

Erratum: Some results on one-relator surface groups

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In [1], a number of results were proved about groups of the form $\pi_1(S)/N(R)$, where S is a closed orientable surface, and $N(R)$ is the normal closure of a single element $R \in \pi_1(S)$.

Unfortunately, one of the results of that paper, Theorem 4.1 (the Freiheitssatz), is not true in the full generality stated there. The purpose of the present note is to explain the gap in the proof, give some examples where the statement of the theorem is false, and to present a conjecture regarding the additional hypotheses necessary for the Freiheitssatz to hold.

The error in the proof of [1, Theorem 4.1] does not affect any of the other results stated in [1].

Elements of $\pi_1(S)$ can be represented as closed curves α in the surface S (usually with self-intersections). Any closed curve in S defines a homology class in $H_1(S)$. Given two closed curves α, β in S , we let $\langle \alpha, \beta \rangle \in \mathbb{Z}$ denote the value of the intersection form on $H_1(S) \times H_1(S)$, applied to the pair of homology classes represented by α and β .

If $R \in \pi_1(S)$ is represented by the closed curve α , then we also use the notation $\pi_1(S)/\alpha$ for $\pi_1(S)/N(R)$.

Given an *embedded* (that is, *simple*) closed curve β in S , I falsely stated in [1, Theorem 4.1] the following assertion, intended as the analogue of Magnus'

Freiheitssatz for one-relator groups [3]:

Assertion 1 *Let S be a closed oriented surface, α a closed curve in S , and β a simple closed curve in S such that α is not homotopic to a curve disjoint from β . Then $\pi_1(S \setminus \beta) \rightarrow \pi_1(S)/\alpha$ is injective.*

The alleged proof of Assertion 1 in [1] is wrong. It seeks to apply Magnus' Freiheitssatz [3] to show that a map $\pi_1(F) \rightarrow \pi_1(F')/\gamma$ is injective, where F is a certain noncompact surface, F' is obtained from F by adjoining an annulus to a pair of circle components of ∂F , and γ is a certain closed curve in F' . This argument is not valid, for the following reason. Although $\pi_1(F)$ and $\pi_1(F')$ are both free, and the inclusion-induced homomorphism $\pi_1(F) \rightarrow \pi_1(F')$ is injective, the image of this homomorphism is not a free factor of $\pi_1(F')$, so Magnus' result does not apply.

Indeed there are some quite simple counterexamples to Assertion 1, as follows:

Example 2 *Let β be a nonseparating, embedded closed curve in the closed orientable surface S , and let α be any closed curve that intersects β transversely in a single point. Then*

$$1 \neq [\alpha, \beta] \in \pi_1(S \setminus \beta) \cap N(\alpha).$$

Example 3 *Let α, β be essential simple closed curves contained in a punctured torus $T \subset S$, such that $\langle \alpha, \beta \rangle \neq 0$. Then*

$$1 \neq \partial T \in \pi_1(S \setminus \beta) \cap N(\alpha).$$

These examples share the following common feature: in each case, the *geometric intersection number* $|\alpha \cap \beta|$ (that is, the minimum number of points of intersection of β with any curve isotopic to α) is equal to the absolute value of $\langle \alpha, \beta \rangle$. In other words, α and β can be isotoped so as to intersect transversely in finitely many points, all of the same sign. This suggests the following

Conjecture 4 *Let α, β, S be as in Assertion 1. Suppose that $|\alpha \cap \beta|$ is strictly greater than the absolute value of $\langle \alpha, \beta \rangle$. Then $\pi_1(S \setminus \beta) \rightarrow \pi_1(S)/\alpha$ is injective.*

There is some evidence in favour of Conjecture 4. Firstly, in [1], a special case of Assertion 1 is proved, using a much simpler (and, more importantly, correct) argument:

Theorem 5 ([1, Proposition 3.10]) *Let S be a closed oriented surface, α a closed curve in S , and β a simple closed curve in S such that α is not homotopic to a curve disjoint from β , and that $\langle \alpha, \beta \rangle = 0$. Then $\pi_1(S \setminus \beta) \rightarrow \pi_1(S)/\alpha$ is injective.*

The proof of this result uses the fact that the infinite cyclic cover of S corresponding to the kernel of the homomorphism $\langle -, \beta \rangle : \pi_1(S) \rightarrow \mathbb{Z}$ can be constructed from a collection F_n , $n \in \mathbb{Z}$ of copies of $F := S \setminus \beta$, with $F_n \cap F_{n+1}$ an annulus. (See [1, Proposition 2.1].)

Combining this fact with a result of Klyachko [2], we can also prove the following special case of Conjecture 4:

Theorem 6 *Let S be a closed oriented surface, α a closed curve in S , and β a simple closed curve in S such that α is not homotopic to a curve that meets β at most once, and that $\langle \alpha, \beta \rangle = \pm 1$. Then $\pi_1(S \setminus \beta) \rightarrow \pi_1(S)/\alpha$ is injective.*

The methods of [1] can also be used to prove the following:

Theorem 7 *Let S be a closed oriented surface, α a closed curve in S , and β_1, β_2 two disjoint simple closed curves in S such that α is not homotopic to a curve disjoint from β_1 or from β_2 . Then $\pi_1(S \setminus (\beta_1 \cup \beta_2)) \rightarrow \pi_1(S)/\alpha$ is injective.*

The proofs of the last two results will appear in a separate article.

References

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- [3] W. Magnus, Über diskontinuierliche Gruppen mit einer definierenden Relation (Der Freiheitssatz), *J. reine angew. Math.* **163** (1930), 141–165.