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Large Components of Bipartite Random Mappings

Jennie Hansen
Department of Actuarial Mathematics and Statistics, Heriot–Watt University, Edinburgh EH14 4AS, UK

Jerzy Jaworski
Department of Discrete Mathematics, Adam Mickiewicz University, 60-769 Poznań, Poland

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ABSTRACT

A bipartite random mapping $T_{K,L}$ of a finite set $V = V_1 \cup V_2$, $|V_1| = K$ and $|V_2| = L$, into itself assigns independently to each $i \in V_1$ its unique image $j \in V_2$ with probability $1/L$ and to each $i \in V_2$ its unique image $j \in V_1$ with probability $1/K$. We study the connected component structure of a random digraph $G(T_{K,L})$, representing $T_{K,L}$, as $K \to \infty$ and $L \to \infty$. We show that, no matter how K and L tend to infinity relative to each other, the joint distribution of the normalized order statistics for the component sizes converges in distribution to the Poisson-Dirichlet distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0$ for every $i \geq 1\}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A bipartite random mapping $T_{K,L}$ of a finite set $V = V_1 \cup V_2$, $V_1 = \{1, 2, \ldots, K\}$ and $V_2 = \{K+1, K+2, \ldots, K+L\}$, into itself assigns independently to each $i \in V_1$ its unique image $j \in V_2$ with probability $1/L$ and to each $i \in V_2$ its unique image $j \in V_1$ with probability $1/K$. The mapping $T_{K,L}$ can be represented by a random bipartite digraph $G(T_{K,L})$ on a set of 'red' labelled vertices corresponding to the set V_1 and a set of 'blue' labelled vertices corresponding to the set V_2. So, for example, $G(T_{K,L})$ has a directed edge from red vertex i to blue vertex j if and only if $T_{K,L}(i) = j$. Each connected component of $G(T_{K,L})$ consists of a bipartite directed cycle with bipartite directed trees connected to the cycle. This model can be viewed as a 'two-dimensional' generalization of the uniform random mapping T_K on a single set of vertices $V = \{1, 2, ..., K\}$ where for each $i, j \in V$, $\Pr\{T_K(i) = j\} = 1/K$ and all assignments are independent. It is worth noting that although $T_{K,L}^2 = T_{K,L} \circ T_{K,L}$ restricted to V_1 (say) is a random mapping from V_1 into V_1, such that for all $i, j \in V_1$, $\Pr\{T_{K,L}^2(i) = j\} = 1/K$, $T_{K,L}^2 \neq T_K$ in distribution since, generally, the choices of the images are not independent, for example for all $i, j, v \in V_1$, $i \neq v$,

$$\Pr\{T_{K,L}^2(i) = j \,|\, T_{K,L}^2(v) = j\} = \frac{1}{K} + \frac{1}{L}\left(1 - \frac{1}{K}\right).$$

This dependence is strongest when K and L are of the same order.

Much is known (see for example the monograph by Kolchin [18]) about the component structure of the random digraph $G(T_K)$ which represents the uniform random mapping T_K. Aldous [1] has shown that the joint distribution of the normalized order statistics for the component sizes in $G(T_K)$ converges to the Poisson-Dirichlet(1/2) distribution. Also, if Y_k denotes the number of components of size k in $G(T_K)$ then the joint distribution of $(Y_1, Y_2, ..., Y_b)$ is close, in the sense of total variation, to the joint distribution of a sequence of independent Poisson random variables when $b = o(K/\log K)$ (see Arratia et.al. [3], [4]) and from this one obtains a functional central limit theorem for the component sizes (see also [10]). An analogous result for the order statistics of the cycle sizes in a random permutation was obtained by Vershik and Shmidt [22] and there is a Poisson approximation result and functional central limit theorem for the cycle sizes in a random permutation (see [2], [8]).

We are interested in determining to what extent such results hold for the component structure of $G(T_{K,L})$ as $K, L \to \infty$. Of course, in the case of bipartite random mappings there are two parameters, so the limit laws obtained may depend on how fast K and L tend to infinity relative to each other. For example, if $L = K^\alpha$ with $\alpha > 1$, then it is easy to verify by a first moment argument that, for example,

$$\Pr\{G(T_{K,K^\alpha}) \quad \text{has a component of size 2 }\} \to 0 \quad \text{as} \quad K \to \infty,$$

whereas the Poisson process approximation for uniform random mappings tells us that

$$\Pr\{G(T_K) \quad \text{has a component of size 2 }\} \to 1 - e^{-c} \quad \text{as} \quad K \to \infty$$

where $c = \lim_{K\to\infty} E(Y_2)$. So there is no Poisson approximation result for the small component sizes of $G(T_{K,K^\alpha})$ when $\alpha > 1$. On the other hand, in this paper we show that the joint distribution of the normalized order statistics for the component sizes converges in distribution to the Poisson-Dirichlet(1/2) distribution no matter how K and L tend to infinity. Our result complements and extends earlier work by Jaworski [12],[13] on asymptotic limit laws for the total number of components in $G(T_{K,L})$. To prove the Poisson-Dirichlet result, we first establish the limiting distribution for the size of the component containing a given vertex and this result may also be of independent interest.

In order to describe these results, we first give a convenient characterization of the Poisson-Dirichlet(θ) distribution which also yields a useful principle for establishing convergence in distribution to the Poisson-Dirichlet(θ) distribution on the simplex

$$\nabla = \Big\{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \quad \text{for every} \quad i \geq 1\Big\}.$$

Let $Z_1, Z_2, Z_3, ...$ be a sequence of i.i.d. random variables such that each Z_i has a Beta(θ) ($\theta > 0$) distribution with density $h(z) = \theta(1-z)^{\theta-1}$ on the unit interval $(0, 1)$. Now define a transformation, f, of the sequence $(Z_1, Z_2, ..)$ such that $f(Z_1, Z_2, ...) = (W_1, W_2, W_3, ..)$ where $W_1 = Z_1$ and $W_n = Z_n(1-Z_1)(1-Z_2)\cdots(1-Z_{n-1})$ for $n > 1$, and observe that $(W_1, W_2, ...) \in \tilde{\nabla} = \{\{x_i\} : x_i \geq 0, \sum x_i \leq 1\}$. Finally, define the map $g : \tilde{\nabla} \to \nabla$ such that $(g\{x_i\})_k$ is the kth largest term in the sequence $\{x_i\} \in \tilde{\nabla}$; then the random sequence $g \circ f(Z_1, Z_2, ...) = (Q_1, Q_2, Q_3, ...) \in \nabla$ has a Poisson-Dirichlet(θ) distribution. The following convergence principle is an important consequence of this characterization: suppose that $(Z_1(n), Z_2(n), ...)$ is a sequence of random variables such that the joint distribution of $(Z_1(n), Z_2(n), ...)$ converges to the joint distribution of the variables $(Z_1, Z_2, ...)$, then the joint distribution of the random sequence $g \circ f(Z_1(n), Z_2(n), ...) = (Q_1(n), Q_2(n), ...)$ converges to the Poisson-Dirichlet(θ) distribution. For further details see Hansen [11] and the references therein.

To see how the convergence principle can be applied to random bipartite mappings, we introduce some additional notation. Recall that K denotes the number of 'red' vertices and L denotes the number of 'blue' vertices, and $N = K + L$ is the total number of vertices in the random digraph $G(T_{K,L})$. In addition, let $\mathcal{C}_1$ denote the component in $G(T_{K,L})$ which contains the red vertex labelled 1. If $\mathcal{C}_1 \neq G(T_{K,L})$, then let $\mathcal{C}_2$ denote the component in $G(T_{K,L}) \setminus \mathcal{C}_1$ which contains the smallest red vertex; otherwise, set $\mathcal{C}_2 = \emptyset$. For $t > 2$ we define $\mathcal{C}_t$ iteratively: If $G(T_{K,L}) \setminus (\mathcal{C}_1 \cup ... \cup \mathcal{C}_{t-1}) \neq \emptyset$, then let $\mathcal{C}_t$ denote the component in $G(T_{K,L}) \setminus (\mathcal{C}_1 \cup ... \cup \mathcal{C}_{t-1})$ which contains the smallest red vertex; otherwise, set $\mathcal{C}_t = \emptyset$. For $t \geq 1$, let $C_t = |\mathcal{C}_t|$ and define the sequence $(X_1(N), X_2(N), \ldots) = (X_1, X_2, \ldots)$ by

$$X_1 = \frac{C_1}{N},\ X_2 = \frac{C_2}{N - C_1},\ \ldots,\ X_t = \frac{C_t}{N - C_1 - C_2 - \ldots - C_{t-1}},\ \ldots$$

where $X_t = 0$ if $N - C_1 - C_2 - \ldots - C_{t-1} = 0$. In Section 3 we show that for each $t \geq 1$ and $0 < a_i < b_i < 1$, $i = 1, 2, \ldots, t$

$$\text{(1.1)} \qquad \lim_{K\to\infty} \Pr\{\, a_1 < X_1 \leq b_1,\ a_2 < X_2 \leq b_2,\ \ldots,\ a_t < X_t \leq b_t \,\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}}\,,$$

under the assumption that $L = L(K) \geq K$. We establish (1.1) by an inductive argument in Section 3. The first step in the induction is established in Section 2, where we prove

Theorem 1. *Suppose that* $L = L(K)$ *and there is a positive constant* η *such that* $0 < \eta K \leq L$ *for all* K*, then for every* $0 < a < b < 1$

$$\Pr\{\, aN < C_1 \leq bN \,\} \to \int_a^b \frac{du}{2\sqrt{1-u}} \qquad as \quad K \to \infty\,.$$

To state our second result, let D_1 denote the size of the largest connected component in $G(T_{K,L})$, let D_2 denote the size of the second largest component and so on. It is easy to check that

$$g \circ f(X_1(N), X_2(N),\ \ldots) = (D_1/N, D_2/N,\ \ldots)\,,$$

so using the convergence principle for the Poisson-Dirichlet distribution, we obtain from (1.1)

Theorem 2. *Let $N = K + L$ and $D_1, D_2, \dots$ be as defined above, then*

$$\left(\frac{D_1}{N}, \frac{D_2}{N}, \dots\right) \xrightarrow{d} \mathcal{PD}(1/2) \quad as \quad L, K \to \infty,$$

where $\mathcal{PD}(1/2)$ denotes the Poisson-Dirichlet(1/2) distribution on the simplex

$$\nabla = \Big\{\{x_i\} : \quad \sum x_i \le 1, \quad x_i \ge x_{i+1} \ge 0 \quad for\ \ every \quad i \ge 1\Big\}.$$

2. THE SIZE OF A CONNECTED COMPONENT

We start with the exact joint probability distribution of the random variables (R_1, B_1), where $R_1 = R_1(i)$ is the number of red vertices (vertices from V_1) in the connected component $\mathcal{C}_1$ of $G(T_{K,L})$ containing a given red vertex i and $B_1 = B_1(i)$ is the number of blue vertices (vertices from V_2) in this component. We will assume that $i = 1$.

