A random mapping with preferential attachment

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August 31, 2007

Abstract

In this paper we investigate the asymptotic structure of a random mapping model with preferential attachment, T_n^ρ, which maps the set $\{1, 2, ..., n\}$ into itself. The model T_n^ρ was introduced in a companion paper [11] and the asymptotic structure of the associated directed graph G_n^ρ which represents the action of T_n^ρ on the set $\{1, 2, ..., n\}$ was investigated in [11] and [12] in the case when the attraction parameter $\rho > 0$ is fixed as $n \to \infty$. In this paper we consider the asymptotic structure of G_n^ρ when the attraction parameter $\rho \equiv \rho(n)$ is a function of n as $n \to \infty$. We show that there are three distinct regimes during the evolution of G_n^ρ: (i) $\rho n \to \infty$ as $n \to \infty$, (ii) $\rho n \to \beta > 0$ as $n \to \infty$, and (iii) $\rho n \to 0$ as $n \to \infty$. It turns out that the asymptotic structure of G_n^ρ is, in some cases, quite different from the asymptotic structure of well-known models such as the uniform random mapping model and models with an attracting center. In particular, in regime (ii) we obtain some interesting new limiting distributions which are related to the incomplete gamma function.

Keywords: random mappings, exchangeable in–degrees, preferential attachment, evolution, component structure.

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‡J. Jaworski acknowledges the generous support by the Marie Curie Intra-European Fellowship No. 501863 (RANDIGRAPH) within the 6th European Community Framework Programme.

1 Introduction

In this paper we investigate a random mapping model, $T_n^\rho : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$, with 'preferential attachment'. This model was first introduced in a companion paper [11] and is defined as follows: For positive parameter ρ and $1 \leq k \leq n$, we define $T_n^\rho(k) = \xi_k^{(\rho,n)}$ where $\xi_1^{(\rho,n)}, \xi_2^{(\rho,n)}, ..., \xi_n^{(\rho,n)}$ is a sequence of random variables whose distributions depend on the evolution of an urn scheme. The distribution of each $\xi_k^{(\rho,n)}$ is determined by a (random) n-tuple of non-negative weights $\vec{a}(k) = (a_1(k), a_2(k), ..., a_n(k))$ where, for $1 \leq j \leq n$, $a_j(k)$ is the 'weight' of the j^{th} urn at the *start* of the k^{th} round of the urn scheme. Specifically, given $\vec{a}(k) = \vec{a} = (a_1, ..., a_n)$, we define

$$\Pr\{\xi_k^{(\rho,n)} = j \mid \vec{a}(k) = \vec{a}\} = \frac{a_j}{\sum_{i=1}^n a_i}.$$

The random weight vectors $\vec{a}(1), \vec{a}(2), \ldots, \vec{a}(n)$ associated with the urn scheme are determined recursively. For $k = 1$, we set $a_1(1) = a_2(1) = \cdots = a_n(1) = \rho > 0$. For $k > 1$, $\vec{a}(k)$ depends on both $\vec{a}(k-1)$ and the value of $\xi_{k-1}^{(\rho,n)}$ as follows: Given that $\xi_{k-1}^{(\rho,n)} = j$, we set $a_j(k) = a_j(k-1) + 1$ and for all other $i \neq j$, we set $a_i(k) = a_i(k-1)$ (i.e. if $\xi_{k-1}^{(\rho,n)} = j$ then a 'ball' with weight 1 is added to the j^{th} urn). We note that since, for $1 \leq k \leq n$, $T_n^\rho(k) = \xi_k^{(\rho,n)}$, and since the (conditional) distribution of $\xi_k^{(\rho,n)}$ depends on the state of the urn scheme at the start of round k, it is clear that k is more likely to be mapped to j if the weight $a_j(k)$ is (relatively) large, i.e. if several of the set $\{1, 2, ..., k-1\}$ have already been mapped to j. We note that T_n^ρ is the random mapping analogue of the random graph models with 'preferential attachment' that have been constructed in order to model the *evolving* structure of complex networks. In a similar way, T_n^ρ may also be useful as a model in some applications of random mapping models.

It is important to note that the preferential model T_n^ρ is fundamentally different from the 'classical' random mapping model $T_{\mathbf{p}(n)}$, special cases of which have been studied extensively since the 1950's and which can be defined as follows: For $n \geq 1$, let $\mathcal{M}_n$ denote the set of all mappings $f : [n] \to [n]$, where $[n] \equiv \{1, 2, ..., n\}$. We note that any mapping $f \in \mathcal{M}_n$ can be represented as a directed graph $G(f)$ on a set of vertices labelled $1, 2, ..., n$, such that there is a directed edge from vertex i to vertex j in $G(f)$ if and only if $f(i) = j$. Now for each $n \geq 1$, let $\mathbf{p}(n) = \{p_{ij}(n) : 1 \leq i, j \leq n\}$ be an array such that $p_{ij}(n) \geq 0$ for $1 \leq i, j \leq n$ and $\sum_{j=1}^n p_{ij}(n) = 1$ for every

$1 \le i \le n$, and let $X_1^n, X_2^n, ..., X_n^n$ be independent random variables such that $\Pr\{X_i^n = j\} = p_{ij}(n)$ for all $1 \le i, j \le n$. Then the random mapping $T_{\mathbf{p}(n)} : [n] \to [n]$ is defined (in terms of the variables $X_1^n, X_2^n, ..., X_n^n$) by

$$T_{\mathbf{p}(n)}(i) = j \quad \text{iff} \quad X_i^n = j \tag{1.1}$$

for all $1 \le i, j \le n$. It follows from (1.1) that the distribution of $T_{\mathbf{p}(n)}$ is given by

$$\Pr\{T_{\mathbf{p}(n)} = f\} = \prod_{i=1}^{n} p_{if(i)}(n)$$

for each $f \in \mathcal{M}_n$. The example of $T_{\mathbf{p}(n)}$ which is best understood is the uniform random mapping, $T_n \equiv T_{\mathbf{p}(n)}$, where $p_{ij}(n) = \frac{1}{n}$ for all $1 \le i, j \le n$ (see for example the monograph by Kolchin [18] and also the references in [11]). Berg, Jaworski, and Mutafchiev (see [5, 13, 15, 16, 17]) have also investigated the structure of the digraph associated with the model $T(n; q) \equiv T_{\mathbf{p}(n,q)}$ where $\mathbf{p}(n, q)$ is given by $p_{ii}(n, q) = q$ for some $0 \le q \le 1$ and all $1 \le i \le n$, and $p_{ij}(n, q) = \frac{1-q}{n-1}$ for all $1 \le i, j \le n$ such that $i \ne j$. In another direction, Stepanov [22], Burtin [7], Mutafchiev [20] and Berg, Mutafchiev [6] have considered the component structure of the digraph associated with the model $T_n(\lambda) \equiv T_{\mathbf{p}(n,\lambda)}$ where $\lambda \ge 1$ is a model parameter and $\mathbf{p}(n, \lambda)$ is given by $p_{i1}(n, \lambda) = \frac{\lambda}{\lambda+n-1}$ and $p_{ij}(n, \lambda) = \frac{1}{\lambda+n-1}$ for all $1 \le i, j \le n$ such that $j \ne 1$. The model $T_n(\lambda)$ is called a random mapping with attracting center and the parameter λ determines the strength of the attraction to the center. The last two models are of special interest because we can consider the evolution of the digraphs representing them, i.e. the changes in the typical structure of the digraph, when the model parameter changes as a function of the number of vertices. The evolution of the digraph representing $T(n; q)$ can be considered as an analogue of the evolution of the classical random graph since the number of edges in the underlying simple graph grows when the parameter q decreases from 1 to 0 (see also [14] for corresponding uniform random graph process). The evolution of random mappings with attracting center is somewhat different since the number of edges in the digraph representing $T_n(\lambda)$ remains fixed as the parameter λ tends to ∞. Finally, Aldous, Miermont, and Pitman (see [1] and [2]) have investigated the asymptotic structure of $G(T_{\mathbf{p}(n)})$, where $\mathbf{p}(n)$ is given by $p_{ij}(n) = p_j(n) > 0$ for all $1 \le i, j \le n$, by using an ingenious coding of the mapping $T_{\mathbf{p}(n)}$ as a stochastic process on the interval $[0, 1]$. Their results are closely related to earlier work on the relationship between random mappings and random forests (see Pitman [21] and references therein). In this model $p_j(n) > 0$ can be viewed as a measure of the relative strength of attraction which is 'assigned' to the vertex j.

Now it follows from the definition of classical model $T_{\mathbf{p}(n)}$ that the variables $X_1^n, X_2^n, \ldots X_n^n$ defined above can be interpreted as the independent 'choices' of the vertices $1, 2, \ldots, n$ in the random digraph $G_{\mathbf{p}(n)} \equiv G(T_{\mathbf{p}(n)})$ (see, in addition, Mutafchiev [19] and Jaworski [13]). In other words, for $1 \leq i, j, k, m \leq n,\ i \neq k$, the events $\{T_{\mathbf{p}(n)}(i) = j\}$ and $\{T_{\mathbf{p}(n)}(k) = m\}$ are independent. In contrast, it is clear from the definition of T_n^ρ that the events $\{T_n^\rho(i) = j\}$ and $\{T_n^\rho(k) = m\}$ are correlated and the strength of the correlation depends on the magnitude of the parameter ρ relative to n. For example, for any $1 \leq j \leq n$ and $\rho > 0$, we have $\Pr\{T_n^\rho(2) = j\} = \frac{1}{n}$ whereas for $j > 1$, $\Pr\{T_n^\rho(2) = j \,|\, T_n^\rho(1) = 1\} = \frac{\rho}{1+n\rho}$. In this paper we are interested in how the relative values of ρ and n determine the structure of $G_n^\rho \equiv G(T_n^\rho)$ and in what ways the structure of G_n^ρ differs from the structure of both the uniform digraph $G_n \equiv G(T_n)$ and the attracting center digraph $G_n(\lambda) \equiv G(T_n(\lambda))$.

