Tree actions of automorphism groups

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Abstract

We introduce conditions on a group action on a tree that are sufficient for the action to extend to the automorphism group. We apply this to two different classes of one-relator groups: certain Baumslag-Solitar groups and one-relator groups with centre. In each case we derive results about the automorphism group, and deduce that there are relatively few outer automorphisms.

1 Introduction

In this note we study the extension of an action of a group G on a tree T to an action of the automorphism group $\mathrm{Aut}(G)$ on T, and exploit such extensions to derive structural information on the automorphism groups of certain one-relator groups.

We call an action of a group G on a tree T *proper* if no edge of T is fixed by the stabiliser of a vertex of T, and *locally tame* if, whenever e, f are edges of T with a common endpoint v, such that the stabiliser of e is contained in that of f, then e, f belong to the same orbit under the stabiliser of v. We prove that, for a group G acting on a tree T with infinite cyclic stabilisers, these conditions are sufficient to ensure that the action of G on T extends to an action of $\mathrm{Aut}(G)$ on T.

Under these conditions, Bass-Serre theory [11] applied to the tree action of $\mathrm{Aut}(G)$ gives information on the structure of $\mathrm{Aut}(G)$. For free products with amalgamation a result of Karrass, Pietrowski and Solitar [7] treats the structure of the automorphism group. This result applies in more generality than ours, but is less specific in its conclusions.

After the work for this paper was complete, we learnt of a paper [9] by M. Pettet, in which it is shown that any proper, locally tame action of a group G on a tree T (Pettet uses the term *edge group incomparability* for local tameness) extends to an action of the subgroup $A \subseteq \mathrm{Aut}(G)$ consisting of those automorphisms that permute the vertex stabilisers. In particular, Pettet's theorem includes as a special case part of our Theorem A below. Indeed, with minor rewording, the second part of our proof of Theorem A gives an alternative proof of Pettet's theorem. Also of interest in this context is a paper of Bass and Jiang [1].

The first class of one-relator groups that we consider are the Baumslag-Solitar groups [2]. Suppose that $G = BS(p,q) = \langle x, t \mid t^{-1}x^p t = x^q \rangle$ is a Baumslag-Solitar group, where p, q are integers, neither of which divides the other. Then it turns out that the natural action of G on a tree is proper and locally tame, so that we get a corresponding action of $\mathrm{Aut}(G)$ on the same tree, and by Bass-Serre theory [11] information on the structure of $\mathrm{Aut}(G)$. In this situation we obtain a presentation for $\mathrm{Aut}(G)$, extending a result of Collins [3] which treats the case where p, q are coprime. We also deduce that the outer automorphism group $\mathrm{Out}(G)$ is a finite dihedral group.

This should be compared with a theorem of Collins and Levin [4], which shows that if $p \geq 2$ properly divides q, then $\mathrm{Aut}(BS(p,q))$ is not finitely generated. Also of interest in this context is a further result of Collins [3] giving a finite presentation for $\mathrm{Aut}(G)$ in the case where $p = 1$ and $|q| > 1$, depending on the prime divisors of $|q|$.

A second class of one-relator groups that we treat are those with non-trivial centre. Here we specifically exclude the free abelian group of rank two, whose behaviour is quite different (and whose automorphisms are well understood). Non-abelian one-relator groups with non-trivial centre are known [10] to have a special structure: any such group G is torsion-free, is generated by two elements, and it acts on a tree T such that each vertex-stabiliser and each edge-stabiliser is infinite cyclic. It turns out that we can choose T such that the action of G is proper and locally tame (except in the case of the Baumslag-Solitar groups $BS(p,p)$, which we discuss separately in Theorem D below) and so it extends to an action of $\mathrm{Aut}(G)$. Again, we can use this action to analyse $\mathrm{Aut}(G)$, to find a presentation, and to compute $\mathrm{Out}(G)$.

Specifically, we can show that $\mathrm{Out}(G)$ is isomorphic to C_2 or $C_2 \times C_2$ if G^{ab} has torsion-free rank 1, and to D_∞ or $C_2 \times D_\infty$ if G^{ab} has torsion-free rank 2.

In particular, the mapping class group of the Klein bottle is the outer automorphism group of $\langle x, y \mid x^2 = y^2 \rangle$, and we recover the result that this is isomorphic to the Klein 4-group.

