Random mappings with exchangeable in-degrees

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Abstract

In this paper we introduce a new random mapping model, $T_n^{\hat{D}}$, which maps the set $\{1, 2, ..., n\}$ into itself. The random mapping $T_n^{\hat{D}}$ is constructed using a collection of exchangeable random variables $\hat{D}_1, ..., \hat{D}_n$ which satisfy $\sum_{i=1}^{n} \hat{D}_i = n$. In the random digraph, $G_n^{\hat{D}}$, which represents the mapping $T_n^{\hat{D}}$, the in-degree sequence for the vertices is given by the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$, and, in some sense, $G_n^{\hat{D}}$ can be viewed as an analogue of the general independent degree models from random graph theory. We show that the distribution of the number of cyclic points, the number of components, and the size of a typical component can be expressed in terms of expectations of various functions of $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. We also consider two special examples of $T_n^{\hat{D}}$ which correspond to random mappings with preferential and anti-preferential attachment, respectively, and determine, for these examples, exact and asymptotic distributions for the statistics mentioned above.

Keywords: random mappings, exchangeable degree sequence, component structure, preferential attachment, anti-preferential attachment.

1 Introduction

The study of random mapping models was initiated independently by several authors in the 1950s (see [7, 17, 18, 24, 29, 38]) and the properties of these

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models have received much attention in the literature. In particular, these models have been useful as models for epidemic processes, and have natural applications in cryptology (see, for example, [8, 9, 10, 14, 16, 19, 27, 28, 31, 32, 35, 37, 40]). To date, the most widely studied models have been special cases of a general model denoted by $T_{\mathbf{p}(n)}$, which can be defined as follows: Let $[n]$ denote the set of integers $\{1, 2, ..., n\}$ and let $\mathcal{M}_n$ denote the set of all mappings from $[n]$ into $[n]$. For each $n \geq 1$, let $\mathbf{p}(n) = \{p_{ij}(n) : 1 \leq i, j \leq n\}$ be an array such that $p_{ij}(n) \geq 0$ for $1 \leq i, j \leq n$ and $\sum_{j=1}^{n} p_{ij}(n) = 1$ for every $1 \leq i \leq n$, and let $X_1^n, X_2^n, ..., X_n^n$ be independent random variables such that $\Pr\{X_i^n = j\} = p_{ij}(n)$ for all $1 \leq i, j \leq n$. Then the random mapping $T_{\mathbf{p}(n)} : [n] \to [n]$ is defined (in terms of the variables $X_1^n, X_2^n, ..., X_n^n$) by

$$T_{\mathbf{p}(n)}(i) = j \quad \text{iff} \quad X_i^n = j \tag{1.1}$$

for all $1 \leq i, j \leq n$. It follows from (1.1) that the distribution of $T_{\mathbf{p}(n)}$ is given by

$$\Pr\{T_{\mathbf{p}(n)} = f\} = \prod_{i=1}^{n} p_{if(i)}(n) \tag{1.2}$$

for each $f \in \mathcal{M}_n$. Any mapping $f \in \mathcal{M}_n$ can be represented as a directed graph $G(f)$ on a set of vertices labelled $1, 2, ..., n$, such that there is a directed edge from vertex i to vertex j in $G(f)$ if and only if $f(i) = j$. So $G_{\mathbf{p}(n)} \equiv G(T_{\mathbf{p}(n)})$ is a random directed graph on a set of vertices labelled $1, 2, ..., n$ which represents the action of the random mapping $T_{\mathbf{p}(n)}$ on $[n]$. We note that since each vertex in $G_{\mathbf{p}(n)}$ has out-degree 1, the components of $G_{\mathbf{p}(n)}$ consist of directed cycles with directed trees attached. Also, it follows from the definition of $T_{\mathbf{p}(n)}$ that the variables $X_1^n, X_2^n, \ldots X_n^n$ can be interpreted as the independent 'choices' of the vertices $1, 2, \ldots, n$ in the random digraph $G_{\mathbf{p}(n)}$ (see, in addition, Mutafchiev [33] and Jaworski [25]).

The example of $T_{\mathbf{p}(n)}$ which is best understood is the uniform random mapping, $T_n \equiv T_{\mathbf{p}(n)}$, where $p_{ij}(n) = \frac{1}{n}$ for all $1 \leq i, j \leq n$. Much is known (see for example the monograph by Kolchin [30]) about the component structure of the random digraph $G_n \equiv G(T_n)$ which represents T_n. Aldous [1] has shown that the joint distribution of the normalized order statistics for the component sizes in G_n converges to the *Poisson-Dirichlet* (1/2) distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \text{ for every } i \geq 1\}$. Also, if M_k denotes the number of components of size k in G_n then the joint distribution of $(M_1, M_2, \ldots, M_b)$ is close, in the sense of total variation, to the joint distribution of a sequence of independent Poisson random variables when $b = o(n/\log n)$ (see Arratia et.al. [5], [6]) and from this result one obtains a functional central limit theorem for the component sizes (see also [20]). The asymptotic distributions of variables such as the number of predecessors and the number of successors of a vertex in G_n are also known (see [9, 10, 14, 27, 28, 31, 32, 35]). In another direction, Berg, Jaworski, and Mutafchiev (see [11, 25, 26, 27, 28]) have investigated the structure of $G_{\mathbf{p}(n)}$ when $\mathbf{p}(n)$ is given by $p_{ii}(n) = q$ for some $0 \leq q \leq 1$ and all $1 \leq i \leq n$, and $p_{ij}(n) = \frac{1-q}{n-1}$ for all $1 \leq i, j \leq n$ such that $i \neq j$. Finally, Aldous, Miermont, and Pitman (see [2] and [3]) have recently investigated the asymptotic structure of $G_{\mathbf{p}(n)}$, where $\mathbf{p}(n)$ is given by $p_{ij}(n) = p_j(n) > 0$ for all $1 \leq i, j \leq n$, by using an ingenious coding of the mapping $T_{\mathbf{p}(n)}$ as a stochastic process on the interval $[0, 1]$. Their results are closely related to earlier work on the relationship between random mappings and random forests (see Pitman [34] and references therein).

The common feature in all the models discussed above is that each vertex in $G_{\mathbf{p}(n)}$ 'chooses' the vertex that it is mapped to independently of the 'choices' made by all other vertices. In this paper we introduce a new random mapping model, $T_n^{\hat{D}}$, in which the vertex 'choices' are not necessarily independent. Before describing the new model, we mention that the definition of $T_n^{\hat{D}}$ is motivated, in part, by developments in the general theory of random graphs. In recent years models for random graphs with a specified degree sequence have received much attention as models for complex networks such as the internet. Loosely speaking, such a random graph on n labelled vertices can be constructed by starting with a collection of i.i.d., non-negative, integer-valued random variables $D_1, D_2, ..., D_n$ and adding edges, at random, to the graph until each vertex i has degree D_i in the constructed random graph. The configuration model has been an important tool for investigating this independent degree model. In another direction random graph models with 'preferential attachment' have been constructed in order to model the *evolving* structure of complex networks. In such models edges are added sequentially to the graph and new edges are more likely to be attached to vertices that already have relatively high degree in the evolving graph. The literature on these new developments in the theory of random graphs is extensive, but a good bibliography is provided by Bonato's survey paper [13].

The new model $T_n^{\hat{D}}$, which is carefully defined in Section 2 below, is the natural analogue of the independent degree model described above. It is constructed by first specifying the in-degrees $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ of the vertices labelled $1, 2, ..., n$, and then selecting a random mapping uniformly from all mappings with the given in-degree sequence $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. After defining the model in Section 2, we show in Section 3 that the distributions of many important random mapping statistics for $T_n^{\hat{D}}$ can be computed as expectations of functions of the (random) in-degree sequence $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. In Section 4 we define both a random mapping model with preferential attachment and with anti-preferential attachment. We show that both of these models are equivalent, under certain distribution assumptions, to the special examples of the general model defined in Section 2. This result is somewhat surprising as there is no such equivalence between preferential/anti-preferential attachment models and the independent degree models of general random graph theory. For these special examples we investigate the distribution of the number of cyclic points, the distribution of the number of components, the probability of connectedness, and the distribution of the size of a typical component using the calculus developed in Section 3.

We adopt the following notation in this paper. If $\mathcal{C}$ is a finite set, then $|\mathcal{C}|$ equals the number of elements in $\mathcal{C}$. If $f(s) = \sum_{k=0}^{\infty} a_k s^k$, then $[s^n]f(s) = a_n$, the coefficient of s^n in the power series expansion of $f(s)$.