Our investigation of the structure of G_n^ρ is based on a result from [11] where we show that the random mapping T_n^ρ has the same distribution as a random mapping model $T_n^{\hat{D}}$ with exchangeable in-degrees. To describe the exchangeable in-degree model we adopt the following notation. For any $f \in \mathcal{M}_n$ and $1 \leq i \leq n$, let $d_i(f)$ denote the in-degree of vertex i in the digraph $G(f)$, and $\vec{d}(f) \equiv (d_1(f), ..., d_n(f))$. Also, for any vector $\vec{d} \equiv (d_1, d_2, ..., d_n)$ of non-negative integers such that $\sum_{i=1}^n d_i = n$, let

$$\mathcal{M}_n(\vec{d}) \equiv \{f \in \mathcal{M}_n : \vec{d}(f) = \vec{d}\}.$$

Now to define $T_n^{\hat{D}}$, we start with a collection of non-negative integer-valued exchangeable random variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ such that $\sum_{i=1}^n \hat{D}_i = n$. Given the event $\left\{\hat{D}_i = d_i,\ i = 1, 2,\ \ldots,\ n\right\}$ (with $\Pr\left\{\hat{D}_i = d_i,\ i = 1, 2, ..., n\right\} > 0$), we define the conditional distribution of $T_n^{\hat{D}}$ by

$$\Pr\{T_n^{\hat{D}} = f \mid \hat{D}_i = d_i, i = 1, 2, ..., n\} = \begin{cases} \frac{\prod_{i=1}^n d_i!}{n!} & \text{if} \quad d_i(f) = d_i, i = 1, 2, ..., n \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

In other words, *given* $(\hat{D}_1, \hat{D}_2, ..., \hat{D}_n) = (d_1, d_2, ..., d_n) = \vec{d}$, $T_n^{\hat{D}}$ is uniformly distributed over $\mathcal{M}_n(\vec{d})$. It follows from (1.2) that for any $f \in \mathcal{M}_n$,

$$\Pr\{T_n^{\hat{D}} = f\} = \frac{\prod_{i=1}^n (d_i(f))!}{n!} \Pr\{\hat{D}_i = d_i(f), 1 \le i \le n\}. \tag{1.3}$$

In [11] and [12] we show that the distributions of many variables which describe features of the structure of the digraph $G_n^{\hat{D}} \equiv G_n(T_n^{\hat{D}})$ such as the number of components in $G_n^{\hat{D}}$, the size of a typical component in $G_n^{\hat{D}}$, etc. can be expressed relatively easily in terms of the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. In particular, this gives us a convenient calculus for investigating the structure of $G_n^{\hat{D}}$.

An important class of examples of $T_n^{\hat{D}}$ can constructed as follows. Suppose that $D_1, D_2, \ldots, D_n$ are i.i.d. non-negative integer-valued random variables with $\Pr\{\sum_{i=1}^n D_i = n\} > 0$, and let $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ be a sequence of random variables with joint distribution is given by

$$\Pr\Big\{\hat{D}_i = d_i, 1 \le i \le n\Big\} = \Pr\Big\{D_i = d_i, 1 \le i \le n \,\Big|\, \sum_{i=1}^n D_i = n\Big\}.$$

Clearly, the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ are exchangeable with $\sum_{i=1}^n \hat{D}_i = n$, so we can use $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ to construct $T_n^{\hat{D}}$ and $G_n^{\hat{D}}$. We note that if the variables $D_1, D_2, ..., D_n$ are i.i.d Poisson variables then $T_n^{\hat{D}}$ has the same distribution as the uniform model T_n, i.e. the uniform model is a special case of the exchangeable in-degree model. In the case where the underlying i.i.d. variables $D_1, D_2, ..., D_n$ have a generalised negative binomial distribution we established in [11] the following result:

Fact 1. *Suppose that $D_1^\rho, D_2^\rho, ...$ are i.i.d. random variables with a generalized negative binomial distribution given by*

$$\Pr\{D_1^\rho = k\} = \frac{\Gamma(k+\rho)}{k!\Gamma(\rho)} \left(\frac{\rho}{1+\rho}\right)^\rho \left(\frac{1}{1+\rho}\right)^k \quad \textit{for} \quad k = 0, 1, ...,$$

where ρ is a positive parameter and $\Gamma(\cdot)$ denotes the usual Gamma function. For $n \ge 1$, let $\hat{D}^\rho(n) = \left(\hat{D}_{1,n}^\rho, \hat{D}_{2,n}^\rho, \ldots, \hat{D}_{n,n}^\rho\right)$ be a sequence of variables with joint distribution given by

$$\Pr\Big\{\hat{D}_{i,n}^\rho = d_i,\ 1 \le i \le n\Big\} = \Pr\Big\{D_i^\rho = d_i,\ 1 \le i \le n \,\Big|\, \sum_{i=1}^n D_i^\rho = n\Big\}.$$

Then for every $n \geq 1$, the random mappings T_n^{ρ} and $T_n^{\hat{D}^{\rho}(n)}$ have the same distribution.

In [11] and [12] we used Fact 1 and the calculus for $G_n^{\hat{D}^{\rho}(n)} \equiv G(T_n^{\hat{D}^{\rho}(n)})$ to obtain exact formulas for the distributions of various variables associated with the structure of G_n^{ρ}. We also investigated the asymptotic distributions of these variables when ρ is fixed and $n \to \infty$. In this paper we investigate the asymptotic distributions of variables associated with the structure of G_n^{ρ} when the parameter $\rho \equiv \rho(n)$ depends on n. In particular, it follows from the definition of T_n^{ρ} that when ρ is *much larger* then n, the distribution of T_n^{ρ} is 'close' to the uniform distribution on $\mathcal{M}_n$. On the other hand, if ρ is *much smaller* than n, then we would expect the digraph G_n^{ρ} to consist of a collection of large 'attracting' components. So, in some sense which will be made precise by the results in this paper, the structure of G_n^{ρ} 'evolves' from the uniform digraph G_n (when $\rho =$'∞') to the 'star' graph as $\rho \to 0$. We note here that our investigation of the 'evolution' of G_n^{ρ} as $\rho \to 0$ is in the same spirit as the results of Stepanov [22] on the evolution of $G_n(\lambda)$ as λ goes from 1 to ∞, where $G_n(1)$ corresponds to the uniform model and $G($'∞'$)$ is a 'star' graph. However, we will see that the evolution of G_n^{ρ} is quite different from the evolution of $G_n(\lambda)$.

Finally, we mention that our investigation of the evolution of G_n^{ρ} is also related to our work on the Poisson–Dirichlet law for combinatorial structures which arise from a cutting process for random mappings (see [10]). It also turns out that exact and asymptotic results for the structure of G_n^{ρ} provide a natural way to introduce some interesting families of discrete distributions (see [12]).

2 Results

To describe our results we begin by introducing some more notation. We say that a vertex $i \in \{1, 2, ..., n\}$ is a cyclic vertex for the mapping $f \in \mathcal{M}_n$ (and for the corresponding digraph $G(f)$) if there is some $k \geq 1$ such that $f^{(k)}(i) = i$, where $f^{(k)}$ is the k^{th} iterate of the function f. We define $X_n(f)$ to be the number of cyclic vertices of $f \in \mathcal{M}_n$ and we let $X_n^{\rho} \equiv X_n(T_n^{\rho})$ denote the number of cyclic vertices in G_n^{ρ}. Given the distribution of X_n^{ρ}, it is straightforward to determine the distribution of N_n^{ρ}, the number of components in G_n^{ρ}, as follows: Let $\mathcal{L}_n^{\rho}$ denote the set of cyclic vertices of

T_n^ρ. It is known (see [11] for details) that *given* $\mathcal{L}_n^\rho = L \subseteq \{1, 2, ..., n\}$, then T_n^ρ restricted to L is a uniform random permutation on the set L. We also note that each connected component in G_n^ρ consists of a directed cycle with directed trees attached. So $N_n^\rho = \ell$ if and only if the mapping T_n^ρ restricted to L consists of ℓ cycles. Hence, for $1 \leq \ell \leq k \leq n$,

$$\Pr\left\{N_n^\rho = \ell \,\middle|\, X_n^\rho = k\right\} = \Pr\left\{N_{\sigma(k)} = \ell\right\}$$

where $\sigma(k)$ is a uniform random permutation on a k-element set and $N_{\sigma(k)}$ denotes the number of cycles in the random permutation $\sigma(k)$. It follows that for $1 \leq \ell \leq n$,

$$\begin{aligned} \Pr\big\{N_n^\rho = \ell\big\} &= \sum_{k=\ell}^{n} \Pr\big\{N_n^\rho = \ell \mid X_n^\rho = k\big\}\Pr\big\{X_n^\rho = k\big\} \\ &= \sum_{k=\ell}^{n} \Pr\big\{N_{\sigma(k)} = \ell\big\}\Pr\big\{X_n^\rho = k\big\}. \end{aligned} \tag{2.1}$$

Now suppose that $\xi_1, \xi_2, \ldots$ is a sequence of independent indicator variables such that, for $i \geq 1$, $\Pr\{\xi_i = 1\} = \frac{1}{i}$. It is well-known (see [8]) that for $k \geq 1$

$$N_{\sigma(k)} \overset{d}{\sim} \sum_{i=1}^{k} \xi_i \,. \tag{2.2}$$

So, by taking the variables $\xi_1, \xi_2, \ldots$ to be independent of X_n^ρ, it follows from (2.1) that

$$N_n^\rho \overset{d}{\sim} \sum_{i=1}^{X_n^{\hat{D}}(\rho,n)} \xi_i \,.$$

Thus, the distribution of N_n^ρ is determined by the distribution of X_n^ρ, which was determined in [11] and is given by

$$\Pr\{X_n^\rho = k\} = \begin{cases} k\rho^k(1+\rho)\frac{(n)_k}{(n\rho+k)_{k+1}} & \text{for } 1 \leq k \leq n-1 \\ \frac{\rho^n n!\,\Gamma(n\rho)}{\Gamma(n+n\rho)} & \text{if } k = n. \end{cases} \tag{2.3}$$

So, to determine the asymptotic distribution of N_n^ρ, we first determine the asymptotic distribution of X_n^ρ in Theorem 1 below.

Theorem 1. *Let X_n^ρ denote the number of cyclic vertices in G_n^ρ.*

(i) *Let R denote a Rayleigh distributed random variable with density given by $f(x) = x\exp(-x^2/2)$ for $x \geq 0$ (and equals 0 otherwise). Suppose that $\rho n \to \infty$ as $n \to \infty$ and let $\phi(n) \equiv \rho n/(1+\rho)$. Then*

$$\frac{X_n^\rho}{\sqrt{\phi(n)}} \quad \xrightarrow{d} R \quad \text{as } n \to \infty,$$

and $E(X_n^\rho) \sim \sqrt{\frac{\pi}{2}\phi(n)}$.