The paper is organised as follows. In section 2 below we prove the theorem about extending group actions on trees to the automorphism group. In section 3 we apply this to Baumslag-Solitar groups, and in section 4 to one-relator groups with centre.

2 Extending tree actions to automorphism groups

We consider situations in which an action of a group G on a tree T extends to an action of its automorphism group $\mathrm{Aut}(G)$ on T.

We let G_z denote the stabiliser in G of a vertex or edge z of T, so that $G_z = \{g \in G \,:\, g(z) = z\}$. We say that a G-tree T is *proper* if G_e is a proper subgroup of G_v whenever e is an edge of T incident at a vertex v. We will say that T is *locally tame* if, whenever e, f are edges of T with a common incident vertex v and $G_e \subseteq G_f$, then $e = g(f)$ for some $g \in G_v$.

Theorem A *Let T be a proper, locally tame G-tree such that every vertex and edge of T has infinite cyclic stabiliser. Then $\mathrm{Aut}(G)$ acts on T in such a way that the diagram*

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr] & & \mathrm{Aut}(G) \arrow[dl] \\
& \mathrm{Aut}(T) &
\end{tikzcd}$$

commutes, where the horizontal map is the canonical one, and the vertical maps are the actions on T.

Proof. We first show that the vertex-stabilisers $\{G_v; v \in VT\}$ are determined, as subgroups of G, by purely group-theoretic properties, from which it follows that they are permuted by $\mathrm{Aut}(G)$. There are two cases to consider. If T consists of a doubly-infinite path, then it follows that G is isomorphic to the the Klein-bottle group $\langle x, y \mid x^2 = y^2 \rangle$, with vertex stabilisers the conjugates of $\langle x \rangle$ and $\langle y \rangle$. But these are precisely the cyclic subgroups of G that contain $Z(G) = \langle x^2 \rangle$, so they are determined by a group-theoretic property, as required.

Suppose then that T is not a doubly infinite path. The vertex stabilisers G_v are commensurable, and we claim that they are the maximal cyclic subgroups of G with commensurator G (again, a group-theoretic property). To see this, let $1 \neq x \in G$ with $\mathrm{Comm}_G(\langle x \rangle) = G$. Now either x fixes a vertex v (in which case $x \in G_v$) or the minimal $\langle x \rangle$–invariant subtree in T is a reduced doubly infinite path P. Now for any $g \in G$ the cyclic subgroup $\langle gxg^{-1} \rangle$ is commensurable with $\langle x \rangle$, and so there exist integers p, q such that $gx^pg^{-1} = x^q$. Now x^q carries P to P, whilst gx^pg^{-1} carries gP to gP. Hence $P = gP$ and so P is G–invariant, and it follows that $P = T$, a contradiction.

Hence in both cases the vertex stabilisers G_v are determined by a group-theoretic property, and are therefore permuted by $\mathrm{Aut}(G)$. Now if $G_v = G_{v'}$ then G_v fixes the T–geodesic from v to v', and this contradicts the fact that T is proper, unless $v = v'$. Therefore each vertex-stabiliser G_v is uniquely determined by v. Moreover $G_{g(v)} = g \cdot G_v \cdot g^{-1}$, and so $\mathrm{Aut}(G)$ permutes the vertices of T in a manner compatible with the G-action on T.

Now suppose $\alpha \in \mathrm{Aut}(G)$. We extend the permutation α of VT to a continuous map $\phi_\alpha : T \to T$ by mapping each edge $[u, v]$ of T homeomorphically onto the geodesic $[\alpha(u), \alpha(v)]$. We cannot (yet) assume that this is a tree automorphism. However it can be made into a surjective graph-map by subdividing the first copy of T, and can be made G-equivariant by twisting the G-action on the second copy of T by α, that is by replacing the given action $\rho : G \to \mathrm{Aut}(T)$ by $\rho \circ \alpha$. To show that ϕ_α is a tree automorphism, it suffices to show that it is locally injective, and it is enough to consider the action of ϕ_α on the neighbourhood of a vertex.