2 The model

In order to define our new random mapping model, we adopt the following notation. For $n \geq 1$, $\mathcal{M}_n$ denotes the set of all mappings $f : [n] \to [n]$, where $[n] \equiv \{1, 2, ..., n\}$, and $G(f)$ denotes, as described in the Introduction, the directed graph on n labelled vertices which represents the mapping $f \in \mathcal{M}_n$. In addition, for $1 \leq i \leq n$, $d_i(f)$ denotes the in-degree of vertex i in the digraph $G(f)$, and we let $\vec{d}(f) \equiv (d_1(f), ..., d_n(f))$. Finally, for any vector $\vec{d} \equiv (d_1, d_2, ..., d_n)$ of non-negative integers such that $\sum_{i=1}^{n} d_i = n$, let

$$\mathcal{M}_n(\vec{d}) \equiv \left\{ f \in \mathcal{M}_n : \vec{d}(f) = \vec{d} \right\}.$$

To define $T_n^{\hat{D}}$, we start with a collection of non-negative integer-valued exchangeable random variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ such that $\sum_{i=1}^{n} \hat{D}_i = n$. Given the event $\{\hat{D}_i = d_i, i = 1, 2, ..., n\}$ (with $\Pr\{\hat{D}_i = d_i, i = 1, 2, ..., n\} > 0$), we define the conditional distribution of $T_n^{\hat{D}}$ by

$$\Pr\{T_n^{\hat{D}} = f \mid \hat{D}_i = d_i,\, 1 \le i \le n\} = \begin{cases} \frac{\prod_{i=1}^n d_i!}{n!} & \text{if} \quad d_i(f) = d_i,\, 1 \le i \le n \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In other words, *given* $(\hat{D}_1, \hat{D}_2, ..., \hat{D}_n) = (d_1, d_2, ..., d_n) = \vec{d}$, $T_n^{\hat{D}}$ is uniformly distributed over $\mathcal{M}_n(\vec{d})$. It follows from (2.1) that for any $f \in \mathcal{M}_n$,

$$\Pr\{T_n^{\hat{D}} = f\} = \frac{\prod_{i=1}^n (d_i(f))!}{n!} \Pr\left\{\hat{D}_i = d_i(f),\, 1 \le i \le n\right\}. \tag{2.2}$$

Given the random mapping $T_n^{\hat{D}}$, let $G_n^{\hat{D}} \equiv G(T_n^{\hat{D}})$ denote the random digraph on n labelled vertices which represents $T_n^{\hat{D}}$. We note that it follows from the exchangeability of the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ and (2.1) that, for any permutation $\sigma : [n] \to [n]$, we have

$$\sigma \circ T_n^{\hat{D}} \overset{d}{\sim} T_n^{\hat{D}} \circ \sigma \overset{d}{\sim} T_n^{\hat{D}}.$$

In other words, the distribution of the corresponding digraph $G_n^{\hat{D}}$ is invariant under re-labelling of the vertices of the graph.

An important class of examples can constructed as follows. Suppose that $D_1, D_2, \ldots, D_n$ are i.i.d. non-negative integer-valued random variables with $\Pr\{\sum_{i=1}^n D_i = n\} > 0$, and let $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ be a sequence of random variables with joint distribution is given by

$$\Pr\left\{\hat{D}_i = d_i, 1 \le i \le n\right\} = \Pr\left\{D_i = d_i, 1 \le i \le n \,\middle|\, \sum_{i=1}^n D_i = n\right\}.$$

Clearly, the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ are exchangeable with $\sum_{i=1}^n \hat{D}_i = n$, so we can use $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ to construct $T_n^{\hat{D}}$ and $G_n^{\hat{D}}$. We note that it is easy to check that if $D_1, D_2, ..., D_n$ are i.i.d. Poisson variables, then $(\hat{D}_1, \hat{D}_2, ..., \hat{D}_n)$ has a multinomial distribution with parameters n and $(1/n, 1/n, ..., 1/n)$ and the corresponding random mapping $T_n^{\hat{D}}$ is just the usual uniform random mapping. In Section 4 below we show that there are interesting interpretations $T_n^{\hat{D}}$ in the cases where the underlying i.i.d. variables $D_1, D_2, ..., D_n$ have **(i)** a generalised negative binomial distribution, and **(ii)** a binomial distribution $Bin(m, p)$.

3 Results

In this section we develop a calculus, in terms of the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$, for determining the distributions of various random variables associated with the structure of $G_n^{\hat{D}}$. The first variable we consider is the number of cyclic vertices in the random digraph $G_n^{\hat{D}}$.

A vertex $i \in [n]$ is a cyclic vertex for the mapping $f \in \mathcal{M}_n$ (and for the corresponding digraph $G(f)$) if there is some $k \geq 1$ such that $f^{(k)}(i) = i$, where $f^{(k)}$ is the k^{th} iterate of the function f. We define $X_n(f)$ to be the number of cyclic vertices of $f \in \mathcal{M}_n$ and we let $X_n^{\hat{D}} \equiv X_n(T_n^{\hat{D}})$ denote the number of cyclic vertices in $G_n^{\hat{D}}$. Then we have

Theorem 1. *For* $1 \leq k \leq n-1$

$$\Pr\left\{X_n^{\hat{D}} = k\right\} = \frac{k}{n-k} E\big((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k\big)$$

and

$$\Pr\left\{X_n^{\hat{D}} = n\right\} = \Pr\left\{\hat{D}_i = 1, 1 \leq i \leq n\right\} = E\big(\hat{D}_1\hat{D}_2 \cdots \hat{D}_n\big).$$

Proof. We begin by considering the case $1 \leq k \leq n-1$. For $f \in \mathcal{M}_n$, let $\mathcal{L}_n(f)$ denote the set of cyclic vertices for the mapping f, and let $\mathcal{L}_n^{\hat{D}} \equiv \mathcal{L}_n(T_n^{\hat{D}})$. Then we have

$$\Pr\{X_n^{\hat{D}} = k\} = \sum_{L \subseteq [n] s.t. |L| = k} \Pr\{\mathcal{L}_n^{\hat{D}} = L\} = \binom{n}{k} \Pr\{\mathcal{L}_n^{\hat{D}} = [k]\}.$$

We note that the second equality holds since the distribution of $G_n^{\hat{D}}$ is invariant under re-labelling its vertices. Next, observe that

$$\begin{aligned}
&\Pr\left\{\mathcal{L}_n^{\hat{D}} = [k]\right\} \\
&\quad = \sum_{\substack{(d_i) \\ s.t. \sum d_i = n}} \Pr\left\{\mathcal{L}_n^{\hat{D}} = [k] \,\Big|\, \hat{D}_i = d_i, 1 \leq i \leq n\right\} \Pr\left\{\hat{D}_i = d_i, 1 \leq i \leq n\right\}.
\end{aligned}$$

Now fix $\vec{d} = (d_1, d_2, ..., d_n)$ such that $\sum_{i=1}^n d_i = n$, then it follows from (2.1) that

$$\Pr\left\{\mathcal{L}_n^{\hat{D}} = [k] \,\Big|\, \hat{D}_i = d_i, 1 \leq i \leq n\right\} = \frac{\big|\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\}\big|}{n!\big(\prod_{i=1}^n d_i!\big)^{-1}}.$$

To determine $\big|\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\}\big|$, we make use of a bijection between the set of sequences

$$\mathcal{S}_{\vec{d}} \equiv \big\{(x_1, x_2, ..., x_n) : \ \big|\{m : x_m = i\}\big| = d_i \ , 1 \le i \le n\big\}$$

and the set of mappings $\mathcal{M}_n(\vec{d})$.

The bijection is defined in terms of a version of the oldest and most well-known tree code, which is due to Prüfer (see [36]). For any mapping $f \in \mathcal{M}(\vec{d}\,)$ it gives, after a series of 'rounds', a corresponding sequence $\mathbf{x}(f) = (x_1, x_2, ..., x_n) \in \mathcal{S}_{\vec{d}}$. Informally, the algorithm works by deleting, one at a time, vertices with in-degree 0 from $G(f)$ and adding the image under f of the deleted vertex to the sequence $\mathbf{x}(f)$. Once all of the vertices with in-degree 0 have been deleted the remaining digraph will consist of directed cycles only. In this case, the algorithm lists the remaining vertices in increasing order and adds the corresponding sequence of their images under f to the end of the code $\mathbf{x}(f)$, and stops.

To describe the algorithm more formally, we introduce the following notation. For any labelled digraph G, define $V_0(G)$ to be the set of vertices of G with in-degree equal to 0. If $V_0(G) \neq \emptyset$, let $v^*(G)$ denote the element of $V_0(G)$ with smallest label.

The algorithm: Given a mapping $f \in \mathcal{M}_n(\vec{d}\,)$, the corresponding code $\mathbf{x}(f) = (x_1, x_2, ..., x_n) \in \mathcal{S}_{\vec{d}}$ is obtained as follows:
Let $G := G(f)$ and $\mathbf{x}(f) = \emptyset$.
Step 1. If $V_0(G) = \emptyset$, then list the vertices of G in the increasing order and add the corresponding sequence of their images under f to $\mathbf{x}(f)$ and STOP. Otherwise go to Step 2.
Step 2. Add $w = f(v^*(G))$ to $\mathbf{x}(f)$.
Set $G := G - \{v^*(G)\}$ (i.e. delete $v^*(G)$ from G) and go to Step 1.

It is clear from the algorithm that if $f \in \mathcal{M}_n(\vec{d})$ then $\mathbf{x}(f) = (x_1, x_2, ..., x_n) \in \mathcal{S}_{\vec{d}}$ since, for $1 \le i \le n$, $|\{m : x_m = i\}| = d_i(f) = d_i$. We also note that if $f, f' \in \mathcal{M}_n(\vec{d})$ and $f \neq f'$, then $\mathbf{x}(f) \neq \mathbf{x}(f')$. To see this, we first observe that if the mapping f restricted to $\mathcal{L}_n(f)$ does not equal the mapping f' restricted to $\mathcal{L}_n(f')$, then it is clear from the algorithm that $\mathbf{x}(f) \neq \mathbf{x}(f')$. On the other hand, suppose that $f|_{\mathcal{L}_n(f)} = f'|_{\mathcal{L}_n(f')}$ and let G and G' denote the state of the algorithm at the start of each round when it is applied to the mappings f and f' respectively. Then since $\vec{d}(f) = \vec{d}(f')$ we must have at the start of the algorithm $V_0(G) = V_0(G') \neq \emptyset$ and $v^*(G) = v^*(G')$.

