,

so Δ could be replaced by a picture which combines with C to form a larger clique. The resulting picture Γ' has fewer cliques than Γ , a contradiction. If a clique has empty boundary label, it can simply be deleted to form a smaller picture.

Hence we can form the clique-quotient $\bar{\Gamma}$. It follows from Theorem 7.1 that $\bar{\Gamma}$ satisfies the small-cancellation condition $C(6)$.

Standard arguments now show that either $\bar{\Gamma}$ (and hence Γ) is empty, or that some arcs of $\bar{\Gamma}$ (and hence of Γ) meet ∂D^2 . In particular this shows that $G_1 \rightarrow G$ and $G_2 \rightarrow G$ are injective. Moreover, if the boundary label of Γ is a word in $\{a, U^{-1}bU\}$, then the $C(6)$ property shows that some clique C of $\bar{\Gamma}$ has at most 3 neighbours. It follows from the proof of Theorem 7.1 that C is connected to ∂D^2 by a class of more than 2ℓ parallel arcs. By Lemma 5.1 it follows that Γ with C removed still has boundary label a word in $\{a, U^{-1}bU\}$. By minimality, this smaller picture (and hence also Γ) has label equal to the identity in H . Hence $H \rightarrow G$ is also injective.

To prove that (1.1) is geometrically Mayer-Vietoris, it suffices to show that any spherical picture is equivalent, via bridge moves, to one in which each component is a single clique. To see this, note that it then follows that, in the space

$$X = K(G_1 * G_2, 1) \cup_{K(C_2 * C_q, 1)} K(H, 1),$$

$\pi_2(X^{(2)})$ is generated by the image of $\pi_2(K(H, 1)^{(2)})$, and hence $\pi_2(X) = 0$. By [12], Theorem 4.2, X is a $K(G, 1)$ -space, and (1.1) is indeed geometrically Mayer-Vietoris.

Suppose then that Γ is a spherical picture that is not equivalent, via bridge moves, to one in which each component is a single clique. Suppose also that, among all such, Γ is minimal in the sense of having the fewest possible cliques. As before, minimality implies that the cliques of Γ are all simply connected. It also implies that Γ is connected. Hence $\bar{\Gamma}$ is a connected spherical picture which satisfies the $C(6)$ condition. It follows that $\bar{\Gamma}$ contains no arcs, and hence consists of at most one vertex. Hence Γ is either empty or consists of a single clique, a contradiction. \square

Proof of Theorem 2. By Theorem 7.1, any clique-picture over G satisfies $C(6)$, and hence a quadratic isoperimetric inequality (see for example [15], p 260, the Area Theorem). In other words, there is a quadratic function f such that any word of length n representing the identity element of G is the boundary label of a picture with at most $f(n)$ cliques.

Note also that there is a bound (as a function of n) on the length of any clique-label arising in such a clique-picture. By Lemma 5.1, no parallel class of arcs connecting two cliques in a reduced clique-picture contains more than $\ell(UaU^{-1}b)$ arcs, so a clique with label of length $n\ell(UaU^{-1}b)$ has degree at least n . Moreover, there is a linear isoperimetric inequality of the form $\ell(\partial\Gamma) \geq \sum_C [\text{degree}(C) - 6]$, where the sum is over all cliques C . Hence no clique can have degree greater than $\ell(\partial\Gamma) + 6f(\ell(\partial\Gamma))$. Since no piece has length greater than $\ell(UaU^{-1}b)$, this gives an upper bound of

$$[\ell(\partial\Gamma) + 6f(\ell(\partial\Gamma))]\ell(UaU^{-1}b)$$

on the length of any clique-label.

Since both the number of cliques and the length of any clique-label are bounded, there are only a finite number of connected graphs that could arise as clique-pictures for words of length less than or equal to that of a given word W . Moreover, any such

graph can be labelled as a clique-picture only in a finite number of ways. For any such potential labelling, we may check whether or not the clique-labels are equal to the identity in H , and whether or not the region-labels are equal to the identity in G_1 or G_2 , using the solution to the word problem in H , G_1 and G_2 respectively. Hence we may obtain an effective list of all words of length $\leq \ell(W)$ that appear as boundary labels of *connected* pictures over G . In particular, we may check, for all cyclic subwords U of W , whether or not Ug belongs to this list for some letter $g \in G_1 \cup G_2$. (Note that this check also uses the solution to the word problem in G_1 and G_2 , and that the letter g , if it exists, is unique by the Freiheitssatz, Theorem 1.) If so, then W is a cyclic conjugate of UV for some V , so $W = 1$ in G if and only if $g^{-1}V = 1$ in G , which we may assume inductively is decidable.

Hence the word problem is soluble for G , as claimed \square

Proof of Theorem 3. Assume, to the contrary, that $V_1 = V_2 = 1$ in G . Amongst all pictures with boundary label V_1 or V_2 , choose one with fewest possible vertices - say Γ' with label V_2 . Form a new picture Γ with boundary label V_1^{-1} by adding a new vertex to Γ' . Then Γ is reduced, for otherwise we may cancel 2 vertices to give a smaller picture than Γ' , contradicting the minimality of choice. Moreover, Γ has precisely one boundary vertex. Hence the corresponding clique-picture $\bar{\Gamma}$ has precisely one boundary clique. By the $C(6)$ property for clique pictures, $\bar{\Gamma}$ must consist of a single clique, so up to conjugacy V_1 is a word $W_1(a, U^{-1}bU)$, where $W_1(x, y)$ is a proper cyclic subword of $W(x, y)^r$ that is equal to the identity in $H = \langle x, y \mid x^p = y^q = W(x, y)^r = 1 \rangle$. Similarly $V_2 = W_2(a, U^{-1}bU)$ (up to conjugacy), where $W(x, y)$ is a cyclic permutation of W_1W_2 . Hence also $W_2(x, y) = 1$ in H . Moreover, one of W_1, W_2 is at most half the length of W^r . But this contradicts the spelling theorem for generalized triangle groups, Theorem 6.1.

This contradiction completes the proof. \square

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