Suppose then that v is a vertex of T. Let x be a generator of the cyclic group G_v. If e is an edge incident at v with (say) $\iota(e) = v$ and G_e generated by x^p, then $(\phi_{\alpha^{-1}} \circ \phi_\alpha)(e)$ is a path in T beginning at v and fixed by x^p. In particular, the first edge f in this path satisifes $G_e \subseteq G_f$, so by local tameness $f = x^s \cdot e$ for some s. If e' is another edge incident at v such that f is the first edge of the path $(\phi_{\alpha^{-1}} \circ \phi_\alpha)(e')$, then a similar argument shows that $f = x^t \cdot e'$ for some t. Hence $e' = x^{s-t} \cdot e$. Now the composite $\phi_{\alpha^{-1}} \circ \phi_\alpha$ is G–equivariant with respect to the given action $\rho : G \to \mathrm{Aut}(T)$. It follows that the first edge of the path

$$(\phi_{\alpha^{-1}} \circ \phi_\alpha)(e') = x^{s-t} \cdot (\phi_{\alpha^{-1}} \circ \phi_\alpha)(e)$$

is $x^{s-t} \cdot f$, so $f = x^{s-t} \cdot f$, so $p \mid (s - t)$, and so $e' = e$.

This completes the proof that each ϕ_α is locally injective, and hence a tree automorphism. It is easy to check that $\alpha \mapsto \phi_\alpha$ is an action of $\mathrm{Aut}(G)$ on T, and that the triangle commutes. $\square$

3 Baumslag-Solitar Groups

Let $G = G_{p,q}$ be the Baumslag-Solitar Group

$$G_{p,q} = \langle x,\ t \mid t^{-1}x^p t = x^q \rangle,$$

with $p > 1, q > 1$. In [3] Collins gave a presentation for $\mathrm{Aut}(G)$ under the assumption that p and q are coprime. Using Theorem A we are able to extend Collins' result to the case when neither of p, q is a multiple of the other. We note that G has automorphisms α, ι given by $\alpha : x \mapsto x\,, t \mapsto xt$ and $\iota : x \mapsto x^{-1}\,, t \mapsto t$ and it is easy to see that neither α nor ι is inner. We denote the inner automorphisms of G determined by x and t by θ and τ respectively.

Theorem B *Let $G = G_{p,q}$ be the Baumslag-Solitar Group*

$$G_{p,q} = \langle\ x,\ t \mid t^{-1}x^p t = x^q\ \rangle,$$

where p, q are integers with $p > 1$ and $q > 1$ and such that neither is a multiple of the other. Then $\mathrm{Aut}(G)$ has a presentation with generators $\theta, \tau, \alpha, \iota$ subject to the relations

$$\begin{array}{rclrclrclrcl} \tau^{-1}\theta^p\tau &=& \theta^q\,, & \alpha^{q-p} &=& \theta^q\,, & \iota^2 &=& 1\,, & \iota\theta &=& \theta^{-1}\iota\,, \\ \iota\tau &=& \tau\iota\,, & \iota\alpha &=& \alpha^{-1}\iota\,, & \alpha\theta &=& \theta\alpha\,, & \alpha\tau\alpha^{-1} &=& \theta\tau, \end{array}$$

and $\mathrm{Out}(G)$ is dihedral of order $2|p-q|$.

Proof. Since G is an HNN group with base group $\langle x \rangle$ and stable letter t, it is the fundamental group of a graph of groups in which the underlying graph has a single vertex and a single edge. If T is the corresponding G-tree, then G acts transitively on VT and on ET with infinite cyclic vertex and edge stabilisers. In particular, there is a vertex v_0 whose stabiliser is $\langle x \rangle$. Now if v is a vertex of T with stabiliser $G_v = \langle y \rangle$, y some conjugate of x, then G_v has two orbits of edges incident at v, with stabilisers $\langle y^p \rangle$ and $\langle y^q \rangle$ respectively. Hence the G-action is proper and locally tame. By Theorem A, $\mathrm{Aut}(G)$ also acts on T in a manner compatible with the G-action.