Therefore, in some round of the algorithm we must have $v^*(G) = v^*(G')$ but $f(v^*(G)) \neq f'(v^*(G'))$ since $f \neq f'$, and hence $\mathbf{x}(f) \neq \mathbf{x}(f')$. It follows that the algorithm gives a bijection between $\mathcal{S}_{\vec{d}}$ and $\mathcal{M}_n(\vec{d})$ since $\left|\mathcal{S}_{\vec{d}}\right| = \frac{n!}{d_1!\cdots d_n!} = \left|\mathcal{M}_n(\vec{d})\right|$. Clearly the inverse of this coding can be used to generate mappings with a given degree sequence, and we note that a similar approach has been used recently by Blitzstein and Diaconis [12] to construct labelled trees with a given degree sequence. Finally, we also note, more generally, that for *any* $f, f' \in \mathcal{M}_n$ such that $f \neq f'$, the corresponding codes $\mathbf{x}(f)$ and $\mathbf{x}(f')$ generated by the algorithm are distinct.

The next lemma identifies a key correspondence between $\mathcal{L}_n(f)$ and a subsequence of $\mathbf{x}(f)$.

Lemma 1. *For each* $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathcal{S}_{\vec{d}}$, *we define* $t(\mathbf{x})$ *as follows:*

$$t(\mathbf{x}) = \min\left\{t : \left|\{x_t, x_{t+1}, ..., x_n\}\right| = n - t + 1\right\}.$$

Then for any $f \in \mathcal{M}_n(\vec{d})$

$$\mathcal{L}_n(f) = \left\{x_{t(\mathbf{x}(f))}, x_{t(\mathbf{x}(f))+1}, ..., x_n\right\}.$$

Proof. Let $f \in \mathcal{M}_n(\vec{d})$ and suppose that $\mathbf{x}(f) = (x_1, x_2, \ldots, x_n)$. If $|\mathcal{L}_n(f)| = n$, then f is a permutation and $\{x_1, x_2, ..., x_n\} = [n]$. So the result holds in this case. Next, if $|\mathcal{L}_n(f)| = m < n$, then it follows from the algorithm that $\{x_{n-m+1}, x_{n-m+2}, ..., x_n\} = \mathcal{L}_n(f)$. So, at the start of the $(n-m)^{st}$ round of the algorithm, the digraph G consists of a permutation of the vertices labelled $x_{n-m+1}, ..., x_n$ with one *non-cyclic* vertex, labelled $v^*(G)$, attached to a cycle. It follows that $x_{n-m} = f(v^*(G)) \in \mathcal{L}_n(f) = \{x_{n-m+1}, ..., x_n\}$. Thus $t(\mathbf{x}(f)) = n - m + 1$, and the result holds in this case. $\square$

It follows from Lemma 1 and the bijection between $\mathcal{M}_n(\vec{d})$ and $\mathcal{S}_{\vec{d}}$ that

$$\left|\left\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\right\}\right|$$

$$= \left|\left\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, ..., x_n\} = [k] \text{ and } x_{n-k} \in [k]\right\}\right|.$$

So routine counting arguments yield

$$\begin{aligned}
&\Pr\left\{\mathcal{L}_n^{\hat{D}} = [k] \,\middle|\, \hat{D}_i = d_i, 1 \le i \le n\right\} \\
&= \frac{\left|\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, ..., x_n\} = [k] \text{ and } x_{n-k} \in [k]\}\right| \times \prod_{i=1}^n d_i!}{n!} \\
&= \binom{n}{k}^{-1} \left(\frac{1}{n-k}\right) \sum_{i=1}^{k} d_1 d_2 \cdots d_{i-1}(d_i - 1) d_i d_{i+1} \cdots d_k
\end{aligned}$$

for $1 \le k \le n-1$. Hence

$$\begin{aligned}
\Pr\left\{\mathcal{L}_n^{\hat{D}} = [k]\right\} &= \binom{n}{k}^{-1} \left(\frac{1}{n-k}\right) \sum_{i=1}^{k} E\big(\hat{D}_1 \cdots (\hat{D}_i - 1)\hat{D}_i \cdots \hat{D}_k\big) \\
&= \binom{n}{k}^{-1} \frac{k}{n-k} E\big((\hat{D}_1 - 1)\hat{D}_1 \hat{D}_2 \cdots \hat{D}_k\big)
\end{aligned}$$

since $\hat{D}_1, \hat{D}_2, ..., \hat{D}_k$ are exchangeable, and the result follows for $1 \le k \le n-1$.

Finally, we consider the case $k = n$. Clearly $X_n^{\hat{D}} = n$ if and only if f is a permutation of $[n]$, and f is a permutation of $[n]$ if and only if $\hat{D}_i = 1$ for all $1 \le i \le n$. Hence

$$\Pr\left\{X_n^{\hat{D}} = n\right\} = \Pr\left\{\hat{D}_i = 1, 1 \le i \le n\right\} = E(\hat{D}_1 \hat{D}_2 \cdots \hat{D}_n)$$

as required, and the theorem is proved. □

Let $N_n^{\hat{D}}$ denote the number of components in $G_n^{\hat{D}}$. Then we have

Corollary 1. *For $1 \le \ell \le n$,*

$$\begin{aligned}
\Pr\{N_n^{\hat{D}} = \ell\} &= \sum_{k=\ell}^{n-1} \frac{|s(k,l)|}{(k-1)!(n-k)} E((\hat{D}_1 - 1)\hat{D}_1 \hat{D}_2 \cdots \hat{D}_k) \\
&\quad + \frac{|s(n,l)|}{n!} \Pr\{\hat{D}_i = 1, 1 \le i \le n\},
\end{aligned}$$

where $s(\cdot\,,\cdot)$ are the Stirling numbers of the first kind.

Proof. Recall that the mapping $T_n^{\hat{D}}$ restricted to $\mathcal{L}_n^{\hat{D}}$, the cyclic vertices of $T_n^{\hat{D}}$, is a permutation of $\mathcal{L}_n^{\hat{D}}$. The corollary follows from the observation that the number of components in $G_n^{\hat{D}}$ equals the number of cycles in the permutation of $\mathcal{L}_n^{\hat{D}}$ by $T_n^{\hat{D}}$. So $N_n^{\hat{D}} = \ell$ if and only if the mapping $T_n^{\hat{D}}$ restricted to $\mathcal{L}_n^{\hat{D}}$ is a permutation with ℓ cycles.

Next, it follows from (2.1) that given the (non-empty) event $\{\hat{D}_i = d_i, i = 1, 2, ..., n\}$, the mapping $T_n^{\hat{D}}$ is uniformly distributed on $\mathcal{M}_n(d_1, d_2, .., d_n)$ and hence $T_n^{\hat{D}}$ restricted to $\mathcal{L}_n^{\hat{D}}$ is a uniform random permutation of $\mathcal{L}_n^{\hat{D}}$. It follows that for $1 \le \ell \le k \le n$,

$$\Pr\left\{N_n^{\hat{D}} = \ell \,\middle|\, X_n^{\hat{D}} = k\right\} = \Pr\left\{N_{\sigma(k)} = \ell\right\} \tag{3.1}$$

where $\sigma(k)$ is a uniform random permutation on a k-element set and $N_{\sigma(k)}$ denotes the number of cycles in the random permutation $\sigma(k)$. It follows that

$$\begin{aligned} \Pr\left\{N_n^{\hat{D}} = \ell\right\} &= \sum_{k=\ell}^{n} \Pr\left\{N_n^{\hat{D}} = \ell \,\middle|\, X_n^{\hat{D}} = k\right\} \Pr\left\{X_n^{\hat{D}} = k\right\} \\ &= \sum_{k=1}^{n} \Pr\left\{N_{\sigma(k)} = \ell\right\} \Pr\left\{X_n^{\hat{D}} = k\right\}. \end{aligned} \tag{3.2}$$

Let $s(\cdot\,,\cdot)$ denote the Stirling numbers of the first kind, then it is well known that there are $|s(k,l)|$ permutations of k-element set with exactly l cycles, i.e.,

$$\Pr\{N_{\sigma(k)} = \ell\} = \frac{|s(k,l)|}{k!},$$

which implies the assertion of the corollary. □

Let $\mathcal{B}_n^{\hat{D}}$ denote the event that the random graph $G_n^{\hat{D}}$ is connected. Then since $\mathcal{B}_n^{\hat{D}} = \{N_n^{\hat{D}} = 1\}$, we obtain the following result immediately from (3.2) and Theorem 1:

Corollary 2.

$$\begin{aligned} \Pr\left\{\mathcal{B}_n^{\hat{D}}\right\} &= \sum_{k=1}^{n} \frac{1}{k} \Pr\left\{X_n^{\hat{D}} = k\right\} \\ &= \sum_{k=1}^{n-1} \frac{E\big((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k\big)}{n-k} + \frac{E(\hat{D}_1\hat{D}_2 \cdots \hat{D}_n)}{n}. \end{aligned}$$

Finally, suppose that $\xi_1, \xi_2, ...$ is a sequence of independent indicator variables such that, for $k \geq 1$, $\Pr\{\xi_k = 1\} = \frac{1}{k}$ and such that $\xi_1, \xi_2, ...$ and $X_n^{\hat{D}}$ are independent. It is well known (see [15]) that for $m \geq 1$

$$N_{\sigma(m)} \overset{d}{\sim} \sum_{k=1}^{m} \xi_k, \tag{3.3}$$

and that $(N_{\sigma(m)} - \log m)/\sqrt{\log m}$ converges in distribution to the standard $N(0,1)$ distribution. It is an easy consequence of (3.1)–(3.3), that

Corollary 3. *For $n \geq 1$,*

$$N_n^{\hat{D}} \overset{d}{\sim} \sum_{k=1}^{X_n^{\hat{D}}} \xi_k. \tag{3.4}$$

We note that (3.4) is useful for investigating the asymptotic distribution of $N_n^{\hat{D}}$.

