A proof of the Scott-Wiegold conjecture on free products of cyclic groups

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Abstract

Problem 5.53 of [18] (contributed by Wiegold, attributed to Scott) asks whether a free product of three (finite) cyclic groups can be normally generated by a single element. We give a proof of the conjectured negative answer, and an application to Dehn surgery on knots: if Dehn surgery on a knot in S^3 gives a connected sum, then all but at most 2 of the connected summands are $\mathbb{Z}$-homology spheres, and hence (by a result of Valdez and Sayari) the number of connected summands is at most 3 .

1 Introduction

A number of problems in combinatorial group theory can be expressed in terms of the non-vanishing of a group given by a certain type of presentation, or more generally the existence of large subgroups in such groups.

For example, the classical Freiheitssatz for one-relator groups [17] says that an n-generator, one-relator group contains a free subgroup of rank $n-1$ (with basis a subset of the generating set). In particular it is nontrivial if $n > 1$. Results of Baumslag and Pride [1], Wilson and Zelmanov [25], and others [5, 11, 15, 23] give other criteria on a presentation for the corresponding group to contain free subgroups.

Several generalisations of the Freiheitssatz exist (see the survey [4] and the references cited there). These deal with a *one-relator product* of groups, that is the quotient of a free product by the normal closure of a single element. Under suitable conditions, each factor of the free product embeds in the group, and so in particular the group is nontrivial.

A good example of this phenomenon is that of a *generalised triangle group*, the free product of two finite cyclic groups by the normal closure of a single element which is a proper power. The Freiheitssatz in this situation is due to Baumslag, Morgan and Shalen [2]. See also [6]. The nontriviality of generalised triangle groups had previously been proved by Boyer [3], and was used to prove that noninteger Dehn surgery on a knot in S^3 can never yield a connected sum [10].

In the present article we consider a 1976 problem of Scott and Wiegold [18, Problem 5.53]:

Question Can a one-relator product of three (or more) cyclic groups ever be trivial?

(Of course, if the answer is affirmative for $n \geq 3$ cyclic groups then it is also affirmative for 3 cyclic groups.) We prove the conjectured negative answer:

Corollary 4.2 *Every one-relator product of three cyclic groups is nontrivial.*

This question, and the more general question of whether a one-relator product of three arbitrary groups can ever be trivial, arose also in [7] and [9] in relation to the possibility of Dehn surgery on a knot in S^3 giving rise to a connected sum of three or more factor manifolds. The Cabling Conjecture of Gonzalez-Acuña and Short [8, Conjecture A] asserts that Dehn surgery on a knot can give a connected sum only when the knot is a cable knot and the surgery slope is that of the cabling annulus. In this case we get a connected sum of a lens space and a manifold obtained by non-integer surgery on a knot, so both factors are prime (see for example [10]). It would follow therefore from the Cabling Conjecture that Dehn surgery on a knot can never produce a connected sum of more than two factors. The Cabling Conjecture remains an open question in general, but has been proved for several classes of knots [12, 13, 16, 19, 20, 22, 26].

In particular, it is still unknown whether a connected sum of three or more summands can be obtained by Dehn surgery on a knot. An easy consequence of our result gives some restrictions on the connected summands that could arise in this unlikely situation:

Corollary 5.2 *If a connected sum of n 3-manifolds $M_1, \ldots, M_n$ can be obtained by Dehn surgery on a knot in S^3, then at least $n-2$ of the M_i are $\mathbb{Z}$-homology spheres.*

Cameron Gordon and Steven Boyer have observed that this result, combined with a result of Valdez [24, Theorem 3] and Sayari [21, proof of Lemma 1.1] gives the following.

Corollary 5.3 *If a connected sum of n 3-manifolds $M_1, \ldots, M_n$ can be obtained by Dehn surgery on a knot in S^3, then $n \leq 3$. If $n = 3$ then two of the M_i are lens spaces and the third is a $\mathbb{Z}$-homology sphere.*

The Scott-Wiegold question easily reduces (by abelianising) to the case where all three cyclic groups are finite, of pairwise coprime orders. A further easy reduction allows us to assume that the orders are (distinct) primes, and it is in this form that the question appears in [18]. The answer to the corresponding question for two cyclic factors can easily be seen to have a negative answer: if G is generated by elements x, y of coprime finite orders such that $xy = 1$, then $x = y = 1$ so $G = 1$. One can also construct more complicated examples. Suppose for example that p, q are coprime integers with $q > 2$ odd. Then

$$G = \langle x, y | x^p = y^q = (xy)^{pq} xy^2 = 1 \rangle$$

is cyclic (by the third relation), and hence trivial as we can see by abelianising the presentation.