(ii) *Suppose that $\rho n \to \beta > 0$ as $n \to \infty$. Then for $k = 1, 2, ...,$*

$$\lim_{n\to\infty} \Pr\big\{X_n^\rho = k\big\} = \frac{k\beta^{k-1}}{(\beta+k)_k} = \frac{\beta^{k-1}}{(\beta+k-1)_{k-1}} - \frac{\beta^k}{(\beta+k)_k}.$$

where $(\beta)_0 \equiv 1$. Moreover

$$\lim_{n\to\infty} E(X_n^\rho) = 1 + \sum_{k=1}^{\infty} \frac{\beta^k}{(\beta+k)_k}.$$

(iii) *Suppose that $\rho n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \Pr\big\{X_n^\rho = 1\big\} = 1$$

Proof. First, suppose that $\rho n \to \infty$ as $n \to \infty$. To show that $X_n^\rho/\sqrt{\phi(n)}$ converges in distribution to R, it is enough to show that for any $a > 0$,

$$\lim_{n\to\infty} \Pr\Big\{\frac{X_n^\rho}{\sqrt{\phi(n)}} \leq a\Big\} = \int_0^a x\exp(-x^2/2)dx. \tag{2.4}$$

So, it follows from (2.3) that

$$\begin{aligned}
\Pr\Big\{\frac{X_n^\rho}{\sqrt{\phi(n)}} \leq a\Big\} &= \sum_{k=1}^{a\sqrt{\phi(n)}} \Pr\Big\{X_n^\rho = k\Big\} \\
&= \sum_{k=1}^{a\sqrt{\phi(n)}} \frac{k}{n}\left(\frac{1+\rho}{\rho}\right)\frac{(1-\frac{1}{n})\cdots(1-\frac{(k-1)}{n})}{(1+\frac{k}{n\rho})\cdots(1+\frac{1}{n\rho})} \\
&= \sum_{k=1}^{a\sqrt{\phi(n)}} \frac{k}{\sqrt{\phi(n)}}\exp\left(\frac{-k^2}{2\phi(n)}\right)\frac{1}{\sqrt{\phi(n)}}\exp\big(\varepsilon(k,n,\rho)\big) \qquad (2.5)
\end{aligned}$$

where $|\varepsilon(k,n,\rho)| \leq \frac{C(a)}{\sqrt{\phi(n)}}$ and $C(a)$ is a constant depends on a but not on n or ρ. Since $\phi(n) \to \infty$ as $n \to \infty$, equation (2.4) now follows from (2.5). Next, we consider

$$E\big(X_n^\rho\big) = \sum_{k=1}^{n} k \Pr\big\{X_n^\rho = k\big\}.$$

First, let $\lambda(n) \equiv \min\{n, \rho n\}$ and observe that all $n \geq 1$, $\phi(n) \leq \lambda(n)$. Now it follows calculations similar to those above that

$$\begin{aligned}
&\sum_{k=1}^{(\lambda(n))^{5/8}} k \Pr\big\{X_n^\rho = k\big\} \\
&= \sqrt{\phi(n)} \sum_{k=1}^{(\lambda(n))^{5/8}} \frac{k^2}{\phi(n)} \exp\left(\frac{-k^2}{2\phi(n)}\right) \frac{1}{\sqrt{\phi(n)}} (\exp(\varepsilon(k,n,\rho)) \\
&\sim \sqrt{\phi(n)} \int_0^{\frac{(\lambda(n))^{5/8}}{(\phi(n))^{1/2}}} x^2 e^{-x^2/2} dx \sim \sqrt{\frac{\pi}{2}\phi(n)}
\end{aligned}$$

where $|\varepsilon(k,n,\rho)| \leq \frac{C}{(\lambda(n))^{1/8}}$ and C is a constant which does not depend on n or ρ. So it is enough to show that

$$\lim_{n\to\infty} \sum_{k>(\lambda(n))^{5/8}} k \Pr\big\{X_n^\rho = k\big\} = 0. \tag{2.6}$$

To bound the sum in (2.6) we divide it into two parts. First, it follows from (2.3) that

$$\begin{aligned}
\sum_{k>(\lambda(n))^{5/8}}^{\lambda(n)} k \Pr\big\{X_n^\rho = k\big\} &= \sum_{k>(\lambda(n))^{5/8}}^{\lambda(n)} \frac{k^2}{\phi(n)} \frac{(1-\frac{1}{n})\cdots(1-\frac{(k-1)}{n})}{(1+\frac{k}{n\rho})\cdots(1+\frac{1}{n\rho})} \\
&\leq \sum_{k>(\lambda(n))^{5/8}}^{\lambda(n)} (\lambda(n))^2 \left(1 + \frac{\lceil k/2 \rceil}{\lambda(n)}\right)^{-\lfloor k/2 \rfloor} \\
&\leq (\lambda(n))^3 \exp\left(-\frac{(\lambda(n))^{1/4}}{4}\right). \qquad (2.7)
\end{aligned}$$

The last inequality above follows from the observation that for $0 < x < 1$, $(1+x)^{-1} \le 1 - x/2 \le \exp(-x/2)$. Next,

$$\begin{aligned}\sum_{k>\lambda(n)} k \Pr\{X_n^\rho = k\} &\le \frac{1}{\phi(n)} \sum_{k>\lambda(n)}^{n-1} k^2 \left(1 + \frac{\lceil k/2 \rceil}{\lambda(n)}\right)^{-\lfloor k/2 \rfloor} + n \Pr\{X_n^\rho = n\} \\ &\le \frac{2}{\phi(n)} \sum_{k>\lambda(n)} k^2 \left(\frac{2}{3}\right)^{k/2} + n \left(\frac{1}{2}\right)^{n/2}. \qquad (2.8)\end{aligned}$$

Since $\phi(n)$, $\lambda(n) \to \infty$ as $n \to \infty$, (2.6) follows from (2.7) and (2.8) and case (i) is proved.
Next, suppose that $\rho n \to \beta > 0$ as $n \to \infty$. The second equality of (ii) is immediate, while the first equality follows from (2.3):

$$\lim_{n\to\infty} \Pr\{X_n^\rho = k\} = \lim_{n\to\infty} \frac{k\rho^k (1+\rho)(n)_k}{(\rho n + k)_{k+1}} = \frac{k\beta^{k-1}}{(\beta+k)_k}. \qquad (2.9)$$

Similarly, it follows from (2.3), (2.9), and dominated convergence that

$$\begin{aligned}\lim_{n\to\infty} E\big(X_n^\rho\big) &= \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{k^2 \rho^{k-1}(1+\rho)(n-1)_{k-1}}{(\rho n + k)_k} + \lim_{n\to\infty} \frac{n\rho^n n! \Gamma(\rho n)}{\Gamma(n+\rho n)} \\ &= \sum_{k=1}^{\infty} \frac{k^2 \beta^{k-1}}{(\beta+k)_k} \\ &= \sum_{k=1}^{\infty} \frac{k\beta^{k-1}}{(\beta+k-1)_{k-1}} - \sum_{k=1}^{\infty} \frac{k\beta^k}{(\beta+k)_k}. \qquad (2.10)\end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \frac{k\beta^{k-1}}{(\beta+k)_k} = 1$$

we can rearrange the right-hand side of (2.10) to obtain the desired limit.
Finally, when $\rho n \to 0$ as $n \to \infty$, again by (2.3), we have

$$\lim_{n\to\infty} \Pr\{X_n^\rho = 1\} = \lim_{n\to\infty} \frac{1+\rho}{n\rho+1} = 1.$$

□

Theorem 2. *Let N_n^ρ denote the number of components in G_n^ρ.*

(i) *Suppose that $\rho n \to \infty$ as $n \to \infty$ and $\phi(n) = \rho n/(1+\rho)$, then*

$$\frac{N_n^\rho - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \xrightarrow{d} Z \qquad \text{as } n \to \infty,$$

where Z is a standard normal $N(0,1)$ variable.

(ii) *Suppose that $\rho n \to \beta > 0$ as $n \to \infty$, and suppose that X^β is a discrete random variable with distribution given by*

$$\Pr\{X^\beta = k\} = \frac{k\beta^{k-1}}{(\beta+k)_k} \quad \text{for} \quad k = 1, 2, \ldots .$$

Also, suppose that $\xi_1, \xi_2, \ldots$ is a sequence of independent indicator variables such that, for $i \geq 1$, $\Pr\{\xi_i = 1\} = \frac{1}{i}$ and such that $\xi_1, \xi_2, \ldots$. and X^β are independent. Then

$$N_n^\rho \xrightarrow{d} N^\beta \equiv \sum_{i=1}^{X^\beta} \xi_i \quad \text{as } n \to \infty.$$

(iii) *Suppose that $\rho n \to 0$ as $n \to \infty$ and let $\mathcal{B}_n^\rho$ denote the event that G_n^ρ is connected. Then*

$$\lim_{n\to\infty} \Pr\{N_n^\rho = 1\} = \lim_{n\to\infty} \Pr\{\mathcal{B}_n^\rho\} = 1\,.$$

Proof. Suppose first that $\rho n \to \infty$ as $n \to \infty$ and $\phi(n) = \rho n/(1+\rho)$, and fix $-\infty < a < \infty$. Then it follows from (2.1) that

$$\begin{aligned}
&\Pr\left\{\frac{N_n^\rho - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \leq a\right\} \\
&= \sum_{m=1}^{n} \Pr\left\{\frac{N_{\sigma(m)} - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \leq a\right\} \Pr\left\{X_n^\rho = m\right\}. \qquad (2.11)
\end{aligned}$$

Now fix $\varepsilon > 0$. Then, it follows from Theorem 1 (i) that there exists $0 < \delta(\varepsilon) < 1 < \gamma(\varepsilon)$ such that

$$\Pr\left\{\delta(\varepsilon)\sqrt{\phi(n)} \leq X_n^\rho \leq \gamma(\varepsilon)\sqrt{\phi(n)}\right\} \geq 1 - \varepsilon \qquad (2.12)$$

for all large n. Also, for $\delta(\varepsilon)\sqrt{\phi(n)} \le m \le \gamma(\varepsilon)\sqrt{\phi(n)}$ we have

$$\frac{N_{\sigma(m)} - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} = \frac{N_{\sigma(m)} - \log m - \varepsilon(m,n)}{\sqrt{\log m + \varepsilon(m,n)}}$$

where

$$|\varepsilon(m,n)| \le \max\left\{|\log\delta(\varepsilon)|, |\log\gamma(\varepsilon)|\right\}.$$

So for all large n and $m \ge \delta(\varepsilon)\sqrt{\phi(n)}$, we have

$$\begin{aligned} \Pr\left\{\frac{N_{\sigma(m)} - \log m}{\sqrt{\log m}} \le a - \varepsilon\right\} \; &\le \; \Pr\left\{\frac{N_{\sigma(m)} - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \le a\right\} \\ &\le \; \Pr\left\{\frac{N_{\sigma(m)} - \log m}{\sqrt{\log m}} \le a + \varepsilon\right\}. \qquad (2.13) \end{aligned}$$

Now it is known (see, for example [8]) that $(N_{\sigma(m)} - \log m)/\sqrt{\log m}$ converges in distribution to the standard normal $N(0,1)$ distribution as $m \to \infty$. So it follows from (2.11), (2.12) and (2.13) that

$$\Phi(a-\varepsilon) - \varepsilon \le \liminf_{n\to\infty} \Pr\left\{\frac{N_n^\rho - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \le a\right\}$$

and

$$\limsup_{n\to\infty} \Pr\left\{\frac{N_n^\rho - \frac{1}{2}\log\phi(n)}{\sqrt{\frac{1}{2}\log\phi(n)}} \le a\right\} \le \Phi(a+\varepsilon) + \varepsilon$$

where $\Phi(\cdot)$ is the distribution function of $N(0,1)$. Let $\varepsilon \to 0$ to obtain part (i).