Lemma 1. *For $k = 0, 1, \dots, K-1$ and $l = 1, \dots, L$ we have*

$$\Pr\{R_1 = k+1,\ B_1 = l\}$$

$$= \binom{K-1}{k}\binom{L}{l}\left(\frac{k+1}{K}\right)^{l-1}\left(1 - \frac{k+1}{K}\right)^{L-l}\left(\frac{l}{L}\right)^{k}\left(1 - \frac{l}{L}\right)^{K-1-k}$$

$$\times \frac{1}{KL}\sum_{j=1}^{min\{l,k+1\}} \frac{(l)_j}{l^j}\frac{(k+1)_j}{(k+1)^j}\,(k+l+1-j)\,.$$

Proof. There are $\binom{K-1}{k}$ ways to choose k red vertices and $\binom{L}{l}$ ways to choose l blue vertices which form the connected component containing the vertex "1". We have $(K-k-1)^{L-l}$ ways to map the remaining $L-l$ blue vertices into the remaining $K-k-1$ red vertices and $(L-l)^{K-k-1}$ ways to map $K-k-1$ red vertices into $L-l$ blue vertices. Finally, there are

$$(k+1)^{l-1} l^k \sum_{j=1}^{min\{l,k+1\}} \frac{(l)_j}{l^j}\frac{(k+1)_j}{(k+1)^j}\,(k+l+1-j)$$

digraphs representing connected bipartite mappings on $k+1$ red and l blue vertices [5], [12]. The result now follows immediately.

□

To prove Theorem 1 we need two more lemmas which are stated and proved below.

Lemma 2. *Fix $\eta > 0$, then for all $K, L > 0$ such that $\eta K \le L \le K^{7/4}$ and for every $0 < a < b < 1$, there is a constant $C(a,b,\eta)$ which only depends on a, b, and η, such that*

$$\left|\Pr\{aN < C_1 \le bN\} - \int_a^b \frac{dx}{2\sqrt{1-x}}\right| \le \frac{C(a,b,\eta)}{K^{1/16}},$$

where $N = K + L$.

Remark. Lemma 2 can be shown to hold more generally. In particular, it can be shown that if $\eta > 0$ and $1 \le \alpha < 2$ are fixed, then for all $K > 0$ and L, $\eta K \le L \le K^\alpha$,

$$\left| \Pr\{aN < C_1 \le bN\} - \int_a^b \frac{dx}{2\sqrt{1-x}} \right| \le \frac{C(a,b,\eta,\alpha)}{K^\zeta}$$

where $\zeta = \min\{\frac{1}{8}, \frac{1}{2} - \frac{\alpha}{4}\}$ and $C(a,b,\eta,\alpha)$ is a constant which only depends on a, b, η, and α. However, for the proof of Theorem 1, it suffices to prove Lemma 2 in the case where $\alpha = 7/4$. The restriction to this case also simplifies the proof of the lemma.

Proof. Throughout the proof $C(a,b,\eta)$ will denote any constant which may depend on a, b, and η but which does not depend on K. Now fix $K > 0$, fix L such that $\eta K \le L \le K^{7/4}$, and suppose m is such that $aN < m \le bN$ where $N = K + L$. Let $x = m/N$ (so $x \in (a, b]$) and let $[Kx]$ and $[Lx]$ be integers such that

$$[Kx] + [Lx] = m \quad \text{and} \quad |[Kx] - Kx| \le 1\,, \quad |[Lx] - Lx| \le 1\,. \tag{2.1}$$

Then

$$\Pr\{\, C_1 = m \,\} = \sum_{-[Lx]<j<[Kx]} \Pr\{R_1 = [Kx] - j,\ B_1 = [Lx] + j\} \tag{2.2}$$

where R_1 is the number of red vertices, and B_1 is the number of blue vertices in connected component $\mathcal{C}_1$. Now we split the above sum into two sums:

$$\text{(i)} \quad \sum_{|j| \le \tau D(x)} \qquad\qquad \text{(ii)} \quad \sum_{|j| > \tau D(x)} \tag{2.3}$$

where

$$D(x) = \frac{KL}{K+L}\sqrt{\frac{x(1-x)}{K+L}} \le C(a,b,\eta)\frac{K}{\sqrt{L}} \tag{2.4}$$

and $\tau = K^{1/16}$.

In order to approximate the sums (2.3) we must investigate the joint distribution of (R_1, B_1). Observe that the expression for $\Pr\{R_1 = k+1,\ B_1 = l\}$, given in Lemma 1, can be split into three factors. The first factor

$$\binom{K-1}{k}\left(\frac{l}{L}\right)^k\left(1 - \frac{l}{L}\right)^{K-1-k} \tag{2.5}$$

represents the probability of the event of k positive outcomes in the binomial distribution with parameters $K-1$ and $\frac{l}{L}$. Similarly, the second one

$$\binom{L}{l}\left(\frac{k+1}{K}\right)^l\left(1 - \frac{k+1}{K}\right)^{L-l} \tag{2.6}$$

represents the probability of the event of l positive outcomes in the binomial distribution with parameters L and $\frac{k+1}{K}$. Approximations for both of these expressions with appropriate error bounds can be obtained from the proof of the de Moivre-Laplace Theorem (see Feller [9], p.182) in the case when $k+1=[Kx]-j$, $l=[Lx]+j$ and $|j|\le\tau D(x)$. In particular,

$$\binom{L}{l}\left(\frac{k+1}{K}\right)^l\left(1-\frac{k+1}{K}\right)^{L-l}=\frac{\exp(-\tilde{y}^2/2)}{\sqrt{2\pi L\frac{k+1}{K}\left(1-\frac{k+1}{K}\right)}}\cdot(1+\tilde{\rho}_j(x))$$

where $\tilde{y}$, the normalized deviation from the mean value, is given by

$$\tilde{y}=\frac{[Lx]+yD(x)-L\Big([Kx]/K-yD(x)/K\Big)}{\sqrt{L\Big([Kx]/K-yD(x)/K\Big)\Big(1-[Kx]/K+yD(x)/K\Big)}}$$

with $y=j/D(x)$. Now let

$$\tilde{\Delta}_j(x)=[Lx]+yD(x)-L\Big([Kx]/K-yD(x)/K\Big),$$

then

$$|\tilde{\rho}_j(x)|\le C(a,b,\eta)\cdot\max\left(\frac{|\tilde{\Delta}_j(x)|^3}{L^2},\frac{|\tilde{\Delta}_j(x)|}{L}\right),$$

and since

$$|\tilde{\Delta}_j(x)|\le C(a,b,\eta)L\tau D(x)/K\le C(a,b,\eta)\tau\sqrt{L}$$

we have

$$|\tilde{\rho}_j(x)|\le C(a,b,\eta)\cdot\frac{\tau^3}{\sqrt{L}}\le C(a,b,\eta)K^{-5/16}.$$

We also have

$$\tilde{y}=y\sqrt{\frac{L}{L+K}}+\tilde{\gamma}_j(x)$$

where $|\tilde{\gamma}_j(x)|\le C(a,b,\eta)K^{-1/8}$. Using the expression for $\tilde{y}$, we obtain

$$\frac{\tilde{y}^2}{2}=\frac{y^2L}{2(L+K)}+y\sqrt{\frac{L}{L+K}}\cdot\tilde{\gamma}_j(x)+\frac{1}{2}(\tilde{\gamma}_j(x))^2=\frac{y^2L}{2(L+K)}+\tilde{\varepsilon}_j(x),$$

where

$$|\tilde{\varepsilon}_j(x)|\le C(a,b,\eta)|\tau\tilde{\gamma}_j(x)|\le C(a,b,\eta)K^{-1/16}.$$

So provided $a<x\le b$ and $|j|\le\tau D(x)$, we have

$$\binom{L}{l}\left(\frac{k+1}{K}\right)^l\left(1-\frac{k+1}{K}\right)^{L-l}=\frac{\exp\left(\frac{-y^2L}{2(K+L)}+\tilde{\varepsilon}_j(x)\right)}{\sqrt{2\pi L\frac{k+1}{K}\left(1-\frac{k+1}{K}\right)}}\cdot\Big(1+\tilde{\rho}_j(x)\Big)\tag{2.7}$$

where

$$|\tilde{\varepsilon}_j(x)|\le C(a,b,\eta)K^{-1/16}\quad\text{and}\quad|\tilde{\rho}_j(x)|\le C(a,b,\eta)K^{-5/16}.$$

Similarly one can show that

$$\binom{K-1}{k}\Big(\frac{l}{L}\Big)^k\Big(1-\frac{l}{L}\Big)^{K-1-k} = \frac{\exp\Big(\frac{-y^2K}{2(K+L)}+\bar{\varepsilon}_j(x)\Big)}{\sqrt{2\pi(K-1)\frac{l}{L}\Big(1-\frac{l}{L}\Big)}}\cdot\Big(1+\bar{\rho}_j(x)\Big) \tag{2.8}$$

where

$$|\bar{\varepsilon}_j(x)| \le C(a,b,\eta)\cdot K^{-1/8} \quad \text{and} \quad |\bar{\rho}_j(x)| \le C(a,b,\eta)K^{-1/8}.$$

Finally, for $k+1=[Kx]-j$ and $l=[Lx]+j$ with $|j|\le \tau D(x)$, one can show, as in the proof of Theorem 7 in [13], that the remaining factor in $\Pr\{R_1=k+1,\ B_1=l\}$ is given by

$$\frac{1}{(k+1)L}\sum_{i=1}^{\min\{k+1,l\}}\frac{(l)_i(k+1)_i}{(l)^i(k+1)^i}(k+l+1-i) = \sqrt{\frac{\pi x(K+L)}{2KL}}\cdot(1+\hat{\varepsilon}(x,j)) \tag{2.9}$$

where

$$\hat{\varepsilon}(x,j) \le \frac{C(a,b,\eta)}{K^{1/8}}\,.$$

Hence by (2.7)-(2.9) we obtain for $a<x=\frac{m}{N}\le b$, and $|j|\le\tau D(x)$

$$\Pr\{R_1=[Kx]-j, B_1=[Lx]+j\} = \Pr\{R_1=[Kx]-yD(x), B_1=[Lx]+yD(x)\}$$

$$= \frac{1}{K+L}\cdot\frac{1}{2\sqrt{1-x}}\,\frac{(K+L)\sqrt{K+L}}{KL}\cdot\frac{1}{\sqrt{2\pi x(1-x)}}\exp\Big(\frac{-y^2}{2}\Big)\cdot(1+\rho_j(x))$$

where $|\rho_j(x)|\le C(a,b,\eta)\cdot K^{-1/16}$.