The objects of study in this paper are therefore group presentations of the form

$$G = \langle x, y, z | x^p = y^q = z^r = W(x, y, z) = 1 \rangle,$$

where p, q, r are distinct primes and W is a word in x, y, z. The problem is to show that $G \neq 1$.

We prove this by adapting the method of Boyer [3] to produce (in the nontrivial case where G is perfect) a nontrivial representation from G to $SO(3)$. Before giving the detailed proof in §4, we first present an easy result on S^1-equivariant homotopy in §2 and an even easier trigonometric calculation in §3, both of which are used in an essential way in the construction of the representation. After proving the Scott-Wiegold Conjecture in §4, we apply it to Dehn surgery in §5, and finish by discussing some potential generalisations in §6.

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2 S^1-equivariant homotopy

If Γ is a (topological) group and X, Y are two spaces equipped with continuous (left) Γ-actions, then a map $f : X \to Y$ is Γ-*equivariant* if $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and $\alpha \in \Gamma$. Two such maps f, g are *equivariantly homotopic* if they are homotopic via a homotopy $H : X \times [0, 1] \to Y$ such that $H(\alpha x, t) = \alpha H(x, t)$ for all $x \in X$, $\alpha \in \Gamma$ and $t \in [0, 1]$. Equivariant homotopy is in general a finer relation than homotopy: two equivariant maps may be homotopic but not equivariantly homotopic.

We will be mainly interested in the case where Γ is the circle group S^1, and the S^1-spaces are the spheres S^2 and S^3, with S^1-actions arising as follows.

We regard S^3 as the group of unit quaternions

$$S^3 = \{a + b\underline{\mathbf{i}} + c\underline{\mathbf{j}} + d\underline{\mathbf{k}} \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

We identify the 2-sphere S^2 with the subset

$$S^2 = \{b\underline{\mathbf{i}} + c\underline{\mathbf{j}} + d\underline{\mathbf{k}} \mid b^2 + c^2 + d^2 = 1\}$$

of S^3, and the circle group S^1 with the subgroup

$$S^1 = \{a + b\underline{\mathbf{i}} \mid a^2 + b^2 = 1\}$$

of S^3.

The conjugacy classes of S^3 are the sets $\{(\cos\theta) + (\sin\theta)\underline{\mathbf{v}} \mid \underline{\mathbf{v}} \in S^2\}$ for $0 \le \theta \le \pi$. For $\theta = 0, \pi$ these are the single element sets $\{\pm 1\}$, otherwise each conjugacy class is topologically a 2-sphere. Then S^3 (and hence also S^1) acts on S^3 by conjugation, and S^2 is invariant under this action. Since S^1 is its own centraliser in S^3, the fixed set of the S^1-action on S^3 is the circle S^1 itself, and the fixed set of the S^1-action on S^2 is $S^1 \cap S^2 = \{\pm\underline{\mathbf{i}}\}$. Note that the inclusion $S^2 \to S^3 \setminus \{\pm 1\}$ is an S^3-equivariant homotopy equivalence, and its inverse is the smooth S^3-equivariant map

$$\psi : (\cos\theta) + (\sin\theta)\underline{\mathbf{v}} \mapsto \underline{\mathbf{v}}, \quad 0 < \theta < \pi. \tag{1}$$

This map ψ will play an important rôle in our proof.

Lemma 2.1 *Let X be a simply-connected space equipped with an S^1-action, and let $f, g : S^2 \to X$ be S^1-equivariant maps. Then f is equivariantly homotopic to g if and only if there are paths in the fixed subspace X^{S^1} joining $f(+\underline{\mathbf{i}})$ to $g(+\underline{\mathbf{i}})$, and $f(-\underline{\mathbf{i}})$ to $g(-\underline{\mathbf{i}})$.*

Proof. Clearly any equivariant homotopy between f and g restricts to an equivariant homotopy between the restrictions of f and g to any S^1-orbit – in particular to paths in X^{S^1} between $f(+\underline{\mathbf{i}})$ and $g(+\underline{\mathbf{i}})$, and between $f(-\underline{\mathbf{i}})$ and $g(-\underline{\mathbf{i}})$.