Now let $\phi \in \mathrm{Aut}(G)$. Since G is transitive on vertices, ϕ is equivalent, modulo inner automorphisms, to one fixing v_0, and hence sending x to $x^{\pm 1}$. After composition with ι if necessary, we may suppose that ϕ fixes x. Then

$\phi(t)^{-1}x^p\phi(t) = x^q = t^{-1}x^pt$, so that $\phi(t)t^{-1}$ is contained in the centraliser $Z_G(\langle x^p\rangle)$ of the edge-stabiliser $G_e = \langle x^p\rangle$. Now $Z_G(\langle x^p\rangle)$ is generated by the stabilisers of the two endpoints of the edge e, namely $\langle x\rangle$ and $\langle txt^{-1}\rangle$. We can thus express $\phi(t)$ as a word in x, t in normal form for the HNN extension G, and such that the t-letters have alternating exponents ± 1, beginning and ending with $+1$. Let m be the t-length of this word, and suppose that β is another automorphism fixing x and such that $\beta(t)$ has t–length n in normal form. Then it is easy to check that $\beta(\phi(t))$ has t–length mn. Putting $\beta = \phi^{-1}$, we see that $m = 1$, and so $\phi(t) = x^kt$ for some integer k. Hence $\phi = \alpha^k$.

This argument shows that $\mathrm{Aut}(G)$ is generated by θ, τ, ι and α. Since $x^pt = tx^q$, we see that $\alpha^{q-p} = \theta^q$ is inner. It is easy to see that α induces an automorphism of order $|p-q|$ on $G^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}_{|p-q|}$: hence in $\mathrm{Out}(G)$ the image of α generates a cyclic subgroup of order $|p-q|$, and since $\iota\alpha = \alpha^{-1}\iota$, we see that $\mathrm{Out}(G)$ is dihedral of order $2|p-q|$. The presentation for $\mathrm{Aut}(G)$ now follows easily. □

4 One-relator groups with centre

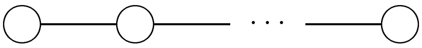
We now consider the automorphism groups of one-relator groups with centre. A well-known theorem of Murasugi [8] asserts that any such group G can be generated by two elements, and that if G is non-abelian then its centre is infinite cyclic. Pietrowski [10] later refined this description, giving a decomposition of G as a fundamental group of a graph of groups. There are two different descriptions, depending on the torsion-free rank of G^{ab}. These are exploited in Theorems C and E below.

Theorem C *Let G be a one-relator group with non-trivial centre such that G^{ab} has torsion-free rank 1. Then either $\mathrm{Out}(G)$ is the cyclic group C_2 of order 2, or $G \cong \langle x, y \mid x^p = y^p\rangle$ for some $p \geq 2$, in which case $\mathrm{Out}(G) \cong C_2 \times C_2$.*

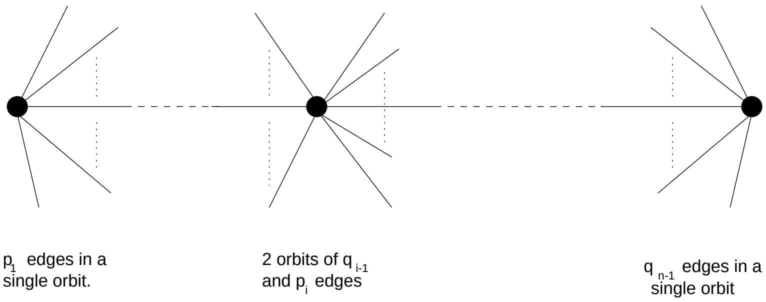
Proof. By [10], G has a presentation

$$G = \langle\ x_1, \dots, x_n \mid x_i^{p_i} = x_{i+1}^{q_i},\ i = 1, \dots, n-1\ \rangle,$$

with $(p_j, q_i) = 1$ for $j > i$. Hence G acts on a tree T properly and locally tamely, with infinite cyclic vertex and edge stabilisers, such that $G\backslash T$ has the form



Note that in T there are two kinds of vertices. Those lying over extremal vertices of $G\backslash T$ have either p_1 or q_{n-1} incident edges and the vertex stabiliser acts transitively on this set of edges. The incident edges at vertices $v \in T$ lying over non-extremal vertices of $G\backslash T$ comprise two orbits of G_v, containing p_i or q_{i-1} edges respectively.