Next suppose that $\rho n \to \beta > 0$ as $n \to \infty$ and let $\xi_1, \xi_2, \ldots$ be a sequence of independent indicator variables as described above which are independent of X^β. Now fix $\ell \ge 1$. Then it follows from (2.1) and (2.2) that for $n \ge \ell$

$$\Pr\left\{N_n^\rho = \ell\right\} = \sum_{k=\ell}^{n} \Pr\left\{\sum_{i=1}^{k} \xi_i = \ell\right\} \Pr\left\{X_n^\rho = k\right\}.$$

Hence

$$\begin{aligned} &\left|\Pr\{N_n^\rho = \ell\} - \Pr\{N^\beta = \ell\}\right| \\ &\le \sum_{k=\ell}^{n} \left|\Pr\{X_n^\rho = k\} - \Pr\{X^\beta = k\}\right| + \sum_{k=n+1}^{\infty} \Pr\{X^\beta = k\}. \qquad (2.14) \end{aligned}$$

Since $X_n^\rho \xrightarrow{d} X^\beta$ as $n \to \infty$, it is straightforward to show that the right-hand side of (2.14) tends to 0 as $n \to \infty$. This establishes part (ii).
Finally it follows from (2.1) and (2.3) that if $\rho n \to\ 0$ as $n \to \infty$, then

$$\begin{aligned}
\lim_{n\to\infty} \Pr\{\mathcal{B}_n^\rho\} &= \lim_{n\to\infty} \Pr\{N_n^\rho = 1\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \Pr\{N_{\sigma(k)} = 1\} \Pr\{X_n^\rho = k\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{\rho^k (1+\rho)(n)_k}{(n\rho + k)_{k+1}} + \lim_{n\to\infty} \frac{\rho^n n!\,\Gamma(n\rho)}{n\Gamma(n+n\rho)} \\
&\geq \lim_{n\to\infty} \frac{1+\rho}{n\rho+1} \ = \ 1.
\end{aligned}$$

□

Next we consider the asymptotic distribution of $C_1^\rho(n)$, the size of the component in G_n^ρ which contains the vertex labelled 1. It is clear from Theorem 2 that when $\rho n \to \infty$ as $n \to \infty$, the distribution of N_n^ρ, when n is large, depends on the *speed* at which $\phi(n) = \rho n/(1+\rho)$ tends to ∞. In particular, when n is large, the number of components in G_n^ρ is (with high probability) of order $\frac{1}{2} \log \phi(n)$. In light of this result, it seems reasonable to suppose that for large n the distribution of $C_1^\rho(n)$ would also depend on $\phi(n)$, but in Theorem 3 we show that this is not the case. We also show that when $\rho n \to \beta$ as $n \to \infty$, where $\beta > 0$, the asymptotic distribution of $C_1^\rho(n)/n$ can be expressed in terms of the incomplete gamma function $\gamma[a, z]$ which is defined by the following "incomplete" integral expression for the Gamma function $\Gamma(a)$

$$\gamma[a, z] = \int_0^z t^{a-1} e^{-t} dt$$

and can also be expressed as

$$\gamma[a, z] = z^a e^{-z} \sum_{k=1}^{\infty} \frac{z^{k-1}}{(a-1+k)_k}\,. \tag{2.15}$$

Given identity (2.15), we note that the limit of the expected value in Theorem 2 (ii) can be expessed in terms of this function. One can also check that when $\beta \to \infty$ we have

$$\gamma[\beta+1, \beta] \sim \frac{1}{2}\,\beta^\beta e^{-\beta} \sqrt{2\pi\beta}\,. \tag{2.16}$$

Theorem 3. *Let $C_1^\rho(n)$ denote the size of the component in G_n^ρ which contains the vertex labelled 1.*

(i) *Suppose that $n\rho \to \infty$ as $n \to \infty$ and let B be Beta(1/2) distributed random variable with density given by $f(u) = \frac{1}{2\sqrt{1-u}}$ on the interval $(0,1)$. Then*
$$\frac{C_1^\rho(n)}{n} \xrightarrow{d} B \quad as \quad n \to \infty\,.$$

(ii) *Suppose that $\rho n \to \beta > 0$ as $n \to \infty$. Then*
$$\begin{aligned}\lim_{n\to\infty} \Pr\{C_1^\rho(n) = n\} &= \lim_{n\to\infty} \Pr\{\mathcal{B}_n^\rho\} \\ &= \sum_{k=1}^{\infty} \frac{\beta^{k-1}}{(\beta+k)_k} = \beta^{-\beta-1} e^\beta\, \gamma[\beta+1,\beta]\end{aligned}$$

where $\gamma[a,z]$ is the incomplete gamma function.

Furthermore, suppose that $0 < x < 1$ is fixed and let $\ell = \lfloor xn \rfloor$. Then
$$\begin{aligned}&\Pr\big\{C_1^\rho(n) = \ell\big\} \\ &\sim \frac{1}{n}\frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)}\frac{1}{x\beta}(1-x)^{(1-x)\beta-1}\left(\frac{e}{\beta}\right)^{x\beta}\gamma[x\beta+1,x\beta]\end{aligned}$$

(iii) *Suppose that $\rho n \to 0$ as $n \to \infty$. Then*
$$\lim_{n\to\infty} \Pr\big\{C_1^\rho = n\big\} = 1$$

Proof. Suppose first that $n\rho \to \infty$ as $n \to \infty$. To prove (i) it is enough to show that for every $0 < a < b < 1$, we have
$$\lim_{n\to\infty} \Pr\Big\{a < \frac{C_1^\rho(n)}{n} \le b\Big\} = \int_a^b \frac{dx}{2\sqrt{1-x}}.$$

Fix $0 < a < b < 1$, then
$$\Pr\Big\{a < \frac{C_1^\rho(n)}{n} \le b\Big\} = \sum_{\ell > an}^{bn} \Pr\Big\{C_1^\rho(n) = \ell\Big\}. \tag{2.17}$$

Now in [11] we established that

$$\Pr\Big\{C_1^\rho(n)=\ell\Big\}=\frac{\ell}{n}\Pr\Big\{\mathcal{B}_\ell^\rho\Big\}\Pr\Big\{\sum_{i=1}^{\ell}D_i^\rho=\ell\;\Big|\;\sum_{i=1}^{n}D_i^\rho=n\Big\} \tag{2.18}$$

where the variables $D_1^\rho,...,D_n^\rho$ are as defined in Fact 1. To evaluate the right hand side of (2.18) we divide the calculation into two parts. First, it follows from the Stirling's formula that provided $an<\ell\le bn$

$$\begin{aligned}\Pr\Big\{\mathcal{B}_\ell^\rho\Big\}&=\sum_{k=1}^{\ell-1}\rho^k(1+\rho)\frac{(\ell)_k}{(\ell\rho+k)_{k+1}}+\frac{\rho^\ell\ell!\Gamma(\ell\rho)}{\ell\Gamma(\ell+\ell\rho)}\\&=\sum_{k=1}^{\ell-1}\frac{1}{\ell}\left(\frac{1+\rho}{\rho}\right)\frac{(1-\frac{1}{\ell})(1-\frac{2}{\ell})\cdots(1-\frac{k-1}{\ell})}{(1+\frac{k}{\ell\rho})(1+\frac{k-1}{\ell\rho})\cdots(1+\frac{1}{\ell\rho})}+\varepsilon_1(\ell,n,\rho)\end{aligned} \tag{2.19}$$

where $|\varepsilon_1(\ell,n,\rho)|\le Ce^{-an}\sqrt{bn}$ and C is a constant which does not depend on a,b, or n. We divide the sum on the right hand side of (2.19) into two parts. As in the proof of Theorem 1 (i), let $\lambda(n)\equiv\min\{n,\rho n\}$ and consider

$$\Sigma(1)\equiv\frac{1}{\ell}\left(\frac{1+\rho}{\rho}\right)\sum_{k>(\lambda(n))^{5/8}}^{\ell-1}\frac{(1-\frac{1}{\ell})(1-\frac{2}{\ell})\cdots(1-\frac{k-1}{\ell})}{(1+\frac{k}{\ell\rho})(1+\frac{k-1}{\ell\rho})\cdots(1+\frac{1}{\ell\rho})}. \tag{2.20}$$

Next, observe that for $k>(\lambda(n))^{5/8}$ we have

$$\begin{aligned}&\frac{(1-\frac{1}{\ell})(1-\frac{2}{\ell})\cdots(1-\frac{k-1}{\ell})}{(1+\frac{k}{\ell\rho})(1+\frac{k-1}{\ell\rho})\cdots(1+\frac{1}{\ell\rho})}\\&\le\frac{(1-\frac{1}{n})\cdots(1-\frac{\lfloor\lambda(n)^{5/8}\rfloor}{n})}{(1+\frac{\lceil\lambda(n)^{5/8}\rceil}{\rho n})\cdots(1+\frac{1}{\rho n})}\left(\frac{1-\frac{\lceil\lambda(n)^{5/8}\rceil}{n}}{1+\frac{\lceil\lambda(n)^{5/8}\rceil}{\rho n}}\right)^{k-\lceil\lambda(n)^{5/8}\rceil}\\&\le\exp(-C(\lambda(n))^{1/4})\left(\frac{1-\frac{\lceil\lambda(n)^{5/8}\rceil}{n}}{1+\frac{\lceil\lambda(n)^{5/8}\rceil}{\rho n}}\right)^{k-\lceil\lambda(n)^{5/8}\rceil}\end{aligned}$$

where C is a constant that does not depend on a,b, or n. Using this bound for the terms in the summation on the right hand side of (2.20), we obtain, after summing over k,

$$\Sigma(1)\le\left(\frac{1+\rho}{\rho\ell}\right)2(\lambda(n))^{3/8}\exp(-C(\lambda(n))^{1/4}). \tag{2.21}$$