It follows that for $a<x=\frac{m}{N}\le b$

$$\begin{aligned}
&\sum_{|j|\le\tau D(x)}\Pr\{R_1=[Kx]-j, B_1=[Lx]+j\}\\
&=\frac{1}{N}\cdot\frac{1}{2\sqrt{1-x}}\sum_{|j|\le\tau D(x)}\frac{1}{D(x)}\cdot\frac{\exp\left(-y^2/2\right)}{\sqrt{2\pi}}\cdot\Big(1+\rho_j(x)\Big)\\
&=\frac{1}{N}\cdot\frac{1}{2\sqrt{1-x}}\cdot(1+\delta_x)
\end{aligned} \tag{2.10}$$

where $|\delta_x|\le C(a,b,\eta)\cdot K^{-1/16}$.

It remains to determine a bound for the second sum in (2.3). Since it is a 'two-sided' sum, we consider one side of the sum; the other case follows by similar calculations. The first step is to note that for all $k=0,1,\dots,K-1$ and $l=1,\dots,L$

$$\Pr\{R_1=k+1, B_1=l\} \le \binom{L}{l}\left(\frac{k+1}{K}\right)^l\left(1-\frac{k+1}{K}\right)^{L-l}$$

since

$$\binom{K-1}{k}\left(\frac{l}{L}\right)^k\left(1-\frac{l}{L}\right)^{K-1-k}\leq 1$$

and (see the end of the proof of Lemma 1)

$$\frac{1}{(k+1)L}\sum_{j=1}^{min\{l,k+1\}}\frac{(l)_j}{l^j}\frac{(k+1)_j}{(k+1)^j}(l+k+1-j)=\frac{l}{L}\Pr\{G(T_{k+1,l})\text{ is connected }\}\leq\frac{l}{L}.$$

It follows that

$$\begin{aligned}&\sum_{j\geq\tau D(x)}\Pr\{R_1=[Kx]-j,\ B_1=[Lx]+j\}\\ &\leq\sum_{l\geq[Lx]+\tau D(x)}\binom{L}{l}\left(\frac{m-l}{K}\right)^l\left(1-\frac{m-l}{K}\right)^{L-l}\end{aligned}\tag{2.11}$$

where $k+1=m-l$. Now the terms in the sum on the right hand side of (2.11) are binomial probabilities, but as we sum over the values of $l\geq[Lx]+\tau D(x)$, the probability of 'success', $\frac{m-l}{K}$, changes as l increases. Nevertheless, we claim that for all $l\geq[Lx]+\tau D(x)$, we have

$$\begin{aligned}&\binom{L}{l+1}\left(\frac{m-l-1}{K}\right)^{l+1}\left(1-\frac{m-l-1}{K}\right)^{L-l-1}\\ &\leq\binom{L}{l}\left(\frac{m-l-1}{K}\right)^{l}\left(1-\frac{m-l-1}{K}\right)^{L-l}\\ &\leq\binom{L}{l}\left(\frac{m-l}{K}\right)^{l}\left(1-\frac{m-l}{K}\right)^{L-l}.\end{aligned}\tag{2.12}$$

The first inequality follows from the unimodality of the binomial distribution. To establish the second inequality, fix l and define the function

$$f_l(w)=\left(\tilde{x}-\frac{w}{K}\right)^l\left(1-\tilde{x}+\frac{w}{K}\right)^{L-l}$$

where $\tilde{x}=\frac{[Kx]}{K}$. It is not hard to show that $f_l'(w)<0$ for $w\geq 0$, so if $l=[Lx]+z$, where z is some integer greater than $\tau D(x)$, then we have

$$\left(\frac{m-l}{K}\right)^l\left(1-\frac{m-l}{K}\right)^{L-l}=f_l(z)\geq f_l(z+1)=\left(\frac{m-l-1}{K}\right)^l\left(1-\frac{m-l-1}{K}\right)^{L-l},$$

since $m-l=[Kx]-z$, and inequality (2.12) is established.

Now let $l_0=[Lx]+\lfloor\tau D(x)\rfloor$ and $k_0+1=m-l_0=[Kx]-\lfloor\tau D(x)\rfloor$ (where $\lfloor x\rfloor$ denotes the greatest integer less than x), then

$$\begin{aligned}&\sum_{l\geq[Lx]+\tau D(x)}\binom{L}{l}\left(\frac{m-l}{K}\right)^l\left(1-\frac{m-l}{K}\right)^{L-l}\leq L\cdot\binom{L}{l_0}\left(\frac{m-l_0}{K}\right)^{l_0}\left(1-\frac{m-l_0}{K}\right)^{L-l_0}\\ &\leq C(a,b,\eta)L\exp\left(\frac{-y^2L}{2(K+L)}\right)\leq C(a,b,\eta)K^{7/4}\exp(-K^{1/16})\end{aligned}\tag{2.13}$$

where $y = \lfloor \tau D(x) \rfloor / D(x) \sim \tau = K^{1/16}$ and the second inequality above follows from (2.7). Therefore

$$\sum_{j \geq \tau D(x)} \Pr\{R_1 = [Kx] - j,\ B_1 = [Lx] + j\} \leq C(a,b,\eta) K^{7/4} \exp(-K^{1/16}) . \tag{2.14}$$

Similarly,

$$\sum_{j \leq -\tau D(x)} \Pr\{R_1 = [Kx] - j,\ B_1 = [Lx] + j\} \leq C(a,b,\eta) K^{7/4} \exp(-K^{1/16}). \tag{2.15}$$

Combining the bounds (2.14)-(2.15) and the approximation (2.10), we obtain

$$\Pr\{C_1 = m\} = \frac{1}{N} \cdot \frac{1}{2\sqrt{1 - m/N}} \cdot (1 + \delta_x) + \gamma_m$$

where

$$|\delta_x| \leq C(a,b,\eta) \cdot K^{-1/16}$$

and

$$\gamma_m = \sum_{|j| \geq \tau D(x)} \Pr\{R_1 = [Kx] - j, B_1 = [Lx] + j\} \leq C(a,b,\eta) K^{7/4} \exp(-K^{1/16}) .$$

Hence

$$\Pr\{aN < C_1 \leq bN\} = \sum_{aN < m \leq bN} \frac{1}{N} \cdot \frac{1}{2\sqrt{1 - m/N}} \cdot (1 + \delta_x) + \sum_{aN < m \leq bN} \gamma_m$$

and it follows that

$$\left| \Pr\{aN < C_1 \leq bN\} - \int_a^b \frac{dx}{2\sqrt{1-x}} \right| \leq \frac{C(a,b,\eta)}{K^{1/16}} .$$

□

To approximate $\Pr\{aN < C_1 \leq bN\}$ in the case where $L = L(K) \geq K^{7/4}$ we take an indirect approach. First, we recall (see [21]) that if T_K is a uniform random mapping of V_1 into V_1, then for any $0 < b < 1$, we have

$$\lim_{K \to \infty} \Pr\left\{ \frac{C_1(T_K)}{K} \leq b \right\} = \int_0^b \frac{dx}{2\sqrt{1-x}} \tag{2.16}$$

where $C_1(T_K)$ denotes the size of the component which contains 1 in the directed graph $G(T_K)$ which represents T_K. Equation (2.16) also holds for the uniform model of random mappings *without* loops, $\hat{T}_K$, for which

$$\Pr\{\hat{T}_K(i) = i\} = 0 \quad \text{and} \quad \Pr\{\hat{T}_K(i) = j\} = 1/(K-1) \quad \text{if} \quad i \neq j ,$$

for any $i, j \in V_1$, and all assignments are independent (see [15]). Now consider the random mapping $T^2 = T^2_{K,L} = T_{K,L} \circ T_{K,L}$ restricted to the red vertices V_1. It is clear that T^2 is not a uniform random mapping on V_1, but if $L = L(K) \geq K^{7/4}$, then Lemma 3 below states that (2.16) holds with $C_1(T_K)$ (or $C_1(\hat{T}_K)$) replaced by $C_1(T^2)$, the size of the connected component containing 1 in $G(T^2)$, the digraph on the red vertices which represents T^2. In the proof of Theorem 1, we show that if $L = L(K) \geq K^{7/4}$, then the variables $\frac{C_1(T^2)}{K}$ and $\frac{C_1(T_{K,L})}{N}$, where $N = K + L$, are 'close' with high probability and hence

$$\Pr\{aN < C_1 \leq bN\} \sim \int_a^b \frac{dx}{2\sqrt{1-x}}.$$

Before proving Lemma 3, we define two random mapping models which are related to $\hat{T}_K$ and whose properties are exploited in the proof of the lemma. To construct the models (see also [14], [15] and [17]) , we start with $G(\hat{T}_K)$, the random digraph which represents the uniform random mapping without loops $\hat{T}_K$ on V_1. To define the first model, fix $0 \leq M \leq K$, and let $G(\hat{T}_K, M)$ denote the random digraph obtained by deleting $K - M$ edges at random from $G(\hat{T}_K)$. To construct the second model, fix $0 \leq q \leq 1$ and let $Q \sim Bin(K, q)$ be a binomial random variable which is independent of $\hat{T}_K$, and let $G(\hat{T}_K; q)$ denote the random digraph obtained by choosing Q vertices at random and deleting the edges in $G(\hat{T}_K)$ which are directed from the chosen vertices. Clearly, given $Q = K - M$, then $G(\hat{T}_K; q)$ has the same distribution as $G(\hat{T}_K, M)$. Actually these two models of random mappings are related to each other in a very similar way as two classical models of random graphs – $G_{n,p}$ and $G_{n,M}$. One can easily adapt the methods which were used to study the relation between them (see [6] and [19]) to get the corresponding results about the equivalence of $G(\hat{T}_K, M)$ and $G(\hat{T}_K; q)$. In particular, if $q \sim 1 - M/K$ then many characteristics of the random digraphs $G(\hat{T}_K, M)$ and $G(\hat{T}_K; q)$ have the same asymptotic distribution as $K \to \infty$.