By Theorem A this action extends to an action of $\mathrm{Aut}(G)$. Let $\phi \in \mathrm{Aut}(G)$, and consider $\phi(x_1)$. There is a unique vertex v of T with $G_v = \langle x_1 \rangle$, and G_v acts transitively on the set of p_1 edges incident at v. Hence $\phi(x_1)$ generates $G_{\phi(v)}$, which acts transitively on the set of p_1 edges incident to $\phi(v)$. The transitivity implies that $\phi(x_1)$ is conjugate to $x_1^{\pm 1}$ or to $x_n^{\pm 1}$.

Now $\phi(x_1^{p_1})$ fixes an edge e of T, such that the stabilisers of the two endpoints of e are $\langle \phi(x_1) \rangle$ and $\langle \phi(x_2) \rangle$ respectively. Hence if $\phi(x_1)$ is conjugate to $x_n^{\pm 1}$, it follows that $q_{n-1} = p_1$ and $p_{n-1} = q_1$. The latter contradicts the coprimality condition, unless $n - 1 \leq 1$. Hence $n = 2$ and $p_1 = q_1 = p$, say. In this case there is an automorphism of order 2 that interchanges x_1 and x_2. Composing ϕ with this automorphism reduces us to the case where $\phi(x_1)$ is conjugate to $x_1^{\pm 1}$.

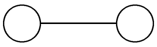
We may now assume that $n \geq 2$ and that $\phi(x_1)$ is conjugate to $x_1^{\pm 1}$. There is a (non-inner) automorphism of order 2 given by $x_i \mapsto x_i^{-1}$ for all i. Composing ϕ with this automorphism if necessary, and then with some inner automorphism, we may assume that ϕ fixes x_1.

For $i = 1, \ldots, n$, let $v_i \in VT$ be the unique vertex with stabiliser $\langle x_i \rangle$. Note that v_{i+1} is adjacent in T to v_i. Suppose inductively that ϕ in fact fixes $x_1, \ldots, x_k$ for some $k < n$. Then, under its action on T, ϕ fixes $v_1, \ldots, v_k$. Let e be the edge of T joining v_k to v_{k+1}. Then $G_e = \langle x_k^{p_k} \rangle = \langle x_{k+1}^{q_k} \rangle$, and $\phi(e) = x_k^m \cdot e$ for some m. Since p_k is coprime to $Q = q_1 \cdots q_{k-1}$, we may assume that m is a multiple of Q. Hence x_k^m centralises $x_1, \ldots, x_k$. Composing ϕ with the inner automorphism induced by x_k^m, we may assume that, in addition to fixing $x_1, \ldots, x_k$, and hence the vertices $v_1, \ldots, v_k$, the automorphism ϕ also fixes the edge e, and hence the vertex v_{k+1}. Hence ϕ fixes x_{k+1}. By induction, we may assume that, modulo inner automorphisms, ϕ fixes $x_1, \ldots, x_n$ and hence is the identity.

This shows that $\mathrm{Out}(G)$ is a homomorphic image of $C_2 \times C_2$ in the case $G = \langle\ x_1, x_2 \mid x_1^p = x_2^p\ \rangle$, and a homomorphic image of C_2 otherwise. To check that, in fact, $\mathrm{Out}(G)$ is isomorphic to $C_2 \times C_2$ or C_2, we observe that $x_i \mapsto x_i^{-1}$ and (in the case $n = 2$ with $p_1 = q_1$) $x_1 \leftrightarrow x_2^{\pm 1}$ induce non-trivial automorphisms of G^{ab}. $\square$

Remark. The action of $\mathrm{Aut}(G)$ on T induces an action of $\mathrm{Out}(G)$ on $G\backslash T$. In most of the cases we consider, this action is trivial, and there results a decomposition of $\mathrm{Aut}(G)$ as the fundamental group of a graph of groups over the underlying graph $G\backslash T$. For example, in Theorem B, $G\backslash T$ has one vertex and one edge, so that $\mathrm{Out}(G)$ acts trivially and $\mathrm{Aut}(G)$ is an HNN extension with base group $\langle \alpha, \iota, \theta \rangle$.