Next, we consider the distribution of the size of a 'typical' component of $G_n^{\hat{D}}$. Let $C_1^{\hat{D}}(n)$ denote the size of the component in $G_n^{\hat{D}}$ which contains the vertex 1, then the distribution of $C_1^{\hat{D}}(n)$ is given by the following theorem.

Theorem 2. *Suppose that $1 \leq \ell \leq n$ and suppose that $\Pr\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\} > 0$. Let $D'_1, D'_2, ..., D'_\ell$ be a sequence of variables with joint distribution given by*

$$\Pr\left\{D'_i = d_i,\ 1 \leq i \leq \ell\right\} = \Pr\left\{\hat{D}_i = d_i,\ 1 \leq i \leq \ell \,\middle|\, \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\},$$

then

$$\Pr\left\{C_1^{\hat{D}}(n) = \ell\right\} = \frac{\ell}{n} \Pr\left\{\mathcal{B}_\ell^{D'}\right\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}.$$

Otherwise, if $\Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} = 0$, *then* $\Pr\left\{C_1^{\hat{D}}(n) = \ell\right\} = 0$.

Proof. Fix $1 \leq \ell \leq n$ and let $\mathcal{C}_1^{\hat{D}}(n)$ denote the vertex set of the component of $G_n^{\hat{D}}$ which contains the vertex 1. Then we have

$$\begin{aligned}
\Pr\left\{C_1^{\hat{D}}(n) = \ell\right\} &= \sum_{\substack{C \subseteq [n] \ s.t.\ 1 \in C \\ and\ |C| = \ell}} \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = C\right\} \\
&= \binom{n-1}{\ell-1} \Pr\left\{\mathcal{C}_1^{\hat{\mathcal{D}}}(n) = [\ell]\right\}. \qquad (3.5)
\end{aligned}$$

The second equality holds since the distribution of $G_n^{\hat{D}}$ (and of $T_n^{\hat{D}}$) is invariant under re-labelling of the vertices. Next, observe that if $\mathcal{C}_1^{\hat{D}}(n) = [\ell]$ then $T_n^{\hat{D}}$ must map $[\ell]$ into $[\ell]$ and $[n] \setminus [\ell]$ into $[n] \setminus [\ell]$. It follows that we must have $\sum_{i=1}^{\ell} \hat{D}_i = \ell$. So if $\Pr\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\} = 0$, then $\Pr\left\{C_1^{\hat{D}}(n) = \ell\right\} = 0$. On the other hand, if $\Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} > 0$, then

$$\Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell]\right\} = \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \,\middle|\, \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}. \quad (3.6)$$

Now fix $\vec{d} = (d_1, ..., d_n)$ such that $\sum_{i=1}^{\ell} d_i = \ell$ and $\sum_{i=1}^{n} d_i = n$, and define $C(d_1, d_2, ..., d_\ell)$ to be the number of mappings $g : [\ell] \to [\ell]$ such that $d_i(g) = d_i$ for $1 \le i \le \ell$ and $G(g)$ is connected. Therefore, since $(n-\ell)!/(d_{\ell+1}! \cdots d_n!)$ is the number of mappings from $[n] \setminus [\ell]$ into $[n] \setminus [\ell]$ with in-degree sequence $(d_{\ell+1}, ..., d_n)$, we have

$$\begin{aligned}
\Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \quad \middle| \quad \hat{D}_i = d_i, 1 \le i \le n\right\} &= \frac{C(d_1, ..., d_\ell)(n-\ell)!}{d_{\ell+1}! \cdots d_n!} \times \frac{d_1! \cdots d_n!}{n!} \\
&= \binom{n}{\ell}^{-1} C(d_1, ..., d_\ell) \times \frac{d_1! \cdots d_\ell!}{\ell!} \\
&= \binom{n}{\ell}^{-1} \Pr\left\{\mathcal{B}_\ell^{\mathcal{D}'} \,\middle|\, D_i' = d_i, 1 \le i \le \ell\right\}. \quad (3.7)
\end{aligned}$$

Finally, for any degree sequence $\vec{d} = (d_1, ..., d_n)$, we define $\vec{d}_\ell = (d_1, ..., d_\ell)$ and $\vec{d}_\ell' = (d_{\ell+1}, ..., d_n)$ (so $\vec{d} = (\vec{d}_\ell, \vec{d}_\ell'))$. Then it follows from (3.7) that

$$\begin{aligned}
&\Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \,\middle|\, \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \\
&= \sum_{\substack{\vec{d} \ s.t. \sum_{i=1}^{\ell} d_i = \ell, \\ \sum_{i=1}^{n} d_i = n}} \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \,\middle|\, \hat{D}_i = d_i, 1 \le i \le n\right\} \\
&\qquad \times \Pr\left\{\hat{D}_i = d_i, 1 \le i \le n \,\middle|\, \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n}{\ell}^{-1} \sum_{\substack{\vec{d}_\ell \ s.t. \\ \sum_{i=1}^{\ell} d_i = \ell}} \sum_{\substack{\vec{d}_\ell{}' \ s.t. \\ \sum_{i=\ell+1}^{n} d_i = n-\ell}} \Pr\left\{ \mathcal{B}_\ell^{\mathcal{D}'} \,\middle|\, D_i' = d_i, 1 \le i \le \ell \right\} \\
&\qquad\qquad \times \Pr\left\{ \hat{D}_i = d_i, 1 \le i \le n \,\middle|\, \sum_{i=1}^{\ell} \hat{D}_i = \ell \right\} \\
&= \binom{n}{\ell}^{-1} \sum_{\substack{\vec{d}_\ell \ s.t. \\ \sum_{i=1}^{\ell} d_i = \ell}} \Pr\left\{ \mathcal{B}_\ell^{\mathcal{D}'} \,\middle|\, D_i' = d_i, 1 \le i \le \ell \right\} \Pr\left\{ D_i' = d_i, 1 \le i \le \ell \right\} \\
&= \binom{n}{\ell}^{-1} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\}. \qquad (3.8)
\end{aligned}
$$

The result now follows from (3.5), (3.6), and (3.8). □

We prove an extension of Theorem 2 in the following special case: Suppose that $D_1, D_2, \ldots$ is a sequence of i.i.d. non-negative integer valued random variables such that for every $n \ge 1$, we have $\Pr\{\sum_{i=1}^{n} D_i = n\} > 0$. For each $n \ge 1$, let $\hat{D}(n) \equiv (\hat{D}_{1,n}, \ldots, \hat{D}_{n,n})$ be a sequence of variables with joint distribution given by

$$
\Pr\left\{ \hat{D}_{i,n} = d_i \text{ for } i = 1, 2, .., n \right\} = \Pr\left\{ D_i = d_i, \quad i = 1, 2, ..., n \,\middle|\, \sum_{i=1}^{n} D_i = n \right\}.
$$

Let $\mathcal{C}_1^{\hat{D}(n)}$ denote the vertex set of the connected component in $G_n^{\hat{D}(n)} \equiv G(T_n^{\hat{D}(n)})$ which contains the vertex labelled 1. For $k > 1$, we define $\mathcal{C}_k^{\hat{D}(n)}$ recursively as follows: If $[n] \setminus (\mathcal{C}_1^{\hat{D}(n)} \cup \cdots \cup \mathcal{C}_{k-1}^{\hat{D}(n)}) \ne \emptyset$, let $\mathcal{C}_k^{\hat{D}(n)}$ denote the vertex set of the connected component in $G_n^{\hat{D}(n)}$ which contains the smallest element of $[n] \setminus (\mathcal{C}_1^{\hat{D}(n)} \cup \cdots \cup \mathcal{C}_{k-1}^{\hat{D}(n)})$; otherwise, set $\mathcal{C}_k^{\hat{D}(n)} = \emptyset$. For all $k \ge 1$, let $C_k^{\hat{D}(n)} = |\mathcal{C}_k^{\hat{D}(n)}|$, then we have

Theorem 3. *Suppose* $1 \le k \le n$ *and* $\ell_1, \ell_2, \ldots, \ell_k$ *are such that* $\ell_i \ge 1$ *for* $i = 1, 2, ..., k$, *and* $\sum_{i=1}^{k} \ell_i \le n$. *Then we have*

$$
\Pr\left\{ C_1^{\hat{D}(n)} = \ell_1, ..., C_k^{\hat{D}(n)} = \ell_k \right\} = \prod_{i=1}^{k} \Pr\{C_1^{\hat{D}(n-t_{i-1})} = \ell_i\}, \qquad (3.9)
$$

where $t_0 = 0$ *and* $t_i \equiv \ell_1 + ... + \ell_i$, $i = 1, 2, ..., k$.