To state the key property of both $G(\hat{T}_K, M)$ and $G(\hat{T}_K; q)$, we need the following definition. A graph property $\mathcal{A}$ is *decreasing* (*increasing*) if given that a graph G has $\mathcal{A}$, then every spanning subgraph (supergraph) of G has this property also. We call such properties *monotone*. In a similar way one can define monotone properties of digraphs. Denote by $A(M)$ the number of digraphs (without loops) on K vertices with M vertices of out-degree 1 and with $K - M$ vertices of out-degree 0, having a property $\mathcal{A}$. One can check that if $\mathcal{A}$ is a decreasing property, then

$$(K-1)(K-M)A(M) \geq (M+1)A(M+1) \tag{2.17}$$

(see [16] for a more general inequality). Notice that (2.17) implies that

$$\Pr\big\{G(\hat{T}_K, M) \text{ has } \mathcal{A}\big\} = \frac{A(M)}{\binom{K}{M}(K-1)^M} \geq \frac{(M+1)A(M+1)}{(K-1)(K-M)\binom{K}{M}(K-1)^M}$$

$$= \frac{A(M+1)}{\binom{K}{M+1}(K-1)^{M+1}} = \Pr\big\{G(\hat{T}_K, M+1) \text{ has } \mathcal{A}\big\} \text{ , i.e.,}$$

The probability that the random digraph $G(\hat{T}_K, M)$ has a decreasing property $\mathcal{A}$ is a non-increasing function of the parameter M.

To prove the corresponding fact for $G(\hat{T}_K;q)$ let us introduce the function

$$f(z) = \Pr\bigl\{G(\hat{T}_K;z) \text{ has } A\bigr\} = \sum_{M=0}^{K} A(M)\, z^{K-M} \left(\frac{1-z}{K-1}\right)^{M} .$$

Then, since

$$f'(z) = \sum_{M=0}^{K-1} A(M)\,(K-M) z^{K-M-1} \left(\frac{1-z}{K-1}\right)^{M} - \sum_{M=1}^{K} A(M)\,\frac{M}{K-1}\, z^{K-M} \left(\frac{1-z}{K-1}\right)^{M-1}$$

$$= \sum_{M=0}^{K-1} A(M)\,(K-M) z^{K-M-1} \left(\frac{1-z}{K-1}\right)^{M} - \sum_{M=0}^{K-1} A(M+1)\,\frac{M+1}{K-1}\, z^{K-M-1} \left(\frac{1-z}{K-1}\right)^{M} ,$$

(2.17) implies that the first derivative of f is non-negative for all z, $0 \le z \le 1$, i.e.,
The probability that the random digraph $G(\hat{T}_K;q)$ has a decreasing property $\mathcal{A}$ is a non-decreasing function of the parameter q.

Let $C_1(M)$ and $C_1(q)$ denote the size of the connected component which contains 1 in the random digraphs $G(\hat{T}_K,M)$ and $G(\hat{T}_K;q)$, respectively. Notice, that '$C_1(M) \le x$' and '$C_1(q) \le x$' are decreasing properties. Therefore, for any M_1, M_2 and $x > 0$, we have

$$0 \le M_1 \le M_2 \le K \quad \Rightarrow \quad \Pr\bigl\{C_1(M_2) \le x\bigr\} \le \Pr\bigl\{C_1(M_1) \le x\bigr\} , \tag{2.18}$$

and for any q_1, q_2

$$0 \le q_1 \le q_2 \le 1 \quad \Rightarrow \quad \Pr\bigl\{C_1(q_1) \le x\bigr\} \le \Pr\bigl\{C_1(q_2) \le x\bigr\} . \tag{2.19}$$

We also need the following result about the size $C_1(q)$ of the connected component of $G(\hat{T}_K;q)$ containing a given vertex (see [15]):

If $q\sqrt{K} \to 0$ as $K \to \infty$, then for every $0 < b < 1$,

$$\lim_{K\to\infty} \Pr\Bigl\{\frac{C_1(q)}{K} \le b\Bigr\} = \int_0^b \frac{dx}{2\sqrt{1-x}} . \tag{2.20}$$

We now proceed to prove

Lemma 3. *Suppose that $L = L(K) \ge K^{7/4}$ for all $K > 0$, then for any $0 < b < 1$,*

$$\lim_{K\to\infty} \Pr\Bigl\{\frac{C_1(T^2)}{K} \le b\Bigr\} = \int_0^b \frac{dx}{2\sqrt{1-x}} .$$

Remark. Lemma 3 can also be shown to hold more generally. In particular, if $\alpha > 3/2$ is fixed and $L = L(K) \ge K^{\alpha}$ for all $K > 0$, then the conclusion of the lemma still holds. We prove the result for the case required in the proof of Theorem 1. The proof of the lemma is also simplified by restricting to the case $\alpha = 7/4$.

Proof. Throughout the proof C will denote any constant which does not depend on K (but which may depend on b). Now let $G(T^2)$ denote the random digraph on V_1 which represents the mapping $T^2 = T^2_{K,L}$ and form the random digraph $G(\tilde{T})$ from $G(T^2)$ by deleting edges as follows. For each $v \in V_1$, delete the edge out of v in $G(T^2)$ if $T^2(v) = v$ or if the vertex $T_{K,L}(v) \in V_2$ has in-degree greater than or equal to 2 in $G(T_{K,L})$, the bipartite digraph which represents $T_{K,L}$, and let $G(\tilde{T})$ denote the resulting random digraph. Let $C_1(\tilde{T})$ denote the connected component in $G(\tilde{T})$ which contains vertex 1. To prove the lemma, it suffices to show that if $L = L(K) \geq K^{7/4}$ for all $K > 0$, then for every $0 < b < 1$

$$\lim_{K\to\infty} \Pr\Big\{\frac{C_1(\tilde{T})}{K} \leq b\Big\} = \int_0^b \frac{dx}{2\sqrt{1-x}} \tag{2.21}$$

and

$$\Big|\frac{C_1(\tilde{T})}{K} - \frac{C_1(T^2)}{K}\Big| \to 0 \quad \text{in probability as } K \to \infty\,. \tag{2.22}$$

Let Z denote the number of vertices in $G(\tilde{T})$ with out-degree 0, then

$$\begin{aligned}
\Pr\Big\{\frac{C_1(\tilde{T})}{K} \leq b\Big\} &= \sum_{j=0}^{K} \Pr\Big\{\frac{C_1(\tilde{T})}{K} \leq b \,\Big|\, Z = j\Big\} \Pr\{Z = j\} \\
&= \sum_{j=0}^{K} \Pr\Big\{\frac{C_1(K-j)}{K} \leq b\Big\} \Pr\{Z = j\}.
\end{aligned}$$

The second equality holds since given $Z = j$, the digraph $G(\tilde{T})$ has the same distribution as $G(\hat{T}_K, K-j)$. Now

$$\begin{aligned}
E(Z) &= K \cdot \Pr\big\{\text{vertex 1 has out-degree } 0\big\} = K \cdot \big(1 - \Pr\big\{\text{vertex 1 has out-degree } 1\big\}\big) \\
&= K \cdot \big(1 - \Pr\big\{\, T^2(1) \neq 1 \text{ and for every } v \in V_1 \setminus \{1\} : \;\; T_{K,L}(v) \neq T_{K,L}(1) \,\big\}\big) \\
&= K\Big(1 - \Big(1 - \frac{1}{L}\Big)^{K-1} \cdot \frac{K-1}{K}\Big) \;\leq\; CK^{1/4},
\end{aligned}$$

so $\Pr\{Z \geq K^{3/8}\} \leq CK^{-1/8}$. It follows from this bound and (2.18) that

$$\begin{aligned}
\Pr\Big\{\frac{C_1(\tilde{T})}{K} \leq b\Big\} &\leq \sum_{j=0}^{\lfloor K^{3/8}\rfloor} \Pr\Big\{\frac{C_1(K-j)}{K} \leq b \,\Big|\, Z = j\Big\} \Pr\{Z = j\} + CK^{-1/8} \\
&\leq \Pr\Big\{\frac{C_1(K - \lfloor K^{3/8}\rfloor)}{K} \leq b\Big\} + CK^{-1/8}.
\end{aligned} \tag{2.23}$$

Next, set $q = K^{-9/16}$, and recall that Q, the number of edges deleted from $G(\hat{T}_K)$ to form $G(\hat{T}_K; q)$, is a $Bin(K, q)$ variable. So Chebyshev's inequality yields

$$\Pr\{Q \leq K^{3/8}\} \leq CK^{-7/16}.$$

This bound and (2.18) imply that

$$\begin{aligned}
\Pr\Big\{\frac{C_1(q)}{K} \leq b\Big\} &\geq \sum_{j > K^{3/8}} \Pr\Big\{\frac{C_1(q)}{K} \leq b \,\Big|\, Q = j\Big\} \Pr\{Q = j\} \\
&= \sum_{j > K^{3/8}} \Pr\Big\{\frac{C_1(K-j)}{K} \leq b\Big\} \Pr\{Q = j\} \\
&\geq \Pr\Big\{\frac{C_1(K - \lfloor K^{3/8}\rfloor)}{K} \leq b\Big\}(1 - CK^{-7/16}).
\end{aligned}$$