In Theorem C, in the generic case $G \not\cong \langle x, y \mid x^p = y^p \rangle$, $\mathrm{Out}(G)$ acts trivially on $G\backslash T$, leading to a decomposition of $\mathrm{Aut}(G)$ as a stem product of finite dihedral groups. But if $G \cong \langle x, y \mid x^p = y^p \rangle$ then $G\backslash T$ is the graph



and in $\mathrm{Out}(G) \cong C_2 \times C_2$, one factor acts trivially and the other by interchanging the two vertices. Similar considerations apply to Theorems D and E below.

Theorem D *Let G be the one-relator group*

$$G = \langle x, t | t^{-1} x^p t = x^{\epsilon p} \rangle, \epsilon = \pm 1, p > 1.$$

If $\epsilon = +1$ then $\mathrm{Out}(G) \cong D_\infty \times C_2$, and if $\epsilon = -1$ then $\mathrm{Out}(G) \cong D_{2p} \times C_2$ (of order $8p$).

Proof. Let $H = \langle x^p \rangle$. If $\epsilon = +1$ then $H = Z(G)$ and so H is characteristic in G. If $\epsilon = -1$ then $H = Z(K)$ where K is the normal closure in G of x and t^2. Of the three subgroups of index 2 in G, only K has the property that

K/G' contains the torsion subgroup of G/G', and so K is characteristic in G. Hence so is $H = Z(K)$.

Now the conjugates of the subgroup $\langle x \rangle$ of G are precisely those subgroups maximal with respect to the property of containing H as a subgroup of finite index. Hence the conjugates of $\langle x \rangle$ are permuted by $\mathrm{Aut}(G)$. Thus if $\phi \in \mathrm{Aut}(G)$ then $\phi(x) = gx^{\pm 1}g^{-1}$ for some $g \in G$. Modulo inner automorphisms, we may assume that $\phi(x) = x^{\pm 1}$.

Since H is characteristic, ϕ induces an automorphism $\overline{\phi}$ of $G/H \cong \mathbb{Z}_p * \mathbb{Z}$ that fixes the $\mathbb{Z}_p$ factor setwise. It follows from a theorem of Fouxe-Rabinovitch [6] that for some integers a, b we have $\overline{\phi}(tH) = x^a t^{\pm 1} x^b H$, and hence for some integer c, $\phi(t) = x^a t^{\pm 1} x^{b+cp}$. Composing with conjugation by x^a, we may assume that $\phi(x) = x^{\pm 1}$ and that for some integer d, $\phi(t) = t^{\pm 1} x^d$.

Now G possesses automorphisms $\beta_1, \beta_2, \beta_3$ defined as follows:

$$\begin{array}{rclcrcl}
\beta_1(x) & = & x^{-1} & , & \beta_1(t) & = & t \\
\beta_2(x) & = & x & , & \beta_2(t) & = & t^{-1} \\
\beta_3(x) & = & x & , & \beta_3(t) & = & tx.
\end{array}$$

The β_i , $(i = 1, 2, 3)$ satisfy, modulo inner automorphisms, the relations

$$\beta_1^2 = \beta_2^2 = (\beta_1\beta_2)^2 = (\beta_1\beta_3)^2 = (\beta_2\beta_3)^2 = 1.$$

The above argument shows that the images of $\beta_1, \beta_2, \beta_3$ generate $\mathrm{Out}(G)$ which is therefore an image of $D_\infty \times C_2$.

Now if $\epsilon = -1$ then $\beta_3^{2p} : t \mapsto tx^{2p} = x^{-p}tx^p$ and hence β_3 has order dividing $2p$ in $\mathrm{Out}(G)$ and $\mathrm{Out}(G)$ is an image of $D_{2p} \times C_2$.

To complete the proof, we verify that the images of the β_i in $\mathrm{Aut}(G/G')$ generate a subgroup isomorphic to $D_\infty \times C_2$ or to $D_{2p} \times C_2$ respectively, and this is straightforward. □

Theorem E *Let G be a one-relator group with non-trivial centre such that G^{ab} has torsion-free rank 2. If G is not free abelian then either* $\mathrm{Out}(G)$ *is the infinite dihedral group D_∞ or* $\mathrm{Out}(G)$ *is the direct product $D_\infty \times C_2$.*

Proof. Pietrowski [10] shows that G has a presentation

$$G = \langle x_1, \ldots, x_n, t \mid t^{-1}x_1t = x_n,\ x_i^{p_i} = x_{i+1}^{q_i}, i = 1, \ldots, n-1 \rangle$$

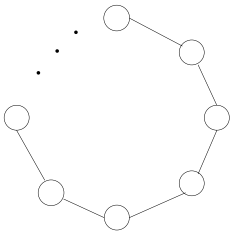
where $p_i, q_i \geq 2$ for every $i = 1, \ldots, n-1$ and $(p_i, q_j) = 1$ for every $i > j$. If $n = 1$ then G is free abelian of rank 2, and so we can assume that $n \geq 2$.