On the other hand, provided $an < \ell \le bn$ and $1 \le k \le (\lambda(n))^{5/8}$, we have

$$\begin{aligned}\frac{(1-\frac{1}{\ell})(1-\frac{2}{\ell})\cdots(1-\frac{k-1}{\ell})}{(1+\frac{k}{\ell\rho})(1+\frac{k-1}{\ell\rho})\cdots(1+\frac{1}{\ell\rho})} &= \frac{\exp\left(\frac{-k(k-1)}{2\ell}+O\left(\frac{1}{a(\lambda(n))^{1/8}}\right)\right)}{\exp\left(\frac{k(k+1)}{2\ell\rho}+O\left(\frac{1}{a(\lambda(n))^{1/8}}\right)\right)} \\ &= \exp\left(-\left(\frac{1+\rho}{\rho}\right)\frac{k^2}{2\ell}\right)(1+\varepsilon_3(k,\ell,\rho))\end{aligned}$$

where $|\varepsilon_3(k,\ell,\rho)| \le C(a)(\lambda(n))^{-1/8}$ and $C(a)$ is a constant which depends on a but not on n or $\lambda(n)$. It follows that

$$\begin{aligned}&\Sigma(2) \equiv \sum_{k=1}^{(\lambda(n))^{5/8}} \frac{1}{\ell}\left(\frac{1+\rho}{\rho}\right)\frac{(1-\frac{1}{\ell})(1-\frac{2}{\ell})\cdots(1-\frac{k-1}{\ell})}{(1+\frac{k}{\ell\rho})(1+\frac{k-1}{\ell\rho})\cdots(1+\frac{1}{\ell\rho})} \\ &= \sqrt{\frac{1+\rho}{\rho\ell}}\sum_{k=1}^{(\lambda(n))^{5/8}}\sqrt{\frac{n}{\ell}}\sqrt{\frac{1+\rho}{\rho}}\exp\left(-\left(\frac{1+\rho}{\rho}\right)\frac{k^2}{2\ell}\right)\frac{1}{\sqrt{n}}(1+\varepsilon_3(k,\ell,\rho)) \\ &= \sqrt{\frac{1+\rho}{\rho\ell}}\int_0^\infty\sqrt{\frac{n}{\ell}}\exp\left(-\frac{y^2n}{2\ell}\right)dy\cdot(1+\varepsilon_4(\ell,n,\rho)) \\ &= \sqrt{\frac{\pi(1+\rho)}{2\rho\ell}}\big(1+\varepsilon_4(\ell,n,\rho)\big) \qquad (2.22)\end{aligned}$$

where $|\varepsilon_4(\ell,n,\rho)| \le C(a)(\lambda(n))^{-1/8}$ and $C(a)$ is a constant which depends on a but not on n or $\lambda(n)$. Combining (2.19), (2.21), and (2.22)

$$\Pr\{\mathcal{B}_\ell^\rho\} = \sqrt{\frac{\pi(1+\rho)}{2\rho\ell}}\Big(1+\varepsilon_5(\ell,n,\rho)\Big)+\varepsilon_1(\ell,n,\rho) \qquad (2.23)$$

where $\big|\varepsilon_5(\ell,n,\rho)\big| \le C(a)(\lambda(n))^{-1/8}$ and $C(a)$ is a constant which depends on a but not on n or $\lambda(n)$.

Next, provided $an < \ell \le bn$, we obtain using Stirling's formula

$$\begin{aligned}&\Pr\Big\{\sum_{i=1}^{\ell} D_i^\rho = \ell \,\Big|\, \sum_{i=1}^{n} D_i^\rho = n\Big\} = \binom{n}{\ell}\frac{\Gamma(\ell+\ell\rho)\Gamma(n-\ell+(n-\ell)\rho)\Gamma(n\rho)}{\Gamma(\ell\rho)\Gamma((n-\ell)\rho)\Gamma(n+n\rho)} \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{\ell(n-\ell)}}\sqrt{\frac{\rho}{1+\rho}}\big(1+\varepsilon_6(\ell,n,\rho)\big) \qquad (2.24)\end{aligned}$$

where $\left|\varepsilon_6(\ell,n,\rho)\right| \le C(a,b)/\lambda(n)$ and $C(a,b)$ is a constant which depends on a and b, but not on $\lambda(n)$. It follows from (2.18), (2.23) and (2.24) that for $an < \ell \le bn$,

$$\Pr\bigl\{C_1^\rho(n)=\ell\bigr\} = \frac{1}{2n}\frac{1}{\sqrt{1-\ell/n}}\bigl(1+\varepsilon_7(\ell,n,\rho)\bigr)+\varepsilon_8(\ell,n,\rho) \tag{2.25}$$

where $\left|\varepsilon_7(\ell,n,\rho)\right| \le C(a,b)(\lambda(n))^{-1/8}$, $\left|\varepsilon_8(\ell,n,\rho)\right| \le C(a,b)e^{-an}$ and $C(a,b)$ is a constant which depends on a and b but does not depend on n or $\lambda(n)$. Part (i) now follows from (2.17) and (2.25).

Suppose now that $\rho n \to \beta > 0$ as $n \to \infty$. First we show that the limiting distribution of $C_1^\rho(n)/n$ has an atom at 1. As in the proof of Theorem 2 (iii), it follows from dominated convergence that

$$\begin{aligned}&\lim_{n\to\infty}\Pr\bigl\{C_1^\rho(n)=n\bigr\} = \lim_{n\to\infty}\Pr\bigl\{\mathcal{B}_n^\rho\bigr\}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\Pr\bigl\{N_{\sigma(k)}=1\bigr\}\Pr\bigl\{X_n^{\hat{D}(\rho,n)}=k\bigr\} = \sum_{k=1}^{\infty}\frac{\beta^{k-1}}{(\beta+k)_k}. \end{aligned}\tag{2.26}$$

Next suppose $0<x<1$ is fixed and $\ell=\lfloor xn\rfloor$. Since $\rho n\to\beta$, we have $\rho\ell\to x\beta$ as $n\to\infty$. So it follows from (2.26), that

$$\lim_{n\to\infty}\Pr\bigl\{\mathcal{B}_\ell^\rho\bigr\} = \sum_{k=1}^{\infty}\frac{(x\beta)^{k-1}}{(x\beta+k)_k}. \tag{2.27}$$

Now, using Stirling's formula, we obtain

$$\begin{aligned}\Pr\Bigl\{\sum_{i=1}^{\ell}D_i^\rho=\ell \Bigm| \sum_{i=1}^{n}D_i^\rho=n\Bigr\} &= \binom{n}{\ell}\frac{\Gamma(\ell+\ell\rho)\Gamma(n-\ell+(n-\ell)\rho)\Gamma(n\rho)}{\Gamma(\ell\rho)\Gamma((n-\ell)\rho)\Gamma(n+n\rho)}\\ &\sim \frac{\ell^{x\beta}(n-\ell)^{(1-x)\beta}}{n^\beta}\frac{n}{\ell(n-\ell)}\frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)}\\ &\sim \frac{1}{xn}\left(\frac{x}{1-x}\right)^{x\beta}(1-x)^{\beta-1}\frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)}\end{aligned}\tag{2.28}$$

It follows from (2.18), (2.27) and (2.28) that

$$\Pr\bigl\{C_1^\rho(n)=\ell\bigr\}\sim\frac{1}{n}\frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)}\left(\frac{x}{1-x}\right)^{x\beta}(1-x)^{\beta-1}\sum_{k=1}^{\infty}\frac{(x\beta)^{k-1}}{(x\beta+k)_k}$$

and (2.15) implies the result immediately.
Finally, part (iii) is equivalent to part (iii) of Theorem 2. $\square$

We can generalise Theorem 3 (i) as follows. For $n > 1$ and $k > 1$, define $\mathcal{C}_k^\rho(n)$ recursively as follows: If $[n] \setminus (\mathcal{C}_1^\rho(n) \cup \cdots \cup \mathcal{C}_{k-1}^\rho(n)) \neq \emptyset$, let $\mathcal{C}_k^\rho(n)$ denote the vertex set of the connected component in G_n^ρ which contains the smallest element of $[n] \setminus (\mathcal{C}_1^\rho(n) \cup \cdots \cup \mathcal{C}_{k-1}^\rho(n))$; otherwise, set $\mathcal{C}_k^\rho(n) = \emptyset$. For all $k \geq 1$, let $C_k^\rho(n) = \left|\mathcal{C}_k^\rho(n)\right|$. It follows from Theorem 3 in [11] that for $1 \leq k \leq n$ and $\ell_1, \ell_2, \ldots, \ell_k$ are such that $\ell_i \geq 1$ for $i = 1, 2, ..., k$, and $\sum_{i=1}^k \ell_i \leq n$, we have

$$\Pr\left\{C_1^\rho(n) = \ell_1, ..., C_k^\rho(n) = \ell_k\right\} = \prod_{i=1}^k \Pr\left\{C_1^\rho(\, n - t_{i-1}) = \ell_i\right\}, \qquad (2.29)$$

where $t_0 = 0$ and $t_i \equiv \ell_1 + ... + \ell_i$, $\;i = 1, 2, ..., k$. Thus from (2.25) and (2.29) we obtain

$$\lim_{n\to\infty} \Pr\left\{a_i < \frac{C_i^\rho(n)}{n - C_1^\rho(n) - \cdots - C_{i-1}^\rho(n)} < b_i, \quad 1 \leq i \leq t\right\}$$
$$= \prod_{i=1}^t \int_{a_i}^{b_i} \frac{1}{2\sqrt{1-x}} dx \,,$$

and hence, using tedious but standard arguments (see, for example [9]), one can show the generalization of Theorem 5 from [11]:

Theorem 4. *Suppose that $\rho n \to \infty$ as $n \to \infty$. Then the joint distribution of the normalized order statistics for the component sizes in G_n^ρ converges to the Poisson-Dirichlet (1/2) distribution on the simplex*

$$\nabla = \left\{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \ \text{ for every } \ i \geq 1\right\}.$$

It follows from Theorem 3 (ii), that when $\rho n \to \beta > 0$ as $n \to \infty$, $C_1^\rho(n)/n$ converges in distribution to a variable Z_β which has an atom at 1 and a density on the interval $(0, 1)$ given by

$$f(x) = \frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)} \left(\frac{x}{1-x}\right)^{x\beta} (1-x)^{\beta-1} \sum_{k=1}^{\infty} \frac{(x\beta)^{k-1}}{(x\beta+k)_k}$$

and we note that numerical calculation of $\int_0^1 f(x)dx$ using Mathematica has confirmed this result. We also note that it follows from Stirling's formula and (2.16) that Z_β converges in distribution as $\beta \to \infty$ to the *Beta(1/2)* distributed random variable B with density $f(x) = 1/(2\sqrt{1-x})$ on $(0,1)$, which was defined in Theorem 3 (i).