Conversely, suppose $\nu, \sigma : [0, 1] \to X^{S^1}$ are paths joining $f(+\underline{\mathbf{i}})$ to $g(+\underline{\mathbf{i}})$, and $f(-\underline{\mathbf{i}})$ to $g(-\underline{\mathbf{i}})$ respectively.

Let γ be the line of longitude $\{\cos(\theta)\underline{\mathbf{i}} + \sin(\theta)\underline{\mathbf{j}},\ 0 \le \theta \le \pi\}$ in S^2. Thus γ is an arc joining $+\underline{\mathbf{i}}$ to $-\underline{\mathbf{i}}$ that meets each S^1-orbit precisely once. Since f and g are equivariant, they are determined entirely by $f(\gamma)$ and $g(\gamma)$ respectively:

$$f(\alpha x) = \alpha f(x), \quad g(\alpha x) = \alpha g(x), \quad \alpha \in S^1, \quad x \in \gamma.$$

Since X is simply connected, the paths ν, σ extend to a homotopy $H : \gamma \times [0, 1] \to X$ between $f(\gamma)$ and $g(\gamma)$, which then extends uniquely to an equivariant homotopy $\tilde{H}$ between f and g by

$$\tilde{H}(\alpha x, t) = \alpha H(x, t).$$

□

Any equivariant map must send fixed points to fixed points. This gives us an easy way to calculate the degree of any S^1-equivariant self-map of S^2.

Corollary 2.2 *The degree of any S^1-equivariant map $f : S^2 \to S^2$ is one of the following:*

(a) 0 *if* $f(\underline{\mathbf{i}}) = f(-\underline{\mathbf{i}}) \in \{\pm\underline{\mathbf{i}}\}$;

(b) $+1$ *if* $f(\underline{\mathbf{i}}) = \underline{\mathbf{i}}$ *and* $f(-\underline{\mathbf{i}}) = -\underline{\mathbf{i}}$*; or*

(c) -1 *if* $f(\underline{\mathbf{i}}) = -\underline{\mathbf{i}}$ *and* $f(-\underline{\mathbf{i}}) = \underline{\mathbf{i}}$.

Proof. As remarked above, since f is equivariant, it maps fixed points to fixed points, that is $f(\pm\underline{\mathbf{i}}) \in \{\pm\underline{\mathbf{i}}\}$. By Lemma 2.1 the degree of f is determined by the values of $f(\pm\underline{\mathbf{i}})$. To compute the degree we compare f to a canonical equivariant map g determined by these values.

(a) If $f(+\underline{\mathbf{i}}) = f(-\underline{\mathbf{i}})$, take g to be the constant map with value $f(+\underline{\mathbf{i}})$.

(b) If $f(\underline{\mathbf{i}}) = \underline{\mathbf{i}}$ and $f(-\underline{\mathbf{i}}) = -\underline{\mathbf{i}}$, take g to be the identity map $S^2 \to S^2$.

(c) If $f(\underline{\mathbf{i}}) = -\underline{\mathbf{i}}$ and $f(-\underline{\mathbf{i}}) = \underline{\mathbf{i}}$, take g to be the reflection $S^2 \to S^2$ defined by

$$b\underline{\mathbf{i}} + c\underline{\mathbf{j}} + d\underline{\mathbf{k}} \mapsto -b\underline{\mathbf{i}} + c\underline{\mathbf{j}} + d\underline{\mathbf{k}}.$$

In all three cases it is easy to check that g is S^1-equivariant, and hence S^1-equivariantly homotopic to f by Lemma 2.1. Moreover the degree of g is (a) 0, (b) $+1$ or (c) -1, as required. □

3 Trigonometry

Our proof of the Scott-Wiegold Conjecture involves finding a nontrivial representation $G \to SO(3) \cong S^3/\{\pm 1\}$, so we need to find elements of S^3 having orders p, q, r respectively when considered as elements of $SO(3)$. For a prime ℓ, an element $\cos\theta + (\sin\theta)\underline{\mathbf{v}} \in S^3$, $\underline{\mathbf{v}} \in S^2$, represents an element of order ℓ in $SO(3)$ if and only if θ is a multiple of π/ℓ but not of π. For our purposes, it is convenient to specify canonical multiples of π/ℓ for each prime ℓ.

Define

$$\theta_\ell = \begin{cases} (\ell-1)\pi/2\ell, & \ell \text{ odd}, \\ \pi/2, & \ell = 2. \end{cases} \tag{2}$$

Then θ_ℓ is a multiple of π/ℓ, as required.