Combining this inequality with (2.23) we obtain the upper bound

$$\Pr\Big\{\frac{C_1(\tilde{T})}{K} \le b\Big\} \le \Pr\Big\{\frac{C_1(q)}{K} \le b\Big\} + CK^{-1/8}$$

and since $\lim_{K\to\infty} q\sqrt{K} = \lim_{K\to\infty} K^{-1/16} = 0$, we have by (2.20)

$$\limsup_{K\to\infty} \Pr\Big\{\frac{C_1(\tilde{T})}{K} \le b\Big\} \le \limsup_{K\to\infty} \Pr\Big\{\frac{C_1(q)}{K} \le b\Big\} = \int_0^b \frac{dx}{2\sqrt{1-x}}. \tag{2.24}$$

To obtain a lower bound, we note that

$$\Pr\Big\{\frac{C_1(\tilde{T})}{K} \le b\Big\} \ge \Pr\Big\{\frac{C_1(K)}{K} \le b\Big\} = \Pr\Big\{\frac{C_1(\hat{T}_K)}{K} \le b\Big\}.$$

It follows that

$$\liminf_{K\to\infty} \Pr\Big\{\frac{C_1(\tilde{T})}{K} \le b\Big\} \ge \lim_{K\to\infty} \Pr\Big\{\frac{C_1(\hat{T}_K)}{K} \le b\Big\} = \int_0^b \frac{dx}{2\sqrt{1-x}}. \tag{2.25}$$

This inequality along with (2.24) establishes (2.21).

To establish (2.22), we first note that each connected component of the random digraph $G(\tilde{T})$ is either an oriented tree rooted at the vertex of out-degree 0 or it consists of exactly one oriented cycle together with oriented rooted trees. Denote by $F_K(\tilde{T})$ the set of vertices in all tree–components of $G(\tilde{T})$ and let $f_K(\tilde{T}) = |F_K(\tilde{T})|$. Now suppose that

$$\big\{v :\, v \in \mathcal{C}_1(\tilde{T})\big\} \cap F_K(\tilde{T}) = \emptyset\,,$$

where $\mathcal{C}_1(\tilde{T})$ is the component in $G(\tilde{T})$ which contains 1. In this case, $\mathcal{C}_1(\tilde{T})$ is not a tree–component and it follows that $\mathcal{C}_1(\tilde{T})$ was formed by cutting edges directed out of non-cyclical vertices in $\mathcal{C}_1(T^2)$, the component in $G(T^2)$ which contains 1. Hence, in this case,

$$C_1(\tilde{T}) \le C_1(T^2) \le C_1(\tilde{T}) + f_K(\tilde{T})\,.$$

In particular, for any fixed $0 < \rho < 1$, $0 < \beta < 1$, $\varepsilon > 0$, and all sufficiently large K, we have

$$\big\{C_1(\tilde{T}) \ge \rho K, f_K(\tilde{T}) \le K^\beta\big\} \subseteq \big\{\mathcal{C}_1(\tilde{T}) \cap F_K(\tilde{T}) = \emptyset\big\} \subseteq \Big\{\Big|\frac{C_1(\tilde{T})}{K} - \frac{C_1(T^2)}{K}\Big| < \varepsilon\Big\}$$

and hence

$$\Pr\{C_1(\tilde{T}) \ge \rho K\} - \Pr\{f_K(\tilde{T}) > K^\beta\} \le \Pr\Big\{\Big|\frac{C_1(\tilde{T})}{K} - \frac{C_1(T^2)}{K}\Big| < \varepsilon\Big\}.$$

To bound $\Pr\{f_K(\tilde{T}) \ge K^\beta\}$ we introduce some additional notation. If there exists an oriented path from i to j (including the null path from i to i) in the digraph $G(\hat{T}_K)$, then i is said to be a *predecessor* of j. For $A \subseteq V_1$, let $P_{\hat{T}_K}(A)$ denote the set of all predecessors in $G(\hat{T}_K)$ of elements in A, and in the special case where $A = \{1, 2, ..., m\}$ for some $1 \le m \le K$, let $P_{\hat{T}_K}(m) = P_{\hat{T}_K}(\{1, 2, ..., m\})$. Note that if $A_1 \subset A_2 \subset V_1$, then

$P_{\hat{T}_K}(A_1) \subset P_{\hat{T}_K}(A_2)$. Moreover, the variables $|P_{\hat{T}_K}(A_1)|$ and $|P_{\hat{T}_K}(A_2)|$ have the same distribution whenever $|A_1| = |A_2|$. Now, given A - the set of vertices $v \in V_1$ which have out-degree 0 in $G(\tilde{T})$, $f_K(\tilde{T}) = |F_K(\tilde{T})|$ has the same distribution as $|P_{\hat{T}_K}(A)|$. So, in particular, $f_K(\tilde{T})$ can be studied as the total number of elements which are eventually infected in the inverse epidemic process (IEP) on the digraph representing the uniform random mapping $\hat{T}_K$ (see [7] and [14]). It follows as in the derivation of (2.23) that

$$\Pr\Big\{f_K(\tilde{T}) \geq K^\beta\Big\} = \sum_{j=0}^{K} \Pr\Big\{f_K(\tilde{T}) \geq K^\beta \,\Big|\, Z = |A| = j\Big\} \Pr\{Z = j\}$$

$$\leq \sum_{j=0}^{\lfloor K^{3/8} \rfloor} \Pr\Big\{|P_{\hat{T}_K}(|A|)| \geq K^\beta \,\Big|\, Z = |A| = j\Big\} \Pr\{Z = j\} + CK^{-1/8}$$

$$\leq \Pr\Big\{|P_{\hat{T}_K}(\lfloor K^{3/8} \rfloor)| \geq K^\beta\Big\} + CK^{-1/8}. \tag{2.26}$$

Now (see [14])

$$E(|P_{\hat{T}_K}(\lfloor K^{3/8} \rfloor)|) \sim K^{3/8}\sqrt{K\pi/2} \leq C \cdot K^{7/8},$$

where C is a constant which does not depend on K. Now let $\beta = 15/16$, then by Markov's inequality we have

$$\Pr\{\big|P_{\hat{T}_K}(\lfloor K^{3/8} \rfloor)\big| \geq K^{15/16}\} \leq C \cdot K^{-1/16}.$$

It follows from this bound and from inequality (2.26) that

$$\liminf_{K\to\infty} \Pr\Big\{\Big|\frac{C_1(\tilde{T})}{K} - \frac{C_1(T^2)}{K}\Big| \leq \varepsilon\Big\} \geq \lim_{K\to\infty} \left(\Pr\Big\{\frac{C_1(\tilde{T})}{K} \geq \rho\Big\} - \Pr\Big\{f_K(\tilde{T}) \geq K^{15/16}\Big\}\right)$$
$$= \int_\rho^1 \frac{dx}{2\sqrt{1-x}}.$$

Letting $\rho \downarrow 0$, we obtain (2.22) and the lemma is proved.

□

We now proceed to prove Theorem 1.

Proof of Theorem 1. Fix $\varepsilon > 0$, then we know from Lemma 2 that there exists $K'_\varepsilon > 0$ such that if $K \geq K'_\varepsilon$ and $\eta K \leq L \leq K^{7/4}$, then

$$\left|\Pr\Big\{a < \frac{C_1}{N} \leq b\Big\} - \int_a^b \frac{dx}{2\sqrt{1-x}}\right| < \varepsilon. \tag{2.27}$$

Now suppose that $L \geq K^{7/4}$ and choose $\delta > 0$ such that

$$\left|\int_{a-\delta}^{b+\delta} \frac{dx}{2\sqrt{1-x}} - \int_a^b \frac{dx}{2\sqrt{1-x}}\right| \leq \frac{\varepsilon}{4} \quad \text{and} \quad \left|\int_{a+\delta}^{b-\delta} \frac{dx}{2\sqrt{1-x}} - \int_a^b \frac{dx}{2\sqrt{1-x}}\right| \leq \frac{\varepsilon}{4}.$$

Observe that

$$\Pr\Big\{a < \frac{C_1}{N} \le b\Big\} \ge \sum_{k>(a+\delta)K}^{\lfloor (b-\delta)K \rfloor} \Pr\Big\{a < \frac{C_1}{N} \le b \,\Big|\, C_1(T^2) = k\Big\} \Pr\{C_1(T^2) = k\}. \tag{2.28}$$

In order to obtain a lower bound for the right hand side of (2.28), we first define two random variables

$$W = \big|\{T_{K,L}(v) : v \in \mathcal{C}_1(T^2)\}\big| \quad \text{and} \quad S = \big|\{T_{K,L}(v) : v \in V_1\}\big|.$$