Proof. The proof is by induction on k. The case $k = 1$ is obvious. We show how the induction step works by proving the result for $k = 2$. The general induction argument is exactly the same, but is notationally messier. We note that in the case $k = 2$, it suffices to show that

$$\Pr\{C_2^{\hat{D}(n)} = \ell_2 \,\big|\, C_1^{\hat{D}(n)} = \ell_1\} = \Pr\{C_1^{\hat{D}(n-\ell_1)} = \ell_2\}. \tag{3.10}$$

We begin by defining, for each $n \geq 1$, the set of connected mappings $\tilde{\mathcal{M}}_n = \{f \in \mathcal{M}_n \;:\; G(f) \text{ is connected}\}$. Now given $g \in \tilde{\mathcal{M}}_{\ell_1}$, we can construct a mapping $f \in \mathcal{M}_n$ such that $\mathcal{C}_1(f) = [\ell_1]$, (where $\mathcal{C}_1(f)$ is the vertex set of the component in $G(f)$ which contains the vertex labelled 1) as follows: Choose $h \in \mathcal{M}_{n-\ell_1}$. Then for $1 \leq i \leq \ell_1$, define $f(i) = g(i)$ and for $\ell_1 + 1 \leq i \leq n$, define $f(i) = h(i - \ell_1)$. In this case, we write $f = (g, h)$. Now fix $g \in \tilde{M}_{\ell_1}$, then it is straightforward to check that

$$\begin{aligned}
&\Pr\left\{T_n^{\hat{D}(n)}\big|_{[\ell_1]} = g,\; \sum_{i=1}^{\ell_1} \hat{D}_{i,n} = \ell_1\right\} \\
&= \frac{\prod_{i=1}^{\ell_1} d_i(g)!(n-\ell_1)!}{n!} \Pr\left\{\hat{D}_{i,n} = d_i(g), 1 \leq i \leq \ell_1,\; \sum_{j=\ell_1+1}^{n-\ell_1} \hat{D}_{j,n} = n - \ell_1\right\}.
\end{aligned}$$

It follows that for *any* $h \in M_{n-\ell_1}$, we have

$$\begin{aligned}
&\Pr\left\{T_n^{\hat{D}(n)} = (g,h) \,\Big|\, T_n^{\hat{D}(n)}\big|_{[\ell_1]} = g,\; \sum_{i=1}^{\ell_1} \hat{D}_{i,n} = \ell_1\right\} \\
= \;& \frac{\Pr\left\{T_n^{\hat{D}(n)} = (g,h)\right\}}{\Pr\left\{T_n^{\hat{D}(n)}\big|_{[\ell_1]} = g,\; \sum_{i=1}^{\ell_1} \hat{D}_{i,n} = \ell_1\right\}} \\
= \;& \frac{\prod_{j=1}^{n-\ell_1} d_j(h)!}{(n-\ell_1)!} \cdot \frac{\Pr\left\{\hat{D}_{i,n} = d_i(g), 1 \leq i \leq \ell_1, \hat{D}_{j+\ell_1,n} = d_j(h), 1 \leq j \leq n-\ell_1\right\}}{\Pr\left\{\hat{D}_{i,n} = d_i(g), 1 \leq i \leq \ell_1,\; \sum_{j=\ell_1+1}^{n-\ell_1} \hat{D}_{j,n} = n-\ell_1\right\}} \\
= \;& \frac{\prod_{j=1}^{n-\ell_1} d_j(h)!}{(n-\ell_1)!} \Pr\left\{\hat{D}_{j,n-\ell_1} = d_j(h), 1 \leq j \leq n-\ell_1\right\} \\
= \;& \Pr\left\{T_{n-\ell_1}^{\hat{D}(n-\ell_1)} = h\right\}.
\end{aligned} \tag{3.11}$$

We note that since the $\{D_1, D_2, ...\}$ are i.i.d., the third equality above follows from the definition of $\hat{D}(n)$ and $\hat{D}(n-\ell_1)$ in terms of the variables

$\{D_1, D_2, ...\}$. It follows from (3.11) and the invariance of the distribution of $G_n^{\hat{D}(n)}$ under re-labelling of the vertices, that *given* $\mathcal{C}_1^{\hat{D}(n)} = C \subseteq [n]$ with $|C| = \ell_1$, then the distribution of $G_n^{\hat{D}(n)}$ *restricted* to the vertices with labels $[n] \setminus C$ is the same (after the obvious re-labelling) as the distribution of $G_{n-\ell_1}^{\hat{D}(n-\ell_1)}$ and hence (3.10) holds. □

We note that in the general case where the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ are exchangeable and $\sum_{i=1}^n \hat{D}_i = n$, a product formula as given in Theorem 3 does *not* hold. Nevertheless, in this case one can obtain, if required, a more complicated analogue of Theorem 3 by generalising the proof of Theorem 2. Theorems 1,2, and 3 and their corollaries illustrate how the distributions of random mapping statistics for $T_n^{\hat{D}}$ can be computed in terms of the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. Similar results for local properties of $T_n^{\hat{D}}$ such as the number of predecessors and the number of successors of a given vertex (or vertices) are obtained in a companion paper [22].

4 Examples

We consider two special examples which correspond, respectively, to a random mapping with 'preferential attachment' and a random mapping with 'anti-preferential attachment'.

A Preferential Attachment Model

In this section we investigate $T_n^\rho : [n] \to [n]$, a random mapping with 'preferential attachment', where ρ is a positive parameter. For $1 \leq k \leq n$, we define $T_n^\rho(k) = \xi_k^{(\rho,n)}$ where $\xi_1^{(\rho,n)}, \xi_2^{(\rho,n)}, ..., \xi_n^{(\rho,n)}$ is a sequence of random variables whose distributions depend on the evolution of an urn scheme. The distribution of each $\xi_k^{(\rho,n)}$ is determined by a (random) n-tuple of non-negative weights $\vec{a}(k) = (a_1(k), a_2(k), ..., a_n(k))$ where, for $1 \leq j \leq n$, $a_j(k)$ is the 'weight' of the j^{th} urn at the *start* of the k^{th} round of the urn scheme. Specifically, given $\vec{a}(k) = \vec{a} = (a_1, ..., a_n)$, we define

$$\Pr\left\{\xi_k^{(\rho,n)} = j \,\middle|\, \vec{a}(k) = \vec{a}\right\} = \frac{a_j}{\sum_{i=1}^n a_i}.$$

The random weight vectors $\vec{a}(1), \vec{a}(2), ..., \vec{a}(n)$ associated with the urn scheme are determined recursively. For $k = 1$, we set $a_1(1) = a_2(1) = \cdots = a_n(1) =$

ρ. For $k > 1$, $\vec{a}(k)$ depends on both $\vec{a}(k-1)$ and the value of $\xi_{k-1}^{(\rho,n)}$ as follows: Given that $\xi_{k-1}^{(\rho,n)} = j$, we set $a_j(k) = a_j(k-1) + 1$ and for all other $i \neq j$, we set $a_i(k) = a_i(k-1)$ (i.e. if $\xi_{k-1}^{(\rho,n)} = j$ then a 'ball' with weight 1 is added to the j^{th} urn).

The random mapping T_n^ρ as defined above is a preferential attachment model in the following sense. Since, for $1 \leq k \leq n$, $T_n^\rho(k) = \xi_k^{(\rho,n)}$, and since the (conditional) distribution of $\xi_k^{(\rho,n)}$ depends on the state of the urn scheme at the start of round k, it is clear that vertex k is more likely to be mapped to vertex j if the weight $a_j(k)$ is (relatively) large, i.e. if several of the vertices $1, 2, ..., k-1$ have already been mapped to vertex j . In the following theorem we establish the distribution of T_n^ρ.

Theorem 4. *Suppose that $D_1^\rho, D_2^\rho, ...$ are i.i.d. random variables with a generalized negative binomial distribution given by*

$$\Pr\{D_1^\rho = k\} = \frac{\Gamma(k+\rho)}{k!\Gamma(\rho)} \left(\frac{\rho}{1+\rho}\right)^\rho \left(\frac{1}{1+\rho}\right)^k \quad \textit{for} \quad k = 0, 1, ...,$$

where ρ is a positive parameter. For $n \geq 1$, let $\hat{D}(\rho, n) = (\hat{D}_{1,n}^\rho, \hat{D}_{2,n}^\rho, ..., \hat{D}_{n,n}^\rho)$ be a sequence of variables with joint distribution given by

$$\Pr\left\{\hat{D}_{i,n}^\rho = d_i, 1 \leq i \leq n\right\} = \Pr\left\{D_i^\rho = d_i, 1 \leq i \leq n \,\middle|\, \sum_{i=1}^n D_i^\rho = n\right\}.$$

Then for every $n \geq 1$, the random mappings T_n^ρ and $T_n^{\hat{D}(\rho,n)}$ have the same distribution.

Proof. To prove the result it is enough to show that for any $n \geq 1$ and any $f \in \mathcal{M}_n$

$$\Pr\{T_n^\rho = f\} = \Pr\left\{T_n^{\hat{D}(\rho,n)} = f\right\}.$$

Suppose that $f \in \mathcal{M}_n$ and that $\vec{d}(f) = (d_1, d_2, ..., d_n)$. It is straightforward to check that

$$\Pr\left\{D_i^\rho = d_i, 1 \leq i \leq n \,\middle|\, \sum_{i=1}^n D_i^\rho = n\right\} = \prod_{i=1}^n \frac{\Gamma(d_i+\rho)}{(d_i)!\Gamma(\rho)} \times \frac{n!\Gamma(n\rho)}{\Gamma(n+n\rho)}$$

since the distribution of $\sum_{i=1}^n D^\rho$ is given by

$$\Pr\left\{\sum_{i=1}^n D_i^\rho = k\right\} = \frac{\Gamma(k+n\rho)}{k!\Gamma(n\rho)}\left(\frac{\rho}{1+\rho}\right)^{n\rho}\left(\frac{1}{1+\rho}\right)^k \tag{4.1}$$

for $k = 0, 1, \ldots$. It follows from the definition of $T_n^{\hat{D}(\rho,n)}$ that

$$\begin{aligned}\Pr\left\{T_n^{\hat{D}(\rho,n)} = f\right\} &= \frac{d_1!\cdots d_n!}{n!} \times \prod_{i=1}^n \frac{\Gamma(d_i+\rho)}{(d_i)!\Gamma(\rho)} \times \frac{n!\Gamma(n\rho)}{\Gamma(n+n\rho)} \\ &= \frac{\prod_{i=1}^n (d_i+\rho-1)_{d_i}}{(n\rho+n-1)_n}\end{aligned}$$

where $(x)_k \equiv x(x-1)\cdots(x-k+1) = \Gamma(x+1)/\Gamma(x-l+1)$ and $(x)_0 = 1$.