It follows from Theorem 1 that as $\rho n \to \infty$, the number of cyclic points, X_n^ρ, is roughly $\sqrt{\phi(n)}$ whereas we see from Theorem 3 that $C_1^\rho(n)$ is always of order n, no matter how slowly $\phi(n)$ tends to ∞. This suggests that it is the *structure* of the typical component which is sensitive to the rate at which $\phi(n)$ tends to ∞, rather than the total *size* of the component. In order to investigate more carefully the dependence of the structure of a typical component on the parameter ρ, we introduce the following notation. For any $f \in \mathcal{M}_n$, let $\mathcal{L}_1(f)$ denote the set of cyclic vertices in the component of $G(f)$ which contains the vertex 1. Define $\ell(f) = |\mathcal{L}_1(f)|$ and define $h(f)$, the height of vertex 1 in $G(f)$, by

$$h(f) = \min\{k \geq 0 : f^{(k)}(1) \in \mathcal{L}(f)\}.$$

We also define, for any $f \in \mathcal{M}_n$, the random mapping tree of vertex 1 in $G(f)$ as follows. Start with the digraph $G(f)$. For every cyclic vertex $v \in G(f)$, delete the directed edge in $G(f)$ from v to $f(v)$. This yields a forest of directed trees on n labelled vertices which are rooted at the cyclic vertices of $G(f)$. We say that the tree in this forest which contains the vertex labelled 1 is the *random mapping tree of vertex 1 in $G(f)$*. Let $\mathcal{Y}_n(f)$ denote the vertex set of this tree and define $Y_n(f) \equiv |\mathcal{Y}_n(f)|$ to be its size. Finally, we define the local structure random variables for G_n^ρ by $\ell_n^\rho \equiv \ell(T_n^\rho)$, $h_n^\rho \equiv h(T_n^\rho)$, and $t_n^\rho \equiv Y_n(T_n^\rho)$.

Theorem 5. *Let ℓ_n^ρ denote the number of cyclic vertices of G_n^ρ in the connected component which contains the vertex labelled 1. Fix $0 < x < \infty$.*

(i) *Suppose that $\rho n \to \infty$ as $n \to \infty$ and let $\phi(n) = \rho n/(1+\rho)$. If $k = \lfloor x\sqrt{\phi(n)} \rfloor$, then*

$$\Pr\{\ell_n^\rho = k\} \sim \frac{1}{\sqrt{\phi(n)}} \int_x^\infty e^{-u^2/2} du.$$

(ii) *Suppose that $\rho n \to \beta > 0$ as $n \to \infty$, then for $k \geq 1$*

$$\lim_{n\to\infty} \Pr\{\ell_n^\rho = k\} = \sum_{j=k}^{\infty} \frac{\beta^{j-1}}{(\beta+j)_j}.$$

(iii) *Suppose that* $\rho n \to 0$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \Pr\big\{\ell_n^\rho = 1\big\} = 1\,.$$

Proof. We sketch the proof. It was shown in [12] that for $\rho > 0$ and for $1 \le k \le n$

$$\begin{aligned}
\Pr\big\{\ell_n^\rho = k\big\} &= \frac{1}{n}\sum_{j=k}^{n-1} \frac{\rho^j(1+\rho)(n)_{j+1}}{(n\rho+j)_{j+1}} + \frac{1}{n}\frac{\rho^k(n)_k}{(n\rho+k-1)_k} \\
&= \sum_{j=k}^{n-1} \frac{(1+\rho)}{n\rho}\frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{j}{n})}{(1+\frac{j}{n\rho})(1+\frac{j-1}{n\rho})\cdots(1+\frac{1}{n\rho})} + \frac{1}{n}\frac{\rho^k(n)_k}{(n\rho+k-1)_k}\,.
\end{aligned}$$

In each of the three cases it is straightforward to establish the asymptotic expression for the above sum by calculations similar to those in the proofs of Theorem 1 and Theorem 2 (iii).

□

Theorem 6. *Let* h_n^ρ *denote the height of vertex 1 in* G_n^ρ*. Fix* $0 < x < \infty$*.*

(i) *Suppose that* $\rho n \to \infty$ *as* $n \to \infty$ *and let* $\phi(n) = \rho n/(1+\rho)$*. If* $k = \lfloor x\sqrt{\phi(n)}\rfloor$*, then*

$$\Pr\big\{h_n^\rho = k\big\} \sim \frac{1}{\sqrt{\phi(n)}}\int_x^\infty e^{-u^2/2}du\,.$$

(ii) *Suppose that* $\rho n \to \beta > 0$ *as* $n \to \infty$*, then for* $k \ge 1$

$$\lim_{n\to\infty} \Pr\big\{h_n^\rho = k\big\} = \sum_{j=k}^{\infty} \frac{\beta^{j-1}}{(\beta+j)_j} \quad \text{and} \quad \lim_{n\to\infty} \Pr\big\{h_n^\rho = 0\big\} = 0\,.$$

(iii) *Suppose that* $\rho n \to \beta > 0$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \Pr\big\{h_n^\rho = 1\big\} = 1\,.$$

Proof. This result follows from the following fact which was proved in [12]: Suppose that $\rho > 0$, then for $1 \le k \le n-1$

$$\Pr\big\{h_n^\rho = k\big\} = \frac{1}{n}\sum_{j=k}^{n-1} \frac{\rho^j(1+\rho)(n)_{j+1}}{(n\rho+j)_{j+1}}$$

and

$$\Pr\{h_n^\rho = 0\} = \frac{1}{n}\sum_{j=1}^{n} \frac{\rho^j (n)_j}{(n\rho + j - 1)_j}.$$

Again, like in the proof of the previous theorem, in all three cases one can easily obtain the asymptotic expressions for the above sums by calculations similar to those in the proofs of Theorem 1 and Theorem 2 (iii).

□

To investigate the distribution of t_n^ρ, we recall that it follows from Fact 1 that $t_n^\rho \overset{d}{\sim} Y^{\hat{D}^\rho(n)}$ and $X_n^\rho \overset{d}{\sim} X^{\hat{D}^\rho(n)}$ where $Y^{\hat{D}^\rho(n)} \equiv Y_n(T_n^{\hat{D}^\rho(n)})$ and $X^{\hat{D}^\rho(n)} \equiv X_n(T_n^{\hat{D}^\rho(n)})$. The distribution of t_n^ρ is obtained from the following lemma in which we obtain the *joint* distribution of $X^{\hat{D}^\rho(n)}$ and $Y^{\hat{D}^\rho(n)}$.

Lemma 1. *Let $Y^{\hat{D}^\rho(n)}$ denote the size of random mapping tree of vertex 1 in $G_n^{\hat{D}^\rho(n)}$.*

(a) *For $1 < \ell < n$ and $1 < k \le n - \ell + 1$*

$$\Pr\{Y^{\hat{D}^\rho(n)} = \ell,\, X^{\hat{D}^\rho(n)} = k\}$$
$$= \frac{k\ell}{n(\ell-1)} E\big(\hat{D}^\rho_{1,\ell}(\hat{D}^\rho_{1,\ell} - 1)\big)\Pr\{X^{\hat{D}^\rho(n-\ell)} = k-1\}\Pr\{\sum_{i=1}^{\ell} \hat{D}^\rho{}_{i,n} = \ell\}.$$

(b) *For $\ell = 1$ and $1 < k \le n$*

$$\Pr\{Y^{\hat{D}^\rho(n)} = 1,\, X^{\hat{D}^\rho(n)} = k\} = \frac{k}{n}\Pr\{X^{\hat{D}^\rho(n-1)} = k-1\}\Pr\{\hat{D}^\rho_{1,n} = 1\}.$$

(c)
$$\Pr\{Y^{\hat{D}^\rho(n)} = n,\, X^{\hat{D}^\rho(n)} = 1\} = \Pr\{X^{\hat{D}^\rho(n)} = 1\}.$$

Proof. Let $\mathcal{Y}^{\hat{D}^\rho(n)} \equiv \mathcal{Y}_n(T_n^{\hat{D}^\rho(n)})$ denote the vertex set of the random mapping tree in $G_n^{\hat{D}^\rho(n)}$ which contains vertex 1 and let $\mathcal{L}^{\hat{D}^\rho(n)}$ denote the set of cyclic vertices in $G_n^{\hat{D}^\rho(n)}$. Also, for $1 \le \ell \le n$ and $1 \le k \le n - \ell + 1$, let

$$\mathcal{E}(k,\ell,n) \equiv \left\{\mathcal{Y}^{\hat{D}^\rho(n)} = [\ell],\, \mathcal{L}^{\hat{D}^\rho(n)} = \{1\} \cup \{\ell+1, \ell+2, \ldots, \ell+k-1\}\right\},$$

where $[\ell] = \{1, 2, \ldots, \ell\}$. We note that it follows from the exchangeability of the variables $\hat{D}^{\rho}_{1,n}, \hat{D}^{\rho}_{2,n}, ..., \hat{D}^{\rho}_{n,n}$ and (1.2) that the distribution of the corresponding digraph $G_n^{\hat{D}^{\rho}(n)}$ is invariant under re-labelling of the vertices of the $G_n^{\hat{D}^{\rho}(n)}$. So, for $1 \leq \ell \leq n$ and $1 \leq k \leq n - \ell + 1$, we have

$$\Pr\big\{Y^{\hat{D}^{\rho}(n)} = \ell,\, X^{\hat{D}^{\rho}(n)} = k\big\} = \ell\binom{n-1}{\ell-1}\binom{n-\ell}{k-1}\Pr\big\{\mathcal{E}(k,\ell,n)\big\} \tag{2.30}$$

where $\binom{n-1}{\ell-1}$ is the number of ways to choose the vertices, other than vertex 1, in the random mapping tree containing 1, ℓ is the number of ways to choose the 'root' for this tree, and $\binom{n-\ell}{k-1}$ is the number of ways to choose the other vertices, in addition to the root of the tree, which form the cyclic vertices of $G_n^{\hat{D}^{\rho}(n)}$. Given the event $\big\{\hat{D}^{\rho}(n) = \vec{d}\big\}$ where $\vec{d} = (d_1, d_2, \ldots, d_n)$ is a vector of non-negative integers such that $\sum_{i=1}^{n} d_i = n$, the event $\mathcal{E}(k,\ell,n)$ occurs only if it is possible to construct a directed tree, rooted at vertex 1, on the vertices $[\ell]$ with in-degree sequence $(d_1 - 1, d_2, \ldots, d_\ell)$. In addition, there must be a directed edge from a cyclic vertex to the root 1 (since 1 must also be a cyclic vertex). It follows that we must have $\sum_{i=1}^{\ell} d_i = \ell$ and hence,

$$\mathcal{E}(k,\ell,n) \subseteq \Big\{\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell\Big\} = \Big\{\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell,\, \sum_{i=\ell+1}^{n}\hat{D}^{\rho}_{i,n}\Big\}.$$

So

$$\Pr\Big\{\mathcal{E}(k,\ell,n)\Big\} = \Pr\Big\{\mathcal{E}(k,\ell,n)\,\Big|\,\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell\Big\}\Pr\Big\{\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell\Big\}. \tag{2.31}$$

Now in the first case, when $1 < \ell < n$ and $2 \leq k \leq n - \ell - 1$ we have

$$\Pr\Big\{\mathcal{E}(k,\ell,n)\,\Big|\,\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell\Big\} \tag{2.32}$$
$$= \sum_{\substack{\vec{d}:\ d_1+\ldots+d_\ell=\ell,\\ d_{\ell+1}+\ldots+d_n=n-\ell,\ d_1\geq 2}} \Pr\Big\{\mathcal{E}(k,\ell,n)\,\Big|\,\hat{D}^{\rho}(n) = \vec{d}\Big\}\Pr\Big\{\hat{D}^{\rho}(n) = \vec{d}\,\Big|\,\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n} = \ell\Big\}.$$

To compute $\Pr\big\{\mathcal{E}(k,\ell,n) \mid \hat{D}^{\rho}(n) = \vec{d}\big\}$ for some fixed $\vec{d}$, recall that given the event $\big\{\hat{D}^{\rho}(n) = \vec{d}\big\}$, the distribution of $T_n^{\hat{D}^{\rho}(n)}$ is uniform on $\mathcal{M}_n(\vec{d})$. So it is enough to count the number of mappings f in $\mathcal{M}_n(\vec{d})$ such that the corresponding digraph $G_n(f)$ has the required structure.