The key observation is that for $(a+\delta)K < k \le (b-\delta)K$, $1 \le s \le K$, and $1 \le w \le k$,

$$\Pr\{aN < C_1 \le bN \,|\, W = w, S = s, C_1(T^2) = k\} = \Pr\{aN < Y + k + w \le bN\},$$

where $Y \sim Bin(L-s, k/K)$. It follows from Chernoff's large deviation bounds for binomial variables (see for example [20] p.39) that for $k \le (b-\delta)K$, $1 \le s \le K$, $1 \le w \le k$ and K large enough,

$$\begin{aligned}\Pr\{Y \le bN - k - w\} &\ge \Pr\{Y \le bL - 2(b-\delta)K\} \ge \Pr\{\tilde{Y} \le bL - 2(b-\delta)K\} \\ &\ge \Pr\{\tilde{Y} \le (b-\delta)L + \log(K) \cdot \sqrt{(b-\delta)(1-b+\delta)L}\} \\ &\ge 1 - \frac{C(b,\delta)}{K}\end{aligned}$$

where $\tilde{Y} \sim Bin(L, b-\delta)$ and $C(b,\delta)$ is a constant which does not depend on K. Similarly, for $(a+\delta)K < k$, $1 \le s \le K$, and $1 \le w \le k$, large deviation bounds yield

$$\begin{aligned}\Pr\{Y > aN - k - w\} &\ge \Pr\{Y > aN\} \ge \Pr\{\hat{Y} > aN\} \\ &\ge \Pr\{\hat{Y} > (a+\delta)(L-K) - \log(K) \cdot \sqrt{(a+\delta)(1-a-\delta)(L-K)}\} \\ &\ge 1 - \frac{C(a,\delta)}{K}\end{aligned}$$

for all large K, where $\hat{Y} \sim Bin(L-K, a+\delta)$. So

$$\Pr\{aN < C_1 \le bN \,|\, W = w, S = s, C_1(T^2) = k\} \ge 1 - \frac{C(a,b,\delta)}{K},$$

for $1 \le s \le K$, $1 \le w \le k$ and hence

$$\Pr\Big\{a < \frac{C_1}{N} \le b \,\Big|\, C_1(T^2) = k\Big\} \ge 1 - \frac{C(a,b,\delta)}{K}$$

for all large K and $(a+\delta)K < k \le (b-\delta)K$. Substituting this bound into (2.28), we obtain

$$\Pr\Big\{a < \frac{C_1}{N} \le b\Big\} \ge \Pr\{(a+\delta)K < C_1(T^2) \le (b-\delta)K\} - \frac{C(a,b,\delta)}{K}. \tag{2.29}$$

To obtain an upper bound, we note that

$$\Pr\Big\{a < \frac{C_1}{N} \le b\Big\} \le \Pr\Big\{a - \delta < \frac{C_1(T^2)}{K} \le b + \delta\Big\} + \Pr\Big\{\frac{C_1}{N} \le b,\ \frac{C_1(T^2)}{K} > b + \delta\Big\}$$
$$+ \Pr\Big\{\frac{C_1(T^2)}{K} \le a - \delta,\ \frac{C_1}{N} > a\Big\}.$$

Large deviation calculations similar to those made above, yield

$$\Pr\Big\{\frac{C_1}{N} \le b,\ \frac{C_1(T^2)}{K} > b + \delta\Big\} \le \frac{C(b,\delta)}{K} \quad \text{and} \quad \Pr\Big\{\frac{C_1(T^2)}{K} \le a - \delta,\ \frac{C_1}{N} > a\Big\} \le \frac{C(a,\delta)}{K},$$

for all large K. It follows that

$$\Pr\Big\{a < \frac{C_1}{N} \le b\Big\} \le \Pr\Big\{a - \delta < \frac{C_1(T^2)}{K} \le b + \delta\Big\} + \frac{C(a,b,\delta)}{K}. \tag{2.30}$$

So it follows from Lemma 3, the choice of $\delta > 0$, and inequalities (2.29) and (2.30) that there exists $K''_\varepsilon > 0$ such that if $K > K''_\varepsilon$ and $L + L(K) \ge K^{7/4}$, then

$$\left|\Pr\Big\{a < \frac{C_1}{N} \le b\Big\} - \int_a^b \frac{dx}{2\sqrt{1-x}}\right| < \varepsilon.$$

Thus (2.27) holds for all $K > K_\varepsilon = K'_\varepsilon \vee K''_\varepsilon$ and $L = L(K) \ge \eta K$ and this completes the proof of the theorem.

□

3. ORDER STATISTICS FOR COMPONENT SIZES

In this section we prove Theorem 2 which gives the limiting distribution of the normalized order statistics for the component sizes of a bipartite random mapping $T_{K,L}$ as $K, L \to \infty$. Before proving this result, we need some additional notation. We denote by R_i the number of red vertices and by B_i the number of blue vertices in the i^{th} connected component $\mathcal{C}_i$. Clearly

$$R_1 + B_1 = C_1\,,\ R_2 + B_2 = C_2\,,\ R_3 + B_3 = C_3\,, \dots .$$

Moreover let $K_1 = K,\ L_1 = L,\ N_1 = K_1 + L_1 = N$ and for $i \ge 2$

$$K_i = K_{i-1} - R_{i-1}; \quad L_i = L_{i-1} - B_{i-1}; \quad N_i = K_i + L_i = N - C_1 - C_2 - \cdots - C_{i-1}$$

and note that for $i \ge 2$, K_i, L_i, N_i are random variables. With this notation we have $X_i = C_i/N_i$. We now proceed to prove Theorem 2.

Proof of Theorem 2. Observe that by symmetry it suffices to prove the result in the case $L = L(K) \ge K$. Now by the convergence principle outlined in Section 1, it is enough to show that for any $t \ge 1$ and any $0 < a_i < b_i < 1$, $i = 1, 2, ..., t$,

$$\lim_{K\to\infty} \Pr\{a_i < X_i \le b_i, \quad i = 1, 2, \dots, t\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{\sqrt{1-u}}. \tag{3.1}$$

To establish (3.1) we divide the proof into two cases.
Case 1. Suppose that for each $K > 0$ we have $K \leq L(K) \leq K^3$. For conciseness, we introduce

$$\mathcal{A}_j = \{a_i < X_i \leq b_i, \quad i = 1, 2, \ldots, j\} \quad \text{for} \quad j = 1, 2, \ldots, t\,,$$

and we write

$$\Pr\{a_i < X_i \leq b_i, \quad i = 1, 2, \ldots, t\} = \Pr\{\mathcal{A}_t\} = \Pr\{\mathcal{B}_t \cap \mathcal{A}_t\} + \Pr\{\mathcal{B}_t^c \cap \mathcal{A}_t\} \tag{3.2}$$

where

$$\mathcal{B}_1 = \{L_1 \geq K_1\}\;,$$

and for $j = 2, \ldots, t$,

$$\mathcal{B}_j = \left\{ L_1 \geq K_1, \quad \frac{1}{2^i} \leq \frac{L_{i+1}}{K_{i+1}}, \quad K \cdot \prod_{s=1}^{i} (1 - b_s - \delta) \leq K_{i+1}, \quad i = 1, 2, \ldots, j-1 \right\}$$

with $\delta := \frac{1}{2} \min\{(1 - b_i) : \, i = 1, 2, ..., t\}$
Observe that

$$\begin{aligned} \Pr\{\mathcal{B}_t \cap \mathcal{A}_t\} = & \prod_{j=1}^{t-1} \Pr\left\{ \frac{1}{2^j} \leq \frac{L_{j+1}}{K_{j+1}}, \; K \cdot \prod_{s=1}^{j} (1 - b_s - \delta) \leq K_{j+1} \,\middle|\, \mathcal{B}_j \cap \mathcal{A}_j \right\} \\ & \times \prod_{i=1}^{t} \Pr\{a_i < X_i \leq b_i \mid \mathcal{B}_i \cap \mathcal{A}_{i-1}\}\,, \end{aligned} \tag{3.3}$$

where $\mathcal{B}_1 \cap \mathcal{A}_0 := \mathcal{B}_1$. The first step is to show that

$$\lim_{K \to \infty} \prod_{i=1}^{t} \Pr\{a_i < X_i \leq b_i \mid \mathcal{B}_i \cap \mathcal{A}_{i-1}\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}}. \tag{3.4}$$

Note that by conditioning on the events $\mathcal{B}_i$ in the terms in the product (3.4) we guarantee that for each $1 \leq i \leq t$, $K_i \to \infty$ as $K \to \infty$ and $\frac{1}{2^{i-1}} K_i \leq L_i$. So by Theorem 1, for $i = 1$,

$$\lim_{K \to \infty} \Pr\left\{ a_1 < X_1 \leq b_1 \,\middle|\, 1 \leq \frac{L}{K} \right\} = \lim_{K, L \to \infty} \Pr\left\{ a_1 < X_1 \leq b_1 \right\} = \int_{a_1}^{b_1} \frac{du}{\sqrt{1-u}}\,. \tag{3.5}$$

For $2 \leq i \leq t$ we exploit the identity

$$\Pr\left\{ a_i < X_i \leq b_i \,\middle|\, K_i = r, \, L_i = b, \, \mathcal{B}_{i-1} \cap \mathcal{A}_{i-1} \right\} = \Pr\left\{ a_i < \frac{C_1(r, b)}{r + b} \leq b_i \right\}$$

where $C_1(r, b)$ is the size of the component which contains the vertex 1 in a random bipartite mapping on r red vertices and b blue vertices. This identity is a straightforward consequence of the independence and uniformity which is built into our model, namely, that each vertex is assigned independently, according to the uniform distribution, to a vertex in the other set. So by Theorem 1, for each $2 \le i \le t$, there exists $K(\varepsilon, i) > 0$ such that if $K > K(\varepsilon, i)$, $K \prod_{s=1}^{i-1}(1 - b_s - \delta) \le r < K$ and $\frac{r}{2^{i-1}} \le b < L$, then

$$\left| \Pr\Big\{ a_i < X_i \le b_i \,\Big|\, K_i = r,\, L_i = b,\, \mathcal{B}_{i-1} \cap \mathcal{A}_{i-1} \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right|$$

$$= \left| \Pr\Big\{ a_i < \frac{C_1(r,b)}{r+b} \le b_i \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right| \le \varepsilon. \tag{3.6}$$

It follows that if $K > \max\{K(\varepsilon, i) : i = 1, 2, ..., t\}$ then

$$\left| \Pr\Big\{ a_i < X_i \le b_i \,\Big|\, \mathcal{B}_i \cap \mathcal{A}_{i-1} \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right| < \varepsilon \quad \text{for } 2 \le i \le t \tag{3.7}$$

and (3.4) now follows from (3.5) and (3.7). Next we show that the events $\mathcal{B}_i$ on which we have conditioned in the calculations above (and which guarantee that we can apply Theorem 1) have high probability. In particular, we claim that

$$\lim_{K \to \infty} \prod_{j=1}^{t-1} \Pr\Big\{ \frac{1}{2^j} \le \frac{L_{j+1}}{K_{j+1}},\ K \prod_{s=1}^{j}(1 - b_s - \delta) \le K_{j+1} \Big| \mathcal{B}_j \cap \mathcal{A}_j \Big\} = 1. \tag{3.8}$$