Next, observe that if $T_n^\rho = f$ and $\vec{d}(f) = (d_1, d_2, ..., d_n)$, then for each $1 \le i \le n$, d_i 'balls' of weight 1 are added to i^{th} urn during the evolution of the urn scheme described above. So it follows from the definition of T_n^ρ in terms of the urn scheme, that

$$\Pr\{T_n^\rho = f\} = \frac{\prod_{i=1}^n (d_i+\rho-1)_{d_i}}{(n\rho+n-1)_n}.$$

The result follows since $n \ge 1$ and $f \in \mathcal{M}_n$ were arbitrary. $\square$

It is clear from Theorem 4 that the order in which a realisation of T_n^ρ is sequentially constructed does not matter. In particular, suppose that $i_1, i_2, ..., i_n$ is a permutation of $[n]$ and for $1 \le k \le n$, let $\tilde{T}_n^\rho(i_k) \equiv \xi_k^{(\rho,n)}$ where the variables $\xi_1^{(\rho,n)}, \xi_2^{(\rho,n)}, ..., \xi_n^{(\rho,n)}$ are as defined above. Then it follows from the proof of Theorem 4 that $\tilde{T}_n^\rho \overset{d}{\sim} T_n^{\hat{D}(\rho,n)} \overset{d}{\sim} T_n^\rho$.

Since $T_n^\rho \overset{d}{\sim} T_n^{\hat{D}(\rho,n)}$, we can investigate the structure of $G_n^\rho \equiv G(T_n^\rho)$ by considering the structure of $G_n^{\hat{D}(\rho,n)}$. In this paper, in order to illustrate the general method, we derive both exact and asymptotic results for X_n^ρ, the number of cyclic vertices in G_n^ρ, and $C_1^\rho(n)$, the size of the component in G_n^ρ which contains vertex 1. By Theorem 1 and Theorem 4 we have, for $1 \le k < n$,

$$\Pr\{X_n^\rho = k\} = \frac{k}{n-k} E\Big((\hat{D}_{1,n}^\rho - 1)\hat{D}_{1,n}^\rho \hat{D}_{2,n}^\rho \cdots \hat{D}_{k,n}^\rho\Big). \tag{4.2}$$

Since

$$E\Big((\hat{D}^{\rho}_{1,n}-1)\hat{D}^{\rho}_{1,n}\hat{D}^{\rho}_{2,n}\cdots\hat{D}^{\rho}_{k,n}\Big)=E\Big((D^{\rho}_1-1)D^{\rho}_1D^{\rho}_2\cdots D^{\rho}_k\,\Big|\,\sum_{i=1}^{n}D^{\rho}_i=n\Big)$$

we have

$$\begin{aligned}
&E\Big((\hat{D}^{\rho}_{1,n}-1)\hat{D}^{\rho}_{1,n}\hat{D}^{\rho}_{2,n}\cdots\hat{D}^{\rho}_{k,n}\Big)\\
=\;&\frac{[s^n]E\Big((D^{\rho}_1-1)D^{\rho}_1 s^{D^{\rho}_1}D^{\rho}_2 s^{D^{\rho}_2}\cdots D^{\rho}_k s^{D^{\rho}_k}s^{D^{\rho}_{k+1}}\cdots s^{D^{\rho}_n}\Big)}{[s^n]E\Big(s^{D^{\rho}_1}\cdots s^{D^{\rho}_n}\Big)}\\
=\;&\frac{[s^n]E\Big((D^{\rho}_1-1)D^{\rho}_1 s^{D^{\rho}_1}\Big)\Big(E\big(D^{\rho}_1 s^{D^{\rho}_1}\big)\Big)^{k-1}\Big(E\big(s^{D^{\rho}_1}\big)\Big)^{n-k}}{[s^n]\Big(E\big(s^{D^{\rho}_1}\big)\Big)^{n}}. \qquad (4.3)
\end{aligned}$$

The last equality holds since the variables $D^{\rho}_1, D^{\rho}_2, ..., D^{\rho}_n$ are independent and identically distributed. Since the identity

$$\frac{1}{(1-u)^{\alpha}}=\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)}u^k$$

holds for all $\alpha>0$ and $|u|<1$, we have

$$E\Big(s^{D^{\rho}_1}\Big)=\left(\frac{\rho}{1+\rho-s}\right)^{\rho},\quad E\Big(D^{\rho}_1 s^{D^{\rho}_1}\Big)=s\left(\frac{\rho}{1+\rho-s}\right)^{\rho+1}$$

and

$$E\Big((D^{\rho}_1-1)D^{\rho}_1 s^{D^{\rho}_1}\Big)=\left(\frac{1+\rho}{\rho}\right)s^2\left(\frac{\rho}{1+\rho-s}\right)^{\rho+2}. \qquad (4.4)$$

It follows from (4.2)-(4.4) and routine calculations that for $1\le k<n$,

$$\begin{aligned}
\Pr\{X^{\rho}_n=k\} &= \frac{k}{n-k}\rho^k(1+\rho)\frac{\Gamma(n\rho)\,n!}{(n-k-1)!\,\Gamma(n\rho+k+1)}\\
&= k\rho^k(1+\rho)\frac{(n)_k}{(n\rho+k)_{k+1}} \qquad (4.5)
\end{aligned}$$

and for $k=n$, we have

$$\Pr\{X^{\rho}_n=n\}=\Pr\{\hat{D}^{\rho}_{i,n}=1, 1\le i\le n\}=\frac{\rho^n n!\,\Gamma(n\rho)}{\Gamma(n+n\rho)}. \qquad (4.6)$$

Next, we consider the limiting distribution of X_n^ρ: Fix $0 < x < \infty$ and suppose that $k = \lfloor x\sqrt{n}\rfloor$, then we have

$$\begin{aligned}\Pr\{X_n^\rho = k\} &= k\rho^k(1+\rho)\frac{(n)_k}{(n\rho+k)_{k+1}} \\ &= \frac{k}{n}\left(\frac{1+\rho}{\rho}\right)\frac{(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})}{(1+\frac{k}{n\rho})\cdots(1+\frac{1}{n\rho})} \\ &\sim \left(\frac{1+\rho}{\rho}\right)x\exp\left(-\frac{(1+\rho)x^2}{2\rho}\right)\frac{1}{\sqrt{n}}. \qquad (4.7)\end{aligned}$$

Hence $X_n^\rho/\sqrt{n}$ converges in distribution to a variable $\tilde{X}_\rho$ with density

$$f_{\tilde{X}_\rho}(x) = \left(\frac{1+\rho}{\rho}\right)x\exp\left(-\frac{(1+\rho)x^2}{2\rho}\right) \text{ for } x \geq 0. \qquad (4.8)$$

We note that by keeping track of the error term in (4.7), it is straightforward to check that $E(X_n^\rho) \sim \sqrt{\frac{\rho\pi n}{2(1+\rho)}}$. In addition, if we let N_n^ρ denote the number of components in G_n^ρ, then using Corollary 3, it is routine to adapt Stepanov's arguments for uniform random mappings (see [39]) to show that $(N_n^\rho - \frac{1}{2}\log n)/\sqrt{\frac{1}{2}\log n}$ converges in distribution to the standard $N(0,1)$ distribution.

We apply Theorem 2 and Theorem 4 to obtain the distribution for $C_1^\rho(n)$:

$$\begin{aligned}\Pr\{C_1^\rho(n) = \ell\} &= \Pr\left\{C_1^{\hat{D}(\rho,n)} = \ell\right\} \\ &= \frac{\ell}{n}\Pr\left\{\mathcal{B}_\ell^{\hat{D}(\rho,\ell)}\right\}\Pr\left\{\sum_{i=1}^{\ell} D_i^\rho = \ell \,\middle|\, \sum_{i=1}^{n} D_i^\rho = n\right\} \qquad (4.9)\end{aligned}$$

for $1 \leq \ell \leq n$. To obtain a local limit theorem for the distribution of $C_1^\rho(n)$, fix $0 < x < \infty$ and suppose that $\ell = \lfloor xn\rfloor$. Then it follows from Theorem 4, Corollary 2, (4.5), and (4.6) that

$$\Pr\left\{\mathcal{B}_\ell^{\hat{D}(\rho,\ell)}\right\} = \sum_{k=1}^{\ell}\frac{1}{k}\Pr\left\{X_\ell^{\hat{D}(\rho,\ell)} = k\right\} = \sum_{k=1}^{\ell-1}\rho^k(1+\rho)\frac{(\ell)_k}{(\ell\rho+k)_{k+1}} + \frac{\rho^\ell \ell!\,\Gamma(\ell\rho)}{\Gamma(\ell+\ell\rho)}$$

$$\sim \frac{1}{\sqrt{\ell}}\int_0^\infty\left(\frac{1+\rho}{\rho}\right)\exp\left(-\frac{(1+\rho)y^2}{2\rho}\right)dy = \sqrt{\frac{1+\rho}{\rho}}\cdot\sqrt{\frac{\pi}{2\ell}}. \qquad (4.10)$$

