Covering Random Points in a Unit Ball

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Abstract

Choose random points $X_1, X_2, X_3, \ldots$ independently from a uniform distribution in a unit ball in $\Re^m$. Call X_n a *dominator* iff distance$(X_n, X_i) \leq 1$ for all $i < n$, i.e. the first n points are all contained in the unit ball that is centered at the n'th point X_n. We prove that, with probability one, only finitely many of the points are dominators.

For the special case $m = 2$, we consider the unit disk graph $\mathcal{G}_n$ determined by n random points $X_1, X_2, \ldots, X_n$ in the unit disk. With asymptotic probability one, $\mathcal{G}_n$ has a connected dominating set consisting of just two points.

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1 Introduction

Every finite set V of points in $\Re^m$, determines a *unit ball graph* $\mathcal{G}(V) = (V, E)$ as follows. The vertex set is V, and an undirected edge $e \in E$ connects vertices $u, v \in V$ iff the distance $d_m(u, v)$ is less than one. (Throughout this paper $d_m(u, v)$ denotes the Euclidean distance between $u, v \in \Re^m$.) If u and v are connected by and edge, we say u and v are *adjacent* vertices in the graph $\mathcal{G}(V)$. The case $m = 2$ is particularly prominent in applications, and in this case $\mathcal{G}(V)$ is called a *unit disk graph*. Unit disk graphs have been used by many authors as simplified mathematical models for the interconnections between hosts in a wireless network, and random unit disk graphs have been used as stochastic models for these networks. For other examples of applications, see Marchette [7]. A recent survey of random unit ball graphs is Penrose [8].

A *dominating set* in any graph $G = (V, E)$ is a subset $\mathcal{C} \subseteq V$ such that every vertex $v \in V$ either is in the set $\mathcal{C}$, or is adjacent to a vertex in $\mathcal{C}$ [5].We say $\mathcal{C}$ is a *connected dominating set* if $\mathcal{C}$ is a dominating set and the subgraph induced by $\mathcal{C}$ is connected. The following question arose naturally in the context of routing algorithms for certain wireless networks [9],[4],[3]. Suppose V is a set of n random points in the unit disk in $\Re^2$, and let $\mathcal{G}_n = \mathcal{G}(V)$ denote the corresponding unit disk graph. How large, typically, is the smallest connected dominating set for $\mathcal{G}_n$? In this paper we show that with asymptotic probability one, the size of the smallest connected dominating set in $\mathcal{G}_n$ is two.

The paper is organized as follows. In Section 2, we prove a general result concerning unit ball graphs formed from random points $X_1, X_2, X_3, \ldots$ chosen independently and uniformly in the unit ball in $\Re^m$. It follows from this general result that 'one point does not suffice': with asymptotic probability one, $\mathcal{G}_n$ does not have a one-point dominating set. In Section 3 we prove a geometric lemma which is required for the proof, in Section 4, of the existence, with high probability, of two-point connected dominating sets in $\mathcal{G}_n$.

2 One Vertex Dominating Sets

Suppose that $X_1, X_2, X_3, \ldots$ is a sequence of random points chosen independently from a uniform distribution in a unit ball in $\Re^m$. Call X_n a *dominator* iff $d_m(X_n, X_i) \leq 1$ for all $i < n$, i.e. all n points are contained in the unit ball that is centered at X_n. Then we prove:

Theorem 1 *With probability one, only finitely many of the points X_n are dominators.*

Proof. Let $\mathcal{A}_n$ be the event that X_n is a dominator. By the Borel-Cantelli lemma, it suffices to prove that $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$. For positive real numbers r and positive integers $m \geq 2$, let $V_m(r)$ be the volume of the a ball of radius r in $\Re^m$, i.e

$$V_m(r) = r^m V_m(1) = \frac{\pi^{\frac{m}{2}} r^m}{\Gamma(\frac{m}{2} + 1)} \tag{1}$$