Since $Z(G) \neq 1$ it is easy to see that $p_1 \dots p_{n-1} = q_1 \dots q_{n-1}$. It follows that neither of p_1, q_{n-1} is a multiple of the other. For suppose that $p_1 = \lambda q_{n-1}$. Then $\lambda p_2 \dots p_{n-1} = q_1 \dots q_{n-2}$. But this is impossible, since $(p_{n-1}, q_i) = 1$ for every $i = 1, \dots, n-2$. A similar argument applies if we suppose that $q_{n-1} = \lambda p_1$.

In order to obtain a locally tame action on a tree, we replace x_n by $t^{-1}x_1t$ and so present G as

$$G = \langle x_1, \dots, x_{n-1}, t | x_{n-1}^{p_{n-1}} = t^{-1}x_1^{q_{n-1}}t,\ x_i^{p_i} = x_{i+1}^{q_i}, i = 1, \dots, n-2 \rangle$$

where $p_i, q_i \geq 2$ for every $i = 1, \dots, n-1$ and $(p_i, q_j) = 1$ for every $i > j$. Hence G acts on a tree T, properly and locally tamely with infinite cyclic vertex and edge stabilizers, such that $G \backslash T$ is a simple circuit



with $n-1$ vertices. By Theorem A, $Aut(G)$ also acts on T in a manner compatible with the G-action.

If $n = 2$ then G is a Baumslag-Solitar group of the form treated in Theorem D. So we can assume that $n > 2$.

Let $\phi \in \mathrm{Aut}(G)$ and let $v_1 \in T$ have stabilizer $\langle x_1 \rangle$. Now $\phi(v_1)$ lies in the G–orbit of some vertex v_k and hence $\phi(x_1)$ is conjugate to some $x_k^{\pm 1}$, $k \in \{1, \dots, n-1\}$. Counting edges incident to v_1 and v_k in their orbits under G_{v_1} and G_{v_k} we see that either $p_1 = p_k$ and $q_{n-1} = q_{k-1}$ or $p_1 = q_{k-1}$ and $q_{n-1} = p_k$.

In the first case, if $k > 1$ then we deduce that $(p_1, q_1) = 1$. Hence q_1 is coprime to each of $p_1, \dots, p_{n-1}$ which contradicts the fact that $p_1 \cdots p_{n-1} = q_1 \cdots q_{n-1}$.

In the second case, cancelling $p_1 = q_{k-1}$ in the equation $p_1 \cdots p_{n-1} = q_1 \cdots q_{n-1}$, we deduce that $p_2 \cdots p_{n-1} = q_1 \cdots q_{k-2} q_k \cdots q_{n-1}$. But q_1 is coprime to each of $p_2, \dots, p_{n-1}$: so this is a contradiction, unless q_1 were cancelled, in which case $k = 2$. Then we have $q_{n-1} = p_2$, and cancelling these, we get $p_3 \cdots p_{n-1} = q_2 \cdots q_{n-2}$. But q_2 is coprime to each of $p_3, \dots, p_{n-1}$, so this is again a contradiction unless q_2 were cancelled, in which case $n = 3$.

Assume first that $n > 3$. Then modulo inner automorphisms, $\phi(x_1) = x_1^{\mp 1}$. Now G admits automorphisms α_1, ι given by

$$\alpha_1 : x_i \mapsto x_i,\ t \mapsto x_1 t$$

$$\iota : x_i \mapsto x_i^{-1},\ t \mapsto t$$

Composing with ι if necessary, we may assume that $\phi(x_1) = x_1$. We now repeat the inductive argument given in the proof of Theorem C to deduce that, by composing with a suitable inner automorphism, we may assume that ϕ in fact fixes $x_1, \dots, x_n$.