Also, since the variables $D_1^\rho, D_2^\rho, ..$ are independent, we have

$$\begin{aligned}
\Pr\left\{\sum_{i=1}^{\ell} D_i^\rho = \ell \,\middle|\, \sum_{i=1}^{n} D_i^\rho = n\right\} &= \frac{\Pr\{\sum_{i=1}^{\ell} D_i^\rho = \ell\}\Pr\{\sum_{i=\ell+1}^{n} D_i^\rho = n-\ell\}}{\Pr\{\sum_{i=1}^{n} D_i^\rho = n\}} \\
&= \binom{n}{\ell}\frac{\Gamma(\ell(1+\rho))\Gamma((n-\ell)(1+\rho))\Gamma(n\rho)}{\Gamma(\ell\rho)\Gamma((n-\ell)\rho)\Gamma(n(1+\rho))} \\
&\sim \sqrt{\frac{\rho}{1+\rho}}\sqrt{\frac{n}{2\pi\ell(n-\ell)}}. \qquad (4.11)
\end{aligned}$$

Substituting (4.10) and (4.11) into (4.9), we obtain

$$\Pr\{C_1^\rho(n) = \ell\} = \Pr\{C_1^\rho(n) = \lfloor xn \rfloor\} \sim \frac{1}{2n\sqrt{1-x}}. \qquad (4.12)$$

So, $\frac{C_1^\rho(n)}{n}$ converges in distribution to Z_1 as $n \to \infty$, where Z_1 has density $f_{Z_1}(u) = \theta(1-u)^{\theta-1}$ on the interval $(0,1)$ with $\theta = 1/2$. It follows immediately from (4.12) and Theorem 3 that for any integer $t \geq 1$ and constants $0 < a_i < b_i < 1$, where $1 \leq i \leq t$,

$$\lim_{n\to\infty} \Pr\left\{a_i < \frac{C_i^{\hat{D}(\rho,n)}}{n - C_1^{\hat{D}(\rho,n)} - \cdots - C_{i-1}^{\hat{D}(\rho,n)}} < b_i, \quad 1 \leq i \leq t\right\}$$

$$= \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{1}{2\sqrt{1-x}} dx.$$

Hence, it follows from standard arguments (see, for example [21]) that:

Theorem 5. *The joint distribution of the normalized order statistics for the component sizes in G_n^ρ converges to the Poisson-Dirichlet (1/2) distribution on the simplex*

$$\nabla = \left\{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \;\; \textit{for every} \;\; i \geq 1\right\}.$$

We note that in the calculations above the parameter $\rho > 0$ is fixed as $n \to \infty$. In a companion paper [23] we also investigate the asymptotic structure of G_n^ρ when $\rho = \rho(n)$ is a function of n. Clearly, if $\rho(n) \to \infty$ as $n \to \infty$, then the distribution $G_n^{\rho(n)}$ is close to the distribution of the digraph G_n corresponding to the uniform model. As we have seen above, for a fixed parameter ρ, the 'fine' structure of the digraph G_n^ρ begins to differ from the structure of the uniform model. We show in [23], when $\rho(n) \to 0$ as $n \to \infty$, that the structure of $G_n^{\rho(n)}$ is significantly different from the structure of the uniform model G_n.

An Anti-Preferential Attachment Model

In this section we define $T_n^m : [n] \to [n]$, a random mapping with 'anti-preferential attachment', where m is a positive integer parameter. For $1 \le k \le n$, we define $T_n^m(k) = \eta_k^{(m,n)}$ where, as in the definition of T_n^ρ, the variables $\eta_1^{(m,n)}, \eta_2^{(m,n)}, .., \eta_n^{(m,n)}$ depend on the evolution of an urn scheme. The distribution of each variable $\eta_k^{(m,n)}$ is determined by a (random) n-tuple of non-negative weights $\vec{b}(k) = (b_1(k), b_2(k), ..., b_n(k))$ where, for $1 \le j \le n$, $b_j(k)$ is the number of balls in the j^{th} urn at the start of the k^{th} round of the urn scheme. Specifically, given $\vec{b}(k) = \vec{b} = (b_1, ..., b_n)$, we define

$$\Pr\left\{\eta_k^{(m,n)} = j \,\middle|\, \vec{b}(k) = \vec{b}\right\} = \frac{b_j}{\sum_{i=1}^n b_i}.$$

The random weight vectors $\vec{b}(1), \vec{b}(2), \ldots, \vec{b}(n)$ associated with the urn scheme are determined recursively. For $k = 1$, we set $b_1(1) = b_2(1) = \cdots = b_n(1) = m$. For $k > 1$, $\vec{b}(k)$ depends on both $\vec{b}(k-1)$ and the value of $\eta_{k-1}^{(m,n)}$ as follows: Given that $\eta_{k-1}^{(m,n)} = j$, we set $b_j(k) = b_j(k-1) - 1$ and for all other $i \neq j$, we set $b_i(k) = b_i(k-1)$ (i.e. if $\eta_{k-1}^{(m,n)} = j$ then a 'ball' is removed from the j^{th} urn).

The random mapping T_n^m as defined above is an anti-preferential attachment model in the following sense. Since, for $1 \le k \le n$, we have $T_n^m(k) = \eta_k^{(m,n)}$, and since the (conditional) distribution of $\eta_k^{(m,n)}$ depends on the state of the urn scheme at the start of round k, it is clear that vertex k is less likely to ' choose' vertex j if the weight $b_j(k)$ is (relatively) small, i.e. if several of the vertices $1, 2, ..., k-1$ have already been mapped to vertex j. It is also clear from the definition of T_n^m that the in-degree of any vertex in the random digraph $G_n^m \equiv G(T_n^m)$ is at most m and in the case $m = 1$, T_n^1 is a (uniform) random permutation. In the following theorem we determine the distribution of T_n^m.

Theorem 6. *Suppose that $D_1^m, D_2^m, ...$ are i.i.d. $Bin(m,p)$ variables where m is a positive integer parameter. Let $\hat{D}(m,n) = (\hat{D}_{1,n}^m, \hat{D}_{2,n}^m, ..., \hat{D}_{n,n}^m)$ be a sequence of variables with joint distribution given by*

$$\Pr\{\hat{D}_{i,n}^m = d_i, 1 \leq i \leq n\} = \Pr\left\{ D_i^m = d_i, 1 \leq i \leq n \,\middle|\, \sum_{i=1}^n D_i^m = n \right\}.$$

Then the random mappings T_n^m and $T_n^{\hat{D}(m,n)}$ have the same distribution.

Proof. To prove the result it is enough to show that for any $n \geq 1$ and any $f \in \mathcal{M}_n$

$$\Pr\{T_n^m = f\} = \Pr\{T_n^{\hat{D}(m,n)} = f\}.$$

Suppose that $f \in \mathcal{M}_n$ and that $\vec{d}(f) = (d_1, d_2, ..., d_n)$. It is straightforward to check that

$$\Pr\left\{ D_i^m = d_i, 1 \leq i \leq n \,\middle|\, \sum_{i=1}^n D_i^m = n \right\} = \frac{\prod_{i=1}^n \binom{m}{d_i}}{\binom{nm}{n}},$$

and hence, from the definition of $T_n^{\hat{D}(m,n)}$, that

$$\Pr\{T_n^{\hat{D}(m,n)} = f\} = \frac{d_1! \cdots d_n!}{n!} \times \frac{\prod_{i=1}^n \binom{m}{d_i}}{\binom{nm}{n}} = \frac{\prod_{i=1}^n (m)_{d_i}}{(nm)_n}.$$

On the other hand, $T_n^m = f$ with $\vec{d}(f) = (d_1, d_2, ..., d_n)$ if and only if for each $1 \leq i \leq n$, d_i balls are removed, in a certain order, from the i^{th} urn during the evolution of the urn model described above. So it follows from the definition of T_n^m in terms of the urn scheme, that

$$\Pr\{T_n^m = f\} = \frac{\prod_{i=1}^n (m)_{d_i}}{(nm)_n}.$$

The result follows since $n \geq 1$ and $f \in \mathcal{M}_n$ were arbitrary. □

Again, it is clear from Theorem 6 that the order in which a realisation of T_n^m is sequentially constructed does not matter. In particular, suppose that $i_1, i_2, ..., i_n$ is a permutation of $[n]$ and for $1 \leq k \leq n$, let $\tilde{T}_n^m(i_k) \equiv \eta_k^{(m,n)}$ where the variables $\eta_1^{(m,n)}, \eta_2^{(m,n)}, ..., \eta_n^{(m,n)}$ are as defined above. Then it follows from the proof of Theorem 6 that $\tilde{T}_n^m \overset{d}{\sim} T_n^{\hat{D}(m,n)} \overset{d}{\sim} T_n^m$.

We apply Theorem 6 to investigate the distributions of the number of cyclic vertices in $G_n^m \equiv G(T_n^m)$ and the size of a typical component in G_n^m. Let X_n^m denote the number of cyclic vertices in the random digraph G_n^m and let $C_1^m(n)$ denote the size of the component in G_n^m which contains the vertex 1. Since $T_n^m \overset{d}{\sim} T_n^{\hat{D}(m,n)}$, we have $G_n^m \overset{d}{\sim} G_n^{\hat{D}(m,n)}$ and $X_n^m \overset{d}{\sim} X_n^{\hat{D}(m,n)}$. So, it follows from Theorem 1 and Theorem 6 (and its proof) that for $m \geq 2$ and $1 \leq k < n$, we have

$$\Pr\{X_n^m = k\} = \Pr\left\{X_n^{\hat{D}(m,n)} = k\right\} = \frac{k}{n-k} E\left((\hat{D}_{1,n}^m - 1)\hat{D}_{1,n}^m \hat{D}_{2,n}^m \cdots \hat{D}_{k,n}\right)$$

$$\begin{aligned}
&= \frac{k}{n-k} \sum_{\vec{d}\ s.t.\ \sum_{i=1}^n d_i = n} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_n}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} \sum_{t=k+1}^{\min(n,km)} \sum_{\substack{\vec{d}\ s.t.\ \sum_{i=1}^k d_i = t \\ and\ \sum_{i=1}^n d_i = n}} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_n}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} \sum_{t=k+1}^{\min(n,km)} \sum_{\vec{d}\ s.t.\ \sum_{i=1}^k d_i = t} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_k} \binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \sum_{t=k+1}^{\min(n,km)} \sum_{\vec{d}\ s.t.\ \sum_{i=1}^k d_i = t} \frac{\binom{m-2}{d_1-2}\binom{m-1}{d_2-1} \cdots \binom{m-1}{d_k-1}\binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \sum_{t=k+1}^{\min(n,km)} \frac{\binom{km-k-1}{t-k-1}\binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \frac{\binom{nm-k-1}{n-k-1}}{\binom{nm}{n}}.
\end{aligned}$$

In the summations above the sum is always taken over those degree sequences for which the binomial coefficients are defined. We also adopt the formal convention that $\binom{0}{0} = 1$. Finally, for $k = n$ and $m \geq 2$, we obtain

$$\Pr\{X_n^m = n\} = \Pr\left\{\hat{D}_{i,n}^m = 1, 1 \leq i \leq n\right\} = \frac{m^n}{\binom{nm}{n}},$$

and when $m = 1$, we have $X_n^1 \equiv n$.