Moreover, we have the following property:

Lemma 3.1 *Let* p, q, r *be distinct primes, and let* $\varepsilon_p, \varepsilon_q, \varepsilon_r \in \{\pm 1\}$*. Then*

$$\varepsilon_p \varepsilon_q \varepsilon_r \sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) < 0.$$

Proof. Note first that $\pi/3 \le \theta_\ell \le \pi/2$ (with both inequalities strict when $\ell > 3$). Hence

$$0 < \theta_p + \theta_q - \theta_r < \pi < \theta_p + \theta_q + \theta_r < 2\pi.$$

The result follows when $\varepsilon_p = \varepsilon_q = +1$. By permuting p, q, r and/or multiplying all of $\varepsilon_p, \varepsilon_q, \varepsilon_r$ by -1, we obtain the result in all the remaining cases. □

Corollary 3.2 *Let $\psi : S^3 \setminus \{\pm 1\} \to S^2$ be the map defined in (1) in §2, and*

$$\beta = \cos(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) + \sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r)\underline{\mathbf{i}} \in S^3,$$

where $\varepsilon_p, \varepsilon_q, \varepsilon_r \in \{\pm 1\}$. Then $\beta \neq \pm 1$ and $\psi(\beta) = -\varepsilon_p\varepsilon_q\varepsilon_r\underline{\mathbf{i}}$.

Proof. Note that $\sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) \neq 0$, by Lemma 3.1, so $\beta \neq \pm 1$. Moreover, $\psi(\beta) = +\underline{\mathbf{i}}$ if $\sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) > 0$, and $\psi(\beta) = -\underline{\mathbf{i}}$ if $\sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) < 0$. The result follows from Lemma 3.1. □

4 Proof of the conjecture

Recall that $G = \langle x, y, z | x^p = y^q = z^r = W(x, y, z) = 1\rangle$. A representation $\rho : G \to H$ is called *essential* if $\rho(x)$, $\rho(y)$ and $\rho(z)$ have orders precisely p, q and r respectively in H.

Theorem 4.1 *Let $G = \langle x, y, z | x^p = y^q = z^r = W(x, y, z) = 1\rangle$, where p, q, r are distinct primes and the exponent sums $e_x(W)$, $e_y(W)$, $e_z(W)$ are coprime to p, q and r respectively. Then there exists an essential representation $G \to SO(3)$.*

Proof. First note that we may replace the relator $W(x, y, z)$ in the presentation of G by $W(x^a, y, z)x^{pb}$ for any $a = 1, 2, \dots, p-1$ and any integer b, and the resulting group will be isomorphic to G via the map $x \mapsto x^a$, $y \mapsto y$, $z \mapsto z$. This move does not change the exponent sums of y and z in W, but changes the exponent sum of x from $e_x(W)$ to $ae_x(W) + pb$. Since $e_x(W)$ is coprime to p, we may choose a and b in such a way that $e_x(W(x^a, y, z)x^{pb}) = 1$. Similarly, we may adjust W so that y and z also appear with exponent sum 1.

Hence we may assume for the purposes of this proof that $e_x(W) = e_y(W) = e_z(W) = 1$.

We identify $SO(3)$ with the quotient group $S^3/\{\pm 1\}$.

If ℓ is a prime number, then $a + b\underline{\mathbf{i}} + c\underline{\mathbf{j}} + d\underline{\mathbf{k}} \in S^3$ represents an element of order ℓ in $SO(3)$ if and only if $a = cos(m\pi/\ell)$ for some $m = 1, 2, \dots, \ell - 1$. In order to construct an essential representation $\rho : G \to SO(3)$, we define

$$\rho(x) = \cos(\theta_p) + \sin(\theta_p)\underline{\mathbf{u}}, \quad \rho(y) = \cos(\theta_q) + \sin(\theta_q)\underline{\mathbf{v}}, \quad \rho(z) = \cos(\theta_r) + \sin(\theta_r)\underline{\mathbf{w}},$$

for suitable $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}} \in S^2$ to be chosen, where θ_p, θ_q and θ_r are as defined in (2) in §3. This choice ensures that $\rho(x)$, $\rho(y)$ and $\rho(z)$ have the correct orders. It remains only to choose $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}} \in S^2$ such that $\rho(W(x, y, z)) = \pm 1$. If such a choice is possible, then we are finished.