Let $\mathcal{T}_\ell(d_1 - 1, d_2, ..., d_\ell)$ denote the set of directed trees on the vertices $\{1, 2, \ldots, \ell\}$ which are rooted at the vertex 1 and have in-degree sequence $(d_1-1, d_2, \ldots, d_\ell)$, and let $\mathcal{M}_{n-\ell}^{k-1}(d_{\ell+1}, d_2, \ldots, d_n)$ denote the set of mappings from $\{\ell+1, \ell+2, \ldots, n\}$ into $\{\ell+1, \ell+2, \ldots, n\}$ with in-degree sequence $(d_{\ell+1}, \ldots, d_n)$ and cyclic vertices $\{\ell+1, \ell+2, \ldots, \ell+k-1\}$. Given $t \in \mathcal{T}_\ell(d_1-1, d_2, \ldots, d_\ell)$ and $g \in \mathcal{M}_{n-\ell}^{k-1}(d_{\ell+1}, \ldots, d_n)$ there are $k-1$ ways to map the root 1 of t to a cyclic vertex of g to create a mapping $f \in \mathcal{M}_n(\vec{d})$ such that $G_n(f)$ has the required structure. Specifically, if i is a cyclic vertex for g, we map 1 to $g(i)$ and i to 1. In addition, by mapping root 1 to itself, the tree t becomes a random mapping component and, with g, this also determines a mapping $f \in \mathcal{M}_n(\vec{d})$ with the required structure. It follows that

$$\Pr\Big\{\mathcal{E}(k,\ell,n) \mid \hat{D}^\rho(n) = \vec{d}\Big\} = \frac{k \cdot \big|\mathcal{T}_\ell(d_1-1, d_2, ..., d_\ell)\big| \cdot \big|\mathcal{M}_{n-\ell}^{k-1}(d_{\ell+1}, ..., d_n)\big|}{n!(\prod_{i=1}^n d_i!)^{-1}}$$

$$= \binom{n}{\ell}^{-1} \frac{k}{\ell(\ell-1)} \cdot \frac{d_1(d_1-1)\big|\mathcal{M}_{n-\ell}^{k-1}(d_{\ell+1}, ..., d_n)\big|}{(n-\ell)!(\prod_{i=\ell+1}^n d_i!)^{-1}} \tag{2.33}$$

where the last equality follows from the identity

$$\big|\mathcal{T}_\ell(d_1-1, d_2, ..., d_\ell)\big| = \frac{(\ell-2)!}{(d_1-2)!d_2! \cdots d_\ell!}$$

(see [12], Lemma 3.1). Next, it follows from the construction of the variables $\hat{D}_{1,n}^\rho, \hat{D}_{2,n}^\rho, ..., \hat{D}_{n,n}^\rho$ and the independence of the variables $D_1^\rho, D_2^\rho, ..$ that

$$\Pr\Big\{\hat{D}^\rho(n) = \vec{d} \,\Big|\, \sum_{i=1}^{\ell} \hat{D}_{i,n}^\rho = \ell\Big\} = \Pr\Big\{\hat{D}^\rho(n) = \vec{d} \,\Big|\, \sum_{i=1}^{\ell} \hat{D}_{i,n}^\rho = \ell,\ \sum_{i=\ell+1}^{n} \hat{D}_{i,n}^\rho = n-\ell\Big\}$$

$$= \Pr\Big\{D_i^\rho = d_i, 1 \le i \le \ell \,\Big|\, \sum_{i=1}^{\ell} D_i^\rho = \ell\Big\}\Pr\Big\{D_i^\rho = d_i,\ \ell+1 \le i \le n \,\Big|\, \sum_{i=\ell+1}^{n} D_i^\rho = n-\ell\Big\}.$$

Hence, it follows from (2.32) and (2.33) that

$$\Pr\Big\{\mathcal{E}(k,\ell,n)\;\Big|\;\sum_{i=1}^{\ell}\hat{D}^{\rho}_{i,n}=\ell\Big\}= \tag{2.34}$$

$$\binom{n}{\ell}^{-1}\frac{k}{\ell(\ell-1)}\sum_{\substack{(d_1,\ldots,d_\ell):\\ d_1+\ldots+d_\ell=\ell,\ d_1\geq 2}} d_1(d_1-1)\Pr\Big\{D^{\rho}_i=d_i, 1\leq i\leq \ell\;\Big|\sum_{i=1}^{\ell}D^{\rho}_i=\ell\Big\}\;\times$$

$$\sum_{\substack{(d_{\ell+1},\ldots,d_n):\\ d_{\ell+1}+\ldots+d_n=n-\ell}}\frac{\big|\mathcal{M}^{k-1}_{n-\ell}(d_{\ell+1},...,d_n)\big|}{(n-\ell)!(\prod_{i=\ell+1}^{n}d_i!)^{-1}}\Pr\Big\{D^{\rho}_i=d_i,\,\ell+1\leq i\leq n\;\Big|\sum_{i=\ell+1}^{n}D^{\rho}_i=n-\ell\Big\}.$$

Now it follows from the construction of the variables $\hat{D}^{\rho}(\ell)=(\hat{D}^{\rho}_{1,\ell},\,\ldots,\,\hat{D}^{\rho}_{\ell,\ell})$ in terms of the conditional distribution of $D^{\rho}_1, D^{\rho}_2,\,\ldots,\,D^{\rho}_{\ell}$ given the event $\sum_{i=1}^{\ell}D^{\rho}_i=\ell$ that

$$\begin{aligned}&\sum_{\substack{(d_1,\ldots,d_\ell):\\ d_1+\ldots+d_\ell=\ell,\ d_1\geq 2}} d_1(d_1-1)\Pr\Big\{D^{\rho}_i=d_i,\,1\leq i\leq \ell\;\Big|\sum_{i=1}^{\ell}D^{\rho}_i=\ell\Big\}\\ =\;& E\Big(\hat{D}^{\rho}_{1,\ell}(\hat{D}^{\rho}_{1,\ell}-1)\Big)\,.\end{aligned} \tag{2.35}$$

Also, by re-labelling the vertices $\ell+1,\,\ell+2,\,\ldots,\,n$ by $1,\,2,\,\ldots,\,n-\ell$, it follows from the construction of $T^{\hat{D}^{\rho}(n-\ell)}_{n-\ell}$ that

$$\begin{aligned}&\sum_{\substack{(d_{\ell+1},\ldots,d_n):\\ d_{\ell+1}+\ldots+d_n=n-\ell}}\frac{\big|\mathcal{M}^{k-1}_{n-\ell}(d_{\ell+1},...,d_n)\big|}{(n-\ell)!(\prod_{i=\ell+1}^{n}d_i!)^{-1}}\Pr\Big\{D^{\rho}_i=d_i,\,\ell+1\leq i\leq n\mid\sum_{i=\ell+1}^{n}D^{\rho}_i=n-\ell\Big\}.\\ =\;&\Pr\big\{\mathcal{L}^{\hat{D}^{\rho}(n-\ell)}=[k-1]\big\}\end{aligned} \tag{2.36}$$

So, now (a) of Lemma 1 follows from (2.30), (2.31) and (2.34) – (2.36).

The proof of (b) of Lemma 1 follows by a similar argument. In the case $\ell=1$ and $2\leq k\leq n$, the random mapping tree which contains vertex 1 consists of only vertex 1, which is also a cyclic vertex. So we must have $\hat{D}^{\rho}_{1,n}=1$ and equation (2.32) becomes

$$\begin{aligned}&\Pr\big\{\mathcal{E}(k,1,n)\mid\hat{D}^{\rho}_{i,n}=1\big\}\\ =&\sum_{\substack{\vec{d}:\ d_1=1,\\ d_2+\ldots+d_n=n-1}}\Pr\big\{\mathcal{E}(k,1,n)\mid\hat{D}^{\rho}(n)=\vec{d}\big\}\Pr\big\{\hat{D}^{\rho}(n)=\vec{d}\,\big|\hat{D}^{\rho}_{1,n}=1\big\}\,.\end{aligned} \tag{2.37}$$

Also, since there is only one random mapping tree of size one with root 1, equation (2.33) becomes

$$\Pr\Big\{\mathcal{E}(k,1,n) \mid \hat{D}^\rho(n) = \vec{d}\Big\} = \frac{k}{n} \cdot \frac{\big|\mathcal{M}_{n-1}^{k-1}(d_2, \ldots, d_n)\big|}{(n-1)!(\prod_{i=2}^{n} d_i!)^{-1}}\,. \tag{2.38}$$

Since

$$\Pr\Big\{\hat{D}^\rho(n) = \vec{d} \mid \hat{D}^\rho_{1,n} = 1\Big\} = \Pr\Big\{D_i^\rho = d_i,\, 2 \le i \le n \mid \sum_{i=2}^{n} D_i^\rho = n-1\Big\}\,,$$

the part (b) of Lemma 1 now follows from (2.30), (2.31) (with $\ell = 1$), (2.37), and (2.38).

Finally, we note that if $X^{\hat{D}^\rho(n)} = 1$ then the digraph $G_n^{\hat{D}^\rho(n)}$ consists of a directed tree rooted at the one cyclic vertex of $G_n^{\hat{D}^\rho(n)}$, i.e.

$$\big\{X^{\hat{D}^\rho(n)} = 1\big\} = \big\{X^{\hat{D}^\rho(n)} = 1,\, Y^{\hat{D}^\rho(n)} = n\big\}\,,$$

and so the part (c) of Lemma 1 holds, too.