To obtain a local limit theorem for X_n^m, $m \geq 2$, fix $0 < x < \infty$ and suppose that $k = \lfloor x\sqrt{n} \rfloor$. Then we have

$$\Pr\{X_n^m = k\} = \frac{k}{n-k} m^k (m-1) \frac{\binom{nm-k-1}{n-k-1}}{\binom{nm}{n}} = \frac{k}{n-k} m^k (m-1) \frac{(n)_{k+1}}{(nm)_{k+1}}$$

$$\sim \left(\frac{m-1}{m}\right) x \exp\left(\frac{-(m-1)x^2}{2m}\right) \frac{1}{\sqrt{n}}. \tag{4.13}$$

It follows that $X_n^m/\sqrt{n}$ converges in distribution to a variable $\tilde{X}_m$ with density

$$f_{\tilde{X}_m}(x) = \left(\frac{m-1}{m}\right) x \exp\left(\frac{-(m-1)x^2}{2m}\right) \quad \text{for } x \geq 0.$$

Again, by keeping track of the error term (4.13), it is straightforward to show $E(X_n^m) \sim \sqrt{\frac{mn\pi}{2(m-1)}}$. Also, if we let N_n^m denote the number of components in G_n^m, then by calculations similar to those given by Stepanov (see [39]), it can be shown that $(N_n^m - \frac{1}{2}\log n)/\sqrt{\frac{1}{2}\log n}$ converges in distribution to a standard $N(0,1)$ distribution.

We apply Theorem 2 and Theorem 6 to obtain the distribution for $C_1^m(n)$:

$$\begin{aligned} \Pr\{C_1^m(n) = \ell\} &= \Pr\left\{C_1^{\hat{D}(m,n)} = \ell\right\} \\ &= \frac{\ell}{n} \Pr\left\{\mathcal{B}_\ell^{\hat{D}(m,\ell)}\right\} \Pr\left\{\sum_{i=1}^{\ell} D_i^m = \ell \,\middle|\, \sum_{i=1}^{n} D_i^m = n\right\} \end{aligned} \tag{4.14}$$

for $1 \leq \ell \leq n$. To obtain a local limit theorem for the distribution of $C_1^m(n)$, fix $0 < x < \infty$ and suppose that $\ell = \lfloor xn \rfloor$. Then it follows from Theorem 6 and Corollary 2, that

$$\Pr\left\{\mathcal{B}_\ell^{\hat{D}(m,\ell)}\right\} = \sum_{k=1}^{\ell} \frac{1}{k} \Pr\left\{X_\ell^{\hat{D}(m,\ell)} = k\right\} = \sum_{k=1}^{\ell-1} \frac{m^k(m-1)}{\ell-k} \frac{\binom{\ell m-k-1}{\ell-k-1}}{\binom{\ell m}{\ell}} + \frac{1}{\ell} \frac{m^\ell}{\binom{\ell m}{\ell}}$$

$$\sim \frac{1}{\sqrt{\ell}} \int_0^\infty \left(\frac{m-1}{m}\right) \exp\left(-\frac{(m-1)y^2}{2m}\right) dy = \sqrt{\frac{m-1}{m}} \sqrt{\frac{\pi}{2\ell}}. \tag{4.15}$$

Since $D_1^m, D_2^m, \ldots$ are i.i.d. $Bin(m,p)$ variables, we also have

$$\begin{aligned}\Pr\left\{\sum_{i=1}^{\ell} D_i^m = \ell \,\Big|\, \sum_{i=1}^{n} D_i^m = n\right\} &= \frac{\binom{m\ell}{\ell}\binom{mn-m\ell}{n-\ell}}{\binom{mn}{n}} \\ &\sim \frac{\sqrt{mn}}{\sqrt{2\pi\ell(m-1)(n-\ell)}}. \qquad (4.16)\end{aligned}$$

Substituting (4.15) and (4.16) into (4.14) we obtain

$$\Pr\{C_1^m(n) = \ell\} \sim \frac{1}{2n\sqrt{1-x}} \quad \text{as } n \to \infty. \qquad (4.17)$$

It follows that as $n \to \infty$, $\frac{C_1^m(n)}{n}$ converges in distribution to Z_1, where Z_1 has $f_{Z_1}(u) = \theta(1-u)^{\theta-1}$ on the interval $(0,1)$ with parameter $\theta = 1/2$. Again, as in the case of the preferential attachment model, we can extend (4.17) to obtain:

Theorem 7. *The joint distribution of the normalized order statistics for the component sizes in G_n^m, $m \geq 2$, converges to the Poisson-Dirichlet (1/2) distribution on the simplex*

$$\nabla = \left\{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \textit{ for every } i \geq 1\right\}.$$

5 Final Remarks

It is interesting to note that for both the preferential and anti-preferential attachment models and for any (fixed) choice of their respective parameters ρ and m, we obtain the Poisson-Dirichlet(1/2) distribution as the limiting distribution of the order statistics of their normalized component sizes. On the other hand, the asymptotic distribution of the number of cyclic vertices in each model depends explicitly on the respective parameters ρ and m. This suggests that the differences between these models are to be found in the 'fine'structure of the components. It is this 'fine' structure of random mapping digraphs that is also of interest in many applications. For example, in applications of random mapping models in cryptology and in epidemic process modelling the distributions of the number of predecessors and of the number of successors of an arbitrary vertex or set of vertices are also of interest. In a companion paper [22], we develop a calculus for computing these distributions based on the underlying variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ and apply the results obtained to both the preferential and anti-preferential models.

One of the main advantages of the random mapping model $T_n^{\hat{D}}$ is that we have a calculus for this model which allows us to determine the distributions of several variables associated with the structure of $G_n^{\hat{D}}$ in terms of expectations of simple functions of $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$. As we have seen above, in the special case where the variables $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ have the same distribution as a collection of i.i.d. variables $D_1, D_2, ..., D_n$ *conditioned* on $\sum_{i=1}^n D_i = n$, it is straightforward to use this calculus to obtain exact and asymptotic distributions for the number of cyclic vertices, the number of components, and the size of a typical component in $G_n^{\hat{D}}$. The calculus for $T_n^{\hat{D}}$ also illustrates the fundamental importance of the distribution of the underlying degree sequence $\hat{D}_1, ..., \hat{D}_n$ to the structure of the random mapping digraph. This suggests that in various modelling applications the key to fitting a random mapping model is to fit the joint distribution of the vertex in-degree data.

As a example of model fitting, we mention the work of Arney and Bender on random mappings with constraints on coalescence [4]. Their work was motivated, in part, by the analysis of shift register data. In order to model a random shift register they put a uniform measure on $\mathcal{M}_n^{\{0,1,2\}}$, the set of all mappings $f : [n] \to [n]$ such that, for every $1 \le i \le n$, $|f^{-1}(i)|$, the number of pre-images of i under f, equals $0, 1$, or 2. So, if $f \in \mathcal{M}_n^{\{0,1,2\}}$, then every vertex in $G_n(f)$ has in-degree equal $0, 1$, or 2. Arney and Bender observed that in some respects their model does not fit the shift register data. In particular, their model predicts $0.293n$ vertices with in-degree 0 whereas the average number of vertices with in-degree 0 in a random shift register is $n/4$. By using the model $T_n^{\hat{D}}$ instead, we can more successfully capture the local structure of the shift register data. Specifically, suppose that $\hat{D}_1, \hat{D}_2, ..., \hat{D}_n$ have the same distribution as n independent $Bin(2, \frac{1}{2})$ variables, $D_1, D_2, ..., D_n$, conditioned on $\sum_{i=1}^n D_i = n$. Then $\Pr\{\hat{D}_1 = 0\} = \frac{1}{4}(1+\frac{1}{2n-2})^{-1}$ and the expected number of vertices with in-degree 0 in $T_n^{\hat{D}}$ is asymptotic to $\frac{n}{4}$. We note that, for example, the asymptotic distribution of the normalised typical component size is the *same* for both the Arney and Bender model and $T_n^{\hat{D}}$. So it is not surprising that Arney and Bender found that their model fit some other features of the shift register data quite well.

Acknowledgement. J. Jaworski acknowledges the generous support by the Marie Curie Intra-European Fellowship No. 501863 (RANDIGRAPH) within

the 6th European Community Framework Programme.
The authors would also like to thank an anonymous referee for carefully reading the first version of this paper and for suggestions which have improved the exposition of the final version.

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