Suppose then that there is no such choice of $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$ and $\underline{\mathbf{w}}$, in other words that

$$W(\cos(\theta_p) + \sin(\theta_p)\underline{\mathbf{u}}, \cos(\theta_q) + \sin(\theta_q)\underline{\mathbf{v}}, \cos(\theta_r) + \sin(\theta_r)\underline{\mathbf{w}}) \in S^3 \setminus \{\pm 1\}$$

for all $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}} \in S^2$. Then we proceed to derive a contradiction as follows. The map $\phi : S^2 \times S^2 \times S^2 \to S^2$ given by

$$\phi(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}}) = \psi(W(\cos(\theta_p) + \sin(\theta_p)\underline{\mathbf{u}}, \cos(\theta_q) + \sin(\theta_q)\underline{\mathbf{v}}, \cos(\theta_r) + \sin(\theta_r)\underline{\mathbf{w}}))$$

is smooth and S^3-equivariant (where S^3 acts on $S^2 \times S^2 \times S^2$ diagonally). For $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in S^2$ this in turn restricts to a smooth map $f = f_{\underline{\mathbf{u}},\underline{\mathbf{v}}} : S^2 \to S^2$, by $f(\underline{\mathbf{w}}) = \phi(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}})$. Since S^2 is path-connected, the homotopy class of f in $\pi_2(S^2) = \mathbb{Z}$ (the degree of f) is independent of the choice of $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$.

Now f is not S^3-equivariant. However, for $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in S^1 \cap S^2 = \{\pm\underline{\mathbf{i}}\}$, the resulting map $f_{\underline{\mathbf{u}},\underline{\mathbf{v}}}$ is S^1-equivariant, and so the degree of $f_{\underline{\mathbf{u}},\underline{\mathbf{v}}}$ is determined by its values on $\pm\underline{\mathbf{i}}$, by Corollary 2.2.

Explicitly, since $e_x(W) = e_y(W) = e_z(W) = 1$ we have, for any $\varepsilon_p, \varepsilon_q, \varepsilon_r \in \{\pm 1\}$,

$$W(\cos(\theta_p) + \varepsilon_p \sin(\theta_p)\underline{\mathbf{i}}, \cos(\theta_q) + \varepsilon_q \sin(\theta_q)\underline{\mathbf{i}}, \cos(\theta_r) + \varepsilon_r \sin(\theta_r)\underline{\mathbf{i}})$$

$$= \cos(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r) + \sin(\varepsilon_p\theta_p + \varepsilon_q\theta_q + \varepsilon_r\theta_r)\underline{\mathbf{i}},$$

so by Corollary 3.2

$$f_{\varepsilon_p\underline{\mathbf{i}},\varepsilon_q\underline{\mathbf{i}}}(\varepsilon_r\underline{\mathbf{i}}) = -\varepsilon_p\varepsilon_q\varepsilon_r\underline{\mathbf{i}},$$

and then by Corollary 2.2 the degree of $f_{\varepsilon_p\underline{\mathbf{i}},\varepsilon_q\underline{\mathbf{i}}}$ is $-\varepsilon_p\varepsilon_q$. But this contradicts the fact that the degree of $f_{\underline{\mathbf{u}},\underline{\mathbf{v}}}$ is independent of the choice of $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$. □

Corollary 4.2 (Scott-Wiegold conjecture) *Every one-relator product of three cyclic groups is nontrivial.*

Proof. As remarked in the Introduction, the problem easily reduces to the case where the cyclic groups concerned are finite, of pairwise distinct prime orders. The resulting one-relator product G has a presentation

$$\langle x, y, z | x^p = y^q = z^r = W(x, y, z) = 1\rangle.$$

If, for example, the exponent sum $e_x(W)$ is divisible by p, then the normal closure of y and z in G has index p, and G is nontrivial. We may therefore assume that $e_x(W)$ is coprime to p. Similarly we may assume that $e_y(W)$ is coprime to q and that $e_z(W)$ is coprime to r. The hypotheses of Theorem 4.1 are therefore satisfied, and so there is an essential representation $\rho : G \to SO(3)$. In particular, G has elements x, y, z of orders p, q, r respectively, so is nontrivial. □

5 Applications

In this final section we note an easy extension of the Scott-Wiegold conjecture, and apply it to Dehn surgery on knots.