Now *given* $\mathcal{B}_j \cap \mathcal{A}_j$, we have for $1 \le j \le t-1$

$$\frac{L_{j+1}}{K_{j+1}} = \frac{L_j - (L_j X_j - d_j)}{K_j - (K_j X_j + d_j)} = \frac{L_j}{K_j} \cdot \frac{1 + d_j/(L_j(1-X_j))}{1 - d_j/(K_j(1-X_j))} \ge \frac{1}{2^{j-1}} \cdot \frac{1 + d_j/(L_j(1-X_j))}{1 - d_j/(K_j(1-X_j))}$$

where $d_j = L_j X_j - B_j = R_j - K_j X_j$, and

$$K_{j+1} = K_j - K_j X_j - d_j \ge K_j \Big(1 - b_j - \frac{d_j}{K_j}\Big) \ge \Big(1 - b_j - \frac{d_j}{K_j}\Big) \cdot K \prod_{s=1}^{j-1}(1 - b_s - \delta)$$

(with the convention that for $j = 1$ the product in the above formula is equal to 1). Hence

$$\Pr\Big\{ \frac{1}{2^j} \le \frac{L_{j+1}}{K_{j+1}}, K \cdot \prod_{s=1}^{j}(1 - b_s - \delta) \le K_{j+1} \Big| \mathcal{B}_j \cap \mathcal{A}_j \Big\}$$

$$\ge \Pr\Big\{ \frac{1}{2} \le \frac{1 + d_j/L_j(1-X_j)}{1 - d_j/K_j(1-X_j)},\ \frac{|d_j|}{K_j} \le \delta \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j \Big\}. \tag{3.9}$$

Also, given the event $\mathcal{B}_j \cap \mathcal{A}_j$, we have $L_j \ge K_j/2^{j-1}$ and $K_j \ge K \prod_{s=1}^{j-1}(1 - b_s - \delta)$, so if $|d_j| \le (K_j)^{2/3}$, then

$$\frac{|d_j|}{L_j(1-X_j)} \le \frac{C(j)}{K^{1/3}} \quad \text{and} \quad \frac{|d_j|}{K_j} \le \frac{|d_j|}{K_j(1-X_j)} \le \frac{C(j)}{K^{1/3}}$$

where $C(j)$ is a constant which depends on $j, b_1, b_2, ..., b_j$. Therefore, given $\mathcal{B}_j \cap \mathcal{A}_j$, if $|d_j| \le (K_j)^{2/3}$, then

$$\frac{1}{2} \le \frac{1 + d_j/L_j(1 - X_j)}{1 - d_j/K_j(1 - X_j)} \quad \text{and} \quad \frac{|d_j|}{K_j} < \delta \,,$$

and

$$\Pr\Big\{\frac{1}{2} \le \frac{1 + d_j/L_j(1 - X_j)}{1 - d_j/K_j(1 - X_j)},\ \frac{|d_j|}{K_j} \le \delta \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j\Big\} \ge \Pr\big\{|d_j| \le (K_j)^{2/3} \,\big|\, \mathcal{B}_j \cap \mathcal{A}_j\big\} \tag{3.10}$$

for all $1 \le j \le t - 1$ and all sufficiently large K.

In the remaining calculations we make use of the identity

$$\Pr\big\{R_j = k,\ B_j = l \,\big|\, K_j = r,\ L_j = b,\ \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\big\} = \Pr\{R_1(r, b) = k,\ B_1(r, b) = l\}$$

where $R_1(r, b)$ is the number of red vertices and $B_1(r, b)$ is the number of blue vertices in the connected component containing the vertex "1" in $G(T_{r,b})$, the digraph which represents the bipartite random mapping $T_{r,b}$ on r red and b blue vertices. In particular, for r, b and m chosen such that $K \prod_{s=1}^{j-1}(1 - b_s - \delta) \le r < K$, $r/2^{j-1} \le b < L$, and $a_j < x = m/(r + b) \le b_j$,

$$\begin{aligned} &\Pr\Big\{|d_j| > r^{2/3},\ C_j = m \,\Big|\, K_j = r,\ L_j = b,\ \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\} \\ &\le \sum_{|i| > r^{2/3} - 1} \Pr\Big\{R_1(r, b) = [rx] - i, B_1(r, b) = [bx] + i\Big\} \\ &\le \sum_{\{k : |[rx] - k - 1| > r^{2/3} - 1\}} \binom{r-1}{k} \left(\frac{m - k - 1}{b}\right)^k \left(1 - \frac{m - k - 1}{b}\right)^{r-k-1} \end{aligned} \tag{3.11}$$

where

$$[rx] + [bx] = m \quad \text{and} \quad |[rx] - rx| \le 1\,, \quad |[bx] - bx| \le 1\,.$$

We note that inequality (3.11) follows from an argument similar to the reasoning which we used to establish inequality (2.11). The right hand side of (3.11) is a 'two-sided' sum. We indicate how to bound one side of the sum. Note that

$$\begin{aligned} &\sum_{k > [rx] + r^{2/3} - 2} \binom{r-1}{k} \left(\frac{m - k - 1}{b}\right)^k \left(1 - \frac{m - k - 1}{b}\right)^{r-k-1} \\ &\le \sum_{k > [rx] + r^{2/3} - 2} \binom{r-1}{k} \left(\tilde{x} - \frac{r^{2/3} - 1}{b}\right)^k \left(1 - \tilde{x} + \frac{r^{2/3} - 1}{b}\right)^{r-k-1} \\ &\le \Pr\left\{\frac{X - E(X)}{\sqrt{Var(X)}} > \frac{r^{2/3} - 1}{\sqrt{Var(X)}}\right\} \le \hat{C}(t) \exp(-\hat{C}(t) K^{1/6}) \end{aligned}$$

where $\tilde{x} = [bx]/b$, $X \sim Bin(r - 1, \tilde{x} - \frac{r^{2/3} - 1}{b})$, and $\hat{C}(t)$ is a constant which only depends on $a_1, a_2, ..., a_t$ and $b_1, b_2, ..., b_t$. The last inequality follows from Chernoff's large deviation bound for the binomial distribution and from the assumption that $r/2^{j-1} \leq b < L$ and $K\prod_{s=1}^{j-1}(1-b_s-\delta) \leq r < K$. Similar calculations yield

$$\sum_{k<[rx]-r^{2/3}} \binom{r-1}{k} \left(\frac{m-k-1}{b}\right)^k \left(1-\frac{m-k-1}{b}\right)^{r-k-1} \leq \hat{C}(t)\exp(-\hat{C}(t)K^{1/6}).$$

Since these bounds are uniform over all r, b, and m satisfying $K\prod_{s=1}^{j-1}(1-b_s-\delta) \leq r < K$, $r/2^{j-1} \leq b < L$, and $a_j < x = m/(r+b) \leq b_j$, we have

$$\begin{aligned}
&\Pr\Big\{|d_j| > r^{2/3},\, a_j < X_j \leq b_j \mid K_j = r,\, L_j = b,\, \mathcal{B}_{j-1}\cap\mathcal{A}_{j-1}\Big\}\\
&= \sum_{m>a_j(r+b)}^{b_j(r+b)} \Pr\Big\{|d_j| > r^{2/3},\, C_j = m \mid K_j = r,\, L_j = b,\, \mathcal{B}_{j-1}\cap\mathcal{A}_{j-1}\Big\}\\
&\leq (r+b)\hat{C}(t)\exp(-\hat{C}(t)K^{1/6}) \leq 2\hat{C}(t)K^3\exp(-\hat{C}(t)K^{1/6}).
\end{aligned}$$

It follows from this inequality and inequality (3.6) that

$$\begin{aligned}
&\Pr\Big\{|d_j| > r^{2/3}\Big| K_j = r,\, L_j = b,\, \mathcal{B}_{j-1}\cap\mathcal{A}_j\Big\}\\
&= \frac{\Pr\Big\{|d_j| > r^{2/3},\, a_j < X_j \leq b_j \Big| K_j = r,\, L_j = b,\, \mathcal{B}_{j-1}\cap\mathcal{A}_{j-1}\Big\}}{\Pr\Big\{a_j < X_j \leq b_j \Big| K_j = r,\, L_j = b,\, \mathcal{B}_{j-1}\cap\mathcal{A}_{j-1}\Big\}} \leq \frac{\hat{C}(t)}{K}
\end{aligned}$$

for $1 \leq j \leq t-1$ and all suficiently large K, provided $K\prod_{s=1}^{j-1}(1-b_s-\delta) \leq r < K$, and $r/2^{j-1} \leq b < L$. Hence

$$\Pr\Big\{|d_j| \leq K_j^{2/3} \,\Big|\, \mathcal{B}_j\cap\mathcal{A}_j\Big\} \geq 1 - \frac{\hat{C}(t)}{K} \tag{3.12}$$

for all $1 \leq j \leq t-1$ and all sufficiently large K. Equation (3.8) now follows from (3.9), (3.10), and (3.12). Finally we obtain

$$\lim_{K\to\infty} \Pr\{\mathcal{B}_t\cap\mathcal{A}_t\} = \prod_{i=1}^{t}\int_{a_1}^{b_i} \frac{du}{2\sqrt{1-u}}. \tag{3.13}$$

from (3.3), (3.4), and (3.8). It remains to show that

$$\lim_{K\to\infty} \Pr\{\mathcal{B}_t^c\cap\mathcal{A}_t\} = 0. \tag{3.14}$$

Observe that

$$\Pr\{\mathcal{B}_t^c\cap\mathcal{A}_t\} \leq \sum_{j=1}^{t-1} \Pr\Big\{\frac{L_{j+1}}{K_{j+1}} < \frac{1}{2^j} \text{ or } K_{j+1} < K\prod_{s=1}^{j}(1-b_s-\delta) \,\Big|\, \mathcal{B}_j\cap\mathcal{A}_j\Big\}\Pr\{\mathcal{B}_j\cap\mathcal{A}_j\}\,,$$

and (3.14) follows immediately from (3.8). Equation (3.1) now follows from (3.2), (3.13) and (3.14).