□

It is straightforward to check that for any $n > 1$

$$E\big(\hat{D}^\rho_{1,n}(\hat{D}^\rho_{1,n} - 1)\big) = \frac{(n-1)(1+\rho)}{n\rho + 1}\,,$$

so we obtain from Fact 1 and Lemma 1

Corollary 1. *Let t_n^ρ denote the size of random mapping tree of vertex 1 in G_n^ρ. For $1 < n$ and $1 < \ell < n$*

$$\Pr\Big\{t_n^\rho = \ell\Big\} = \frac{\ell(1+\rho)}{n(\ell\rho+1)}\big(1 + E(X_{n-\ell}^\rho)\big)\Pr\Big\{\sum_{i=1}^{\ell} D_i^\rho = \ell \;\Big|\; \sum_{i=1}^{n} D_i^\rho = n\Big\}.$$

Also

$$\Pr\Big\{t_n^\rho = 1\Big\} = \frac{1}{n}\big(1 + E(X_{n-1}^\rho)\big)\Pr\Big\{D_1^\rho = 1 \;\Big|\; \sum_{i=1}^{n} D_i^\rho = n\Big\}$$

and

$$\Pr\Big\{t_n^\rho = n\Big\} = \frac{1+\rho}{n\rho+1}\,.$$

This leads to our last result concerning the asymptotic distribution of t_n^ρ.

Theorem 7. *Let t_n^ρ denote the size of random mapping tree of vertex 1 in G_n^ρ. Fix $0 < x < 1$.*

(i) *Suppose that $\rho n \to \infty$ as $n \to \infty$, then*

$$\Pr\{t_n^\rho = \lfloor xn \rfloor\} \sim \frac{1}{n}\frac{1}{2\sqrt{x}}\,.$$

(ii) *Suppose that $\rho n \to \beta > 0$ as $n \to \infty$, then*

$$\Pr\{t_n^\rho = \lfloor xn \rfloor\} \sim \frac{1}{n} f(x)\,,$$

where

$$f(x) =$$
$$\frac{\Gamma(\beta)}{\Gamma(x\beta)\Gamma((1-x)\beta)}\frac{1}{x\beta+1}\left(\frac{x}{1-x}\right)^{x\beta}(1-x)^{\beta-1}\left(2+\sum_{k=1}^{\infty}\frac{((1-x)\beta)^k}{((1-x)\beta+k)_k}\right)$$

and

$$\lim_{n\to\infty}\Pr\{t_n^\rho = n\} = \frac{1}{\beta+1}\,.$$

(iii) *Suppose that $\rho n \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty}\Pr\{t_n^\rho = n\} = 1\,.$$

Proof. In the case where $\rho n \to \infty$ as $n \to \infty$ and $\ell = \lfloor xn \rfloor$, (i) follows from Theorem 1 (i), (2.24), and Corollary 1.

Now suppose that $\rho n \to \beta > 0$ as $n \to \infty$ and $\ell = \lfloor xn \rfloor$. It follows from Theorem 1 (ii)

$$\lim_{n\to\infty} E(X_{n-\ell}^\rho) = 1 + \sum_{k=1}^{\infty}\frac{((1-x)\beta)^k}{((1-x)\beta+k)_k}\,. \tag{2.39}$$

So, (ii) now follows from Corollary 1, (2.28) and (2.39). Lastly, it is clear that

$$\lim_{n\to\infty}\Pr\{t_n^\rho = n\} = \lim_{n\to\infty}\frac{1+\rho}{n\rho+1} = \frac{1}{\beta+1}\,.$$

Part (iii) follows immediately from the above.

□

It follows from Theorem 7, that when $\rho n \to \beta > 0$ as $n \to \infty$, t_n^ρ/n converges in distribution to a random variable W_β which has an atom at 1 and a density on the interval $(0,1)$ given by $f(x)$ for $0 < x < 1$ defined in Theorem 7 (ii). Again, we have confirmed this result by numerical calculation of $\int_0^1 f(x)dx$ using Mathematica.

3 Final Remarks

Given the results above, we can now describe the 'evolution' of the structure of G_n^ρ when n is *large* and $\rho \to 0$. Roughly speaking, when $\rho =$'∞', G_n^ρ is just the uniform random mapping digraph. Then, as long as $\frac{1}{n} = o(\rho)$, the *structure* of a typical component in G_n^ρ changes as ρ decreases but the (asymptotic) distribution of the *total* size of the typical component remains the same. In particular, with high probability, the typical component in G_n^ρ has $O(\sqrt{\phi(n)})$ cyclic vertices with directed trees attached to the cycle, and since the size of the typical component is $O(n)$, some of these trees are 'large'. On the other hand, the height of a typical vertex in the component is also $O(\sqrt{\phi(n)})$, so the large trees in the typical component become shorter and 'bushier' as $\rho \to 0$. It also follows from Theorems 2 and 4 that with high probability the number of components in G_n^ρ is $O(\frac{1}{2}\log(\phi(n)))$ whereas the (asymptotic) joint distribution of the order statistics of the normalised component sizes is Poisson-Dirichlet(1/2) as long as $\frac{1}{n} = o(\rho)$. Hence, as $\rho \to 0$, the number of components of size $o(n)$ in G_n^ρ must be decreasing - i.e. the 'small' components of G_n^ρ 'coagulate' to form larger components as $\rho \to 0$. Finally, it follows from Theorem 4 that $\Pr\{\mathcal{B}_n^\rho\} \sim 0$ as long as $\frac{1}{n} = o(\rho)$.

Next, a 'phase transition' occurs in the structure of G_n^ρ when $\rho = O(\frac{1}{n})$. In this phase, with high probability, G_n^ρ has $O(1)$ components and all components are of size $O(n)$. On the other hand, with high probability, the height of a typical vertex is $O(1)$ and the typical component consists of a very small cycle with short, large trees attached. In addition, at this threshold, we have $\Pr\{\mathcal{B}_n^\rho\} > 0$ and $\Pr\{\mathcal{B}_n^\rho\} \to 1$ as ρ decreases to 0. We also note that in this phase, we have obtained some limiting distributions which are new in the context of random mapping models. Finally, once $\rho = o(\frac{1}{n})$, G_n^ρ is a 'star' graph with 1 cyclic vertex and all other vertices attached to this cyclic vertex.

It is interesting to compare the evolution of the structure of G_n^ρ as $\rho \downarrow 0$ to the evolution of the structure of $G_n(\lambda)$, the random mapping with attracting center at 1, as $\lambda \uparrow \infty$. First, recall that $G_n(1)$ is just the uniform random mapping. Now provided that $\lambda = o(\sqrt{n})$, the (asymptotic) structure of $G_n(\lambda)$ is the same as the (asymptotic) structure of the uniform model G_n. In particular, the asymptotic distribution of the normalized size of the 'attracting' component in $G_n(\lambda)$ is the same as the asymptotic distribution of the normalised size of the typical component in G_n (i.e. it is a *Beta(1/2)*) distribution). When $\lambda = O(\sqrt{n})$ the attracting component begins to 'grow' and the asymptotic distribution of its normalised size is no longer *Beta(1/2)*. In this phase there are still $O(n)$ vertices outside the attracting component. We also note that the random mapping $T_n(\lambda)$ *restricted* to the vertices outside the attracting component is just a uniform random mapping on that subset of vertices. Once $\sqrt{n} = o(\lambda)$ but still $\lambda = o(n)$ the number of vertices outside the attracting component is $O(n^2/\lambda^2)$ and the attracting component is the dominant component. Finally, once $\lambda = O(n)$ with high probability there are only $O(1)$ vertices outside the attracting component and when $n = o(\lambda)$, $G_n(\lambda)$ is a 'star' graph.

So the evolution of both G_n^ρ and $G_n(\lambda)$ starts at the uniform random mapping and ends with a 'star' graph. However because the structure of G_n^ρ is determined by a preferential attachment process, there is no 'favoured' vertex in G_n^ρ at the start of the attachment process. In particular, as edges are added to G_n^ρ there is 'competition' between the vertices to become the 'most attractive' vertex. As a result, we do not see the dominance of a single component in G_n^ρ until the phase $\rho = o(\frac{1}{n})$. We also note that, roughly speaking, the phase $\lambda = O(\sqrt{n})$ in the evolution of $G_n(\lambda)$ is comparable to the phase $\rho = O(\frac{1}{\sqrt{n}})$ in the evolution of G_n^ρ. At this stage the Poisson-Dirichlet limit law no longer holds for $G_n(\lambda)$ whereas it continues to hold for G_n^ρ until the phase $\rho = O(\frac{1}{n})$. Finally, another important difference is that G_n^ρ is, in some sense, homogeneous whereas $G_n(\lambda)$ is not, i.e. if we choose a vertex v at random and remove its component from G_n^ρ then conditioned on m, the size of the removed component, the remaining subgraph will have the same structure as G_{n-m}^ρ. The 'homogeneity' of G_n^ρ may partly explain the persistence of the Poisson-Dirichlet limit law in the evolution of G_n^ρ.

The persistence of the Poisson-Dirichlet(1/2) limit law for G_n^ρ until the phase $\rho = O(\frac{1}{n})$ is intriguing in other respects. In particular, we obtain the Poisson-Dirichlet limit law even when $\frac{1}{2}\log(\phi(n))$ (i.e. the asymptotic average number of components in G_n^ρ) tends to ∞ but $\phi(n) = o(n)$. This is very different from the usual structure for logarithmic combinatorial structures where the number of components in the structure is asymptotic to $\theta \log n$ and the order statistics for the normalised component sizes have a Poisson-Dirichlet(θ) limit distribution (see [3], [4] and references therein). It is also interesting to compare our results for G_n^ρ to those obtained in [10] via a cutting process for random mappings. In particular, we show in [10] that once the order of the number of cyclic vertices in a random mapping is *greater* than $\sqrt{n}$, the limit law for the order statistics of the normalized component sizes is Poisson-Dirichlet(1) (rather than Poisson-Dirichlet(1/2)). In contrast, our results for G_n^ρ show that even when G_n^ρ has $o(\sqrt{n})$ cyclic vertices the Poisson-Dirichlet(1/2) limit law can still hold.

Finally, we mention some possible directions for future work on random mappings with preferential attachment. First, we note that our results do not yield much information about the distribution of number component of size $o(n)$. However, in light of Theorems 2 and 4, it seems unlikely that we can approximate the joint distribution of numbers of 'small' components by a logarithmic structure. In a different direction, it would be interesting to use the 'regression' approach (see, for example [5] and [6]) to study the structure of a connected component *conditioned* on the size of the component. Lastly, we note that the preferential attachment process ends when n directed edges have been added to the digraph on n vertices. By considering the number of edges as an additional parameter in the attachment process and by letting the number of directed edges grow one can investigate the hitting time of connectedness, the time that the first cycle appears, etc. Note that in this case the attachment process does not need to finish once the the random mapping digraph has been constructed. This approach was successfully developed in [14] for the uniform graph process associated with uniform random mappings.

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