Corollary 5.1 *If a one-relator product of n groups $G_1, \dots, G_n$ is trivial, then at least $n-2$ of the G_i are perfect.*

Proof. By abelianisation, we can see that

$$(G_1 * \cdots * G_n)^{ab} \cong G_1^{ab} \oplus \cdots \oplus G_n^{ab}$$

is cyclic, so each G_i^{ab} is cyclic. Suppose by way of contradiction that G_i^{ab} is nontrivial for $i = 1, 2, 3$. Then the free product $G_1^{ab} * G_2^{ab} * G_3^{ab}$ of three cyclic groups is a homomorphic image of $G_1 * \cdots * G_n$, via some epimorphism ϕ, say. Since $G_1 * \cdots * G_n$ is the normal closure of some element W, it follows that $G_1^{ab} * G_2^{ab} * G_3^{ab}$ is the normal closure of $\phi(W)$, contradicting Corollary 4.2. □

Corollary 5.2 *If a connected sum of n 3-manifolds $M_1, \dots, M_n$ can be obtained by Dehn surgery on a knot in S^3, then at least $n-2$ of the M_i are $\mathbb{Z}$-homology spheres.*

Proof. By hypothesis, there is a knot exterior $M = S^3 \setminus N(k)$ such that each of S^3 and $M_1 \# \cdots \# M_n$ is obtained from M by a Dehn filling (attaching a solid torus to the torus boundary of M). The effect of Dehn filling on the fundamental group is to quotient out the normal closure of a single element (represented by the boundary of a meridional disc of the solid torus). Since $\pi_1(S^3)$ is trivial, $\pi_1(M)$ is the normal closure of a single element, and hence so is $\pi_1(M_1 \# \cdots \# M_n) = \pi_1(M_1) * \cdots * \pi_1(M)$, being a homomorphic image of $\pi_1(M)$. By Corollary 5.1, at least $n-2$ of the groups $\pi_1(M_i)$ are perfect. But the M_i are closed orientable 3-manifolds, so any that has perfect fundamental group is a $\mathbb{Z}$-homology sphere. □

Steven Boyer and Cameron Gordon have independently made the following observation. Corollary 5.2 complements a result of Valdez [24, Theorem 3] and Sayari [21, proof of Lemma 1.1]: if $M_1, \dots, M_n$ are as in Corollary 5.2, then at least $n-1$ of the M_i are lens spaces. Combining the two results gives a universal bound for the number of prime factors in a 3-manifold obtained by Dehn surgery on a knot:

Corollary 5.3 *If a connected sum of n 3-manifolds $M_1, \dots, M_n$ can be obtained by Dehn surgery on a knot in S^3, then $n \leq 3$. If $n = 3$ then two of the M_i are lens spaces and the third is a $\mathbb{Z}$-homology sphere.*

6 Generalisations

Here we discuss two possible generalisations of the Scott-Wiegold conjecture which do not follow from the arguments of this paper. The first of these was proposed in [7] and [9].

Conjecture 1 *A one-relator product of three nontrivial groups can never be trivial.*

Corollary 5.1 reduces this conjecture to the case where at least one of the groups is perfect. Since our proof involved the explicit construction of a nontrivial representation into a linear group, it will not generalise to the extreme situation where, for example, none of the factor groups in the one-relator product admit any nontrivial (finite-dimensional) representations into linear groups. An example of such a group is Higman's group [14]

$$\langle a, b, c, d \mid a^2b = ba,\ b^2c = cb,\ c^2d = dc,\ d^2a = ad \rangle.$$

The second was suggested to me by Hamish Short.

Conjecture 2 *A one-relator product of* $2n + 1$ *cyclic groups cannot be the normal closure of* n *elements.*

Corollary 4.2 is the case $n = 1$ of this conjecture. If we try to use the same method to prove the case $n = 2$, we find ourselves considering a certain smooth map

$$S^2 \times S^2 \times S^2 \times S^2 \times S^2 \to S^3 \times S^3$$

and trying to prove that one of the points $(\pm 1, \pm 1)$ belongs to the image. There is no obvious (to me) homotopy invariant that can be used to do so (by analogy with the class in $\pi_2(S^2)$ that was used in the proof of Theorem 4.1).

Since a one-relator product of two cyclic groups can be trivial, it follows that an n-relator product of $2n$ cyclic groups can be also be trivial. Thus Conjecture 2 would be the strongest possible result of this nature for cyclic groups.

Conjectures 1 and 2 are both special cases of a natural common generalisation, which I state here on the basis of no evidence, and little hope of proof using existing methods.

Conjecture 3 *A one-relator product of* $2n+1$ *nontrivial groups cannot be the normal closure of* n *elements.*

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