Case 2: Now suppose that for each $K > 0$, we have $L = L(K) \geq K^3$. In this case we take an indirect approach. As in Section 2, $T^2 = T_{K,L} \circ T_{K,L}$, T_K denotes the uniform random mapping of V_1 into V_1, and $G(T^2)$ and $G(T_K)$ denote the random digraphs on K vertices which represent the random mappings T^2 and T_K, respectively. We also extend our notation as follows. For any mapping f from V_1 into V_1, let $\mathcal{C}_1(f)$ denote the component in $G(f)$ which contains the vertex labelled 1. If $\mathcal{C}_1(f) \neq G(f)$, then let $\mathcal{C}_2(f)$ denote the component in $G(f) \setminus \mathcal{C}_1(f)$ which contains the smallest vertex; otherwise, set $\mathcal{C}_2(f) = \emptyset$. For $t > 2$ we define $\mathcal{C}_t(f)$ iteratively: If $G(f) \setminus (\mathcal{C}_1(f) \cup ... \cup \mathcal{C}_{t-1}(f)) \neq \emptyset$, then let $\mathcal{C}_t(f)$ denote the component in $G(f) \setminus (\mathcal{C}_1(f) \cup ... \cup \mathcal{C}_{t-1}(f))$ which contains the smallest vertex; otherwise, set $\mathcal{C}_t(f) = \emptyset$. For $t \geq 1$, let $C_t(f) = |\mathcal{C}_t(f)|$ and define the sequence $(X_1(f), X_2(f), \ldots)$ by

$$X_1(f) = \frac{C_1(f)}{K_1(f)},\ X_2(f) = \frac{C_2(f)}{K_2(f)},\ \ldots,\ X_t(f) = \frac{C_t(f)}{K_t(f)},\ \ldots$$

where $K_1(f) = K$, $K_i(f) = K_{i-1}(f) - C_{i-1}(f) = K - C_1(f) - C_2(f) - ... - C_{i-1}(f)$ for $i \geq 2$, and $X_i(f) = 0$ if $K_i(f) = 0$.

Now let

$$\mathcal{H}_K = \{\, T_{K,L}(v) \neq T_{K,L}(w) \text{ for all } v, w \in V_1, v \neq w \,\},$$

then it is easy to show that *given* $\mathcal{H}_K$, the random mapping T^2 on V_1 has the same distribution as the uniform random mapping T_K. In particular, for any $t \geq 1$ and $0 < a_i < b_i < 1$, $i = 1, 2, ..., t$,

(3.15)

$$\Pr\left\{a_i < X_i(T^2) \leq b_i : i = 1, 2, ..., t \,|\, \mathcal{H}_K\right\} = \Pr\left\{a_i < X_i(T_K) \leq b_i : i = 1, 2, ..., t\right\}.$$

Furthermore, provided $L = L(K) \geq K^3$,

$$\Pr\{\mathcal{H}_K\} = \frac{(L)_K}{L^K} \geq 1 - \frac{C}{K} \tag{3.16}$$

where C is a constant which does not depend on K. It follows from (3.15) and (3.16) that for any $t \geq 1$ and $0 < a_i < b_i < 1$, $i = 1, 2, ..., t$,

$$\Pr\left\{a_i < X_i(T_K) \leq b_i : i = 1, 2, ..., t\right\}\left(1 - \frac{C}{K}\right) \leq \Pr\left\{a_i < X_i(T^2) \leq b_i : i = 1, 2, ..., t\right\}$$

$$\leq \Pr\left\{a_i < X_i(T_K) \leq b_i : i = 1, 2, ..., t\right\} + \frac{C}{K}.$$

For the uniform random mapping T_K (see [1])

$$\lim_{K\to\infty} \Pr\{\, a_i < X_i(T_K) \leq b_i \,:\, i = 1, 2, ..., t \,\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}}\,, \tag{3.17}$$

so it follows that

$$\lim_{K\to\infty} \Pr\left\{a_i < X_i(T^2) \leq b_i : i = 1, 2, ..., t\right\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}}. \tag{3.18}$$

Thus, to obtain (3.1) in the case when $L = L(K) \geq K^3$ it suffices to show that for every $t \geq 1$ and every $\varepsilon > 0$,

$$\lim_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon \,:\, i = 1, 2, ..., t\} = 1. \tag{3.19}$$

Fix $t \geq 1$, $\varepsilon > 0$, and $\delta > 0$ and choose $\rho = \rho(\delta) > 0$ such that

$$\left(\int_\rho^{1-\rho} \frac{du}{2\sqrt{1-u}}\right)^t > 1 - \delta.$$

and let

$$\mathcal{A}_K(\rho) = \big\{\, \rho \leq X_i(T^2) \leq 1 - \rho \,:\, i = 1, 2, ..., t \,\big\}.$$

We note that if $\rho \leq X_i(T^2) \leq 1 - \rho$ for $i = 1, 2, ..., t$, then by induction

$$\rho^{i-1} K \leq K_i \leq (1-\rho)^{i-1} K \quad \text{and} \quad \rho^i K \leq C_i(T^2) \leq (1-\rho)^i K \tag{3.20}$$

for $1 \leq i \leq t$. Also, it follows from (3.15) and (3.17) that

$$\lim_{K\to\infty} \Pr\{\mathcal{A}_K(\rho) \,|\, \mathcal{H}_K\} = \left(\int_\rho^{1-\rho} \frac{du}{2\sqrt{1-u}}\right)^t > 1 - \delta,$$

and hence

$$\begin{aligned} &\limsup_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon \,:\, i = 1, 2, ..., t\} \\ &\geq \limsup_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon\,,\, i = 1, 2, ..., t \,|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\} \Pr\{\mathcal{A}_K(\rho) \,|\, \mathcal{H}_K\} \Pr\{\mathcal{H}_K\} \end{aligned}$$

$$> \limsup_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon\,,\, i = 1, 2, ..., t \,|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\}(1 - \delta). \tag{3.21}$$

We claim that

$$\limsup_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon\,,\, i = 1, 2, ..., t \,|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\} = 1. \tag{3.22}$$

To prove (3.22) we define variables $Y_i = C_i - 2C_i(T^2)$ for $1 \leq i \leq t$. Using this notation, we have

$$X_i = \frac{2C_i(T^2)/L + Y_i/L}{1 + K/L - 2(C_1(T^2) + ... + C_{i-1}(T^2))/L - Y_1/L - Y_2/L - ... - Y_{i-1}/L}$$

and

$$X_i(T^2) = \frac{C_i(T^2)/K}{1 - C_1(T^2)/K - ... - C_{i-1}(T^2)/K}.$$

So if $L = L(K) \geq K^3$, $\rho \leq X_i(T^2) \leq 1 - \rho$, and

$$\Big|\frac{Y_i}{L} - \frac{C_i(T^2)}{K}\Big| < \frac{2}{L^{1/3}} < \frac{2}{K} \quad \text{for } 1 \leq i \leq t,$$

then it follows from these inequalitites and from (3.20) that

$$|X_i - X_i(T^2)| < \frac{C(\rho, t)}{K} \quad \text{for } 1 \le i \le t,$$

where $C(\rho, t)$ is a constant which depends only on ρ and t. Hence

$$\Pr\big\{|X_i(T^2) - X_i| < \varepsilon\,,\, i = 1, 2, ..., t \,|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\big\}$$

$$\ge \Pr\left\{\Big|\frac{Y_i}{L} - \frac{C_i(T^2)}{K}\Big| < \frac{2}{L^{1/3}},\, i = 1, 2, ..., t \,|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\right\} \tag{3.23}$$

for all sufficiently large K.

Now fix $K > 0$ and integers $0 < r_1, r_2, ..., r_t < K$ such that if $C_i(T^2) = r_i$ for $1 \le i \le t$, then $\rho \le X_i(T^2) \le 1 - \rho$ for $1 \le i \le t$, i.e.,

$$\rho \le X_i(T^2) = \frac{r_i/K}{1 - r_1/K - r_2/K - ... - r_{i-1}/K} \le 1 - \rho \tag{3.24}$$

for every $i = 1, 2, ..., t$. Now *given* $\mathcal{H}_K$ and $C_i(T^2) = r_i$ for $1 \le i \le t$, we have

$$C_i = 2r_i + Y_i \quad \text{for} \quad 1 \le i \le t,$$

where the marginal distribution of each Y_i is $Bin(L - r_i, \frac{r_i}{K})$. It follows from Chebyshev's inequality that

$$\begin{aligned}
&\Pr\left\{\Big|\frac{Y_i}{L} - \frac{r_i}{K}\Big| < \frac{2}{L^{1/3}},\, i = 1, 2, ..., t \,\Big|\, C_i(T^2) = r_i,\, i = 1, 2, ..., t,\, \mathcal{H}_K\right\} \\
&\ge \Pr\left\{\Big|Y_i - \frac{Lr_i}{K} - \frac{r_i^2}{K}\Big| < L^{2/3},\, i = 1, 2, ..., t \,\Big|\, C_i(T^2) = r_i,\, i = 1, 2, ..., t,\, \mathcal{H}_K\right\} \\
&\ge 1 - \sum_{i=1}^{t} \frac{(L - r_i)}{4L^{4/3}} \ge 1 - \frac{t}{4L^{1/3}} \ge 1 - \frac{t}{4K}.
\end{aligned}$$

Since this inequality holds for all $0 < r_1, r_2, ..., r_t < K$ satisfying (3.24), we obtain

$$\Pr\left\{\Big|\frac{Y_i}{L} - \frac{C_i(T^2)}{K}\Big| < \frac{2}{L^{1/3}},\, i = 1, 2, ..., t \,\Big|\, \mathcal{A}_K(\rho)\,,\, \mathcal{H}_K\right\} \ge 1 - \frac{t}{4K}. \tag{3.25}$$

Equation (3.22) now follows from (3.23) and (3.25) and we obtain

$$\limsup_{K\to\infty} \Pr\{|X_i(T^2) - X_i| < \varepsilon \,:\, i = 1, 2, ..., t\} > 1 - \delta$$

from (3.21) and (3.22). Letting $\delta \downarrow 0$, we obtain (3.19) and the theorem is proved in this case.

It follows from Case 1 and Case 2, that (3.1) holds under the hypotheses of the theorem and the proof is complete.

$\square$

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