Then

$$\phi(t)t^{-1}x_1 t\phi(t)^{-1} = \phi(t)x_n\phi(t)^{-1} = \phi(tx_nt^{-1}) = \phi(x_1) = x_1.$$

Hence $\phi(t)t^{-1} \in Z_G(x_1) = \langle x_1 \rangle$. Therefore $\phi(t) = x_1^k t$ for some $k \in \mathbb{Z}$ and $\phi = \alpha_1^k$. Hence, $\mathrm{Aut}(G)$ is generated by α_1, ι and the inner automorphisms.

Clearly, ι and $\alpha_1 \circ \iota$ have order 2 in $\mathrm{Aut}(G)$, and so $\mathrm{Out}(G)$ is a homomorphic image of D_∞. To see that in fact $\mathrm{Out}(G) \cong D_\infty$, note that G^{ab} is free abelian of rank 2 (since it is 2-generated by Murasugi's theorem [8]), and that α_1 and ι induce the automorphisms $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $GL(2, \mathbb{Z})$. Since A_1, A_2 generate an infinite dihedral subgroup of $GL(2, \mathbb{Z})$, it follows that $\mathrm{Out}(G) \cong D_\infty$.

Now the above argument also applies when $n = 3$ and $k = 1$. So finally we consider the case $n = 3, k = 2$ with $p_1 = q_1$ and $p_2 = q_2$. Then G has a presentation

$$G = \langle x_1, x_2, x_3, t | t^{-1}x_1t = x_3, x_1^{p_1} = x_2^{p_1}, x_2^{p_2} = x_3^{p_2} \rangle$$

and so admits an automorphism σ given by

$$\sigma : x_1 \leftrightarrow x_2\,, x_3 \mapsto tx_2t^{-1}\,, t \mapsto t^{-1}.$$

Given our automorphism ϕ with $\phi(x_1) = x_2^{\pm 1}$ we can compose with σ and with ι if necessary to assume that $\phi(x_1) = x_1$, and then proceed as above. We conclude that $\mathrm{Aut}(G)$ is generated by α_1, ι, σ and the inner automorphisms, and it is then easy to check that $\mathrm{Out}(G) \cong D_\infty \times C_2$, with the C_2 factor generated by the image of σ. $\square$

Corollary. *Let G be a one-relator group with non-trivial centre. Then the virtual cohomological dimension* $\mathrm{vcd}(\mathrm{Aut}(G))$ *of* $\mathrm{Aut}(G)$ *is at most* 2.

Proof. If G is cyclic then $\mathrm{Aut}(G)$ is finite, whilst if $G \cong \mathbb{Z} \times \mathbb{Z}$ then $\mathrm{Aut}(G) \cong GL(2, \mathbb{Z})$ and so $\mathrm{vcd}(\mathrm{Aut}(G)) = 1$. We may therefore assume that G is non-abelian.

If G^{ab} has torsion-free rank 1 then Theorem C applies. As in the proof of Theorem C, G acts on a tree T with infinite cyclic stabilisers and finite quotient, and so $G/Z(G)$ acts on T with finite stabilisers and finite quotient. It follows (see for example [5], chapter IV.1) that $G/Z(G)$ is free-by-finite. Since $\mathrm{Out}(G)$ is finite we have $\mathrm{vcd}(\mathrm{Aut}(G)) = \mathrm{vcd}(G/Z(G)) = 1$.

If G^{ab} has torsion-free rank 2 then Theorem E applies. Again $\mathrm{vcd}(G/Z(G)) = 1$, but now $\mathrm{vcd}(\mathrm{Out}(G)) = 1$ and so $\mathrm{vcd}(\mathrm{Aut}(G)) = 2$. $\square$

Remark. A similar result holds for the Baumslag-Solitar groups $G = BS(p, q)$ studied in this paper. Here $\mathrm{cd}(G) = 2$ and if neither of p, q divides the other, or if $p = -q > 1$, then $\mathrm{Out}(G)$ is finite by Theorems A and D, so $\mathrm{vcd}(\mathrm{Aut}(G)) = 2$. If $p = q > 1$ then G has a centre and G^{ab} is free abelian of rank 2, and the proof of the corollary shows that again $\mathrm{vcd}(\mathrm{Aut}(G)) = 2$.

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