Let $L(r)$ denote the volume of the intersection of two unit balls in $\Re^m$ whose centers are at distance r from each other. If the the distance from the point X_n to the origin is r, then the conditional probability that the i'th point X_i is within distance one of X_n is $\frac{L(r)}{V_m(1)}$. The distance between the origin and the random point X_n is a random variable with density $f(r) = \frac{V_m^{'}(r)}{V_m(1)} = mr^{m-1}$. Hence

$$\Pr(X_n \text{ is a dominator}) = \int_0^1 f(r)\Big(\frac{L(r)}{V_m(1)}\Big)^{n-1} dr.$$

We split the integral into two. Let $\xi = \frac{4(\log n)V_m(1)}{(n-1)V_{m-1}(1)}$. Then

$$\Pr(X_n \text{ is a dominator}) = I_1 + I_2, \tag{2}$$

where

$$I_1 = m \int_0^\xi r^{m-1}\Big(\frac{L(r)}{V_m(1)}\Big)^{n-1} dr \tag{3}$$

and

$$I_2 = m \int_\xi^1 r^{m-1}\Big(\frac{L(r)}{V_m(1)}\Big)^{n-1} dr. \tag{4}$$

For the first piece, we use the trivial estimate $\frac{L(r)}{V_m(1)} \leq 1$: for $m \geq 2$,

$$I_1 \leq m \int_0^\xi r^{m-1} dr = \xi^m = O(\frac{\log^2 n}{n^2}). \tag{5}$$

To estimate I_2, we use the following "well-known"formula for $L(r)$:

$$L(r) = 2 \int_{r/2}^1 V_{m-1}(\sqrt{1-x^2})dx = 2V_{m-1}(1) \int_{r/2}^1 (1-x^2)^{\frac{m-1}{2}} dx \tag{6}$$

It is intuitively obvious that $L(r)$ is decreasing, and this is easily confirmed by differentiating the right side of (6) to obtain

$$L^{'}(r) = -V_{m-1}(1) \cdot \left(1 - \frac{r^2}{4}\right)^{\frac{m-1}{2}} \leq 0 \tag{7}$$

for $0 \leq r \leq 1$. By differentiating again, we also obtain

$$L^{''}(r) = V_{m-1}(1) \cdot \frac{(m-1)r}{8} \left(1 - \frac{r^2}{4}\right)^{\frac{m-1}{2}} \geq 0 \tag{8}$$

for $0 \le r \le 1$. Since $L(r) \le L(\xi)$ for all $r \ge \xi$, and since f is a density function, we have

$$I_2 \le \left(\frac{L(\xi)}{V_m(1)}\right)^{n-1} \int_{\xi}^{1} f(r)dr \le \left(\frac{L(\xi)}{V_m(1)}\right)^{n-1} \tag{9}$$

To estimate the right side of (9), we note that it follows from (7) and (8) that there is some $0 < c_\xi < \xi$ such that

$$L(\xi) = L(0) + L^{'}(c_\xi)\xi = V_m(1) - V_{m-1}(1) \cdot \left(1 - \frac{c_\xi^2}{4}\right)^{\frac{m-1}{2}} \cdot \xi \tag{10}$$

Since $0 < c_\xi < \xi = o(1)$, we have $(1 - \frac{c_\xi^2}{4})^{\frac{m-1}{2}} > \frac{1}{2}$ for all sufficiently large n. So it follows from (10) that

$$L(\xi) \le V_m(1) - \frac{V_{m-1}(1)\xi}{2} \tag{11}$$

for all sufficiently large n. Putting (11) back into the right side of (9), we get

$$I_2 \le \left(1 - \frac{\xi V_{m-1}(1)}{2V_m(1)}\right)^{n-1} = O(\frac{1}{n^2}). \tag{12}$$

Combining our estimates (5) and (12) for I_1 and I_2 respectively, we conclude that, for some positive constant c, and all sufficiently large n, $\Pr(\mathcal{A}_n) < \frac{c \log^2 n}{n^2}$. Hence $\sum_n \Pr(\mathcal{A}_n)$ converges. □

Now let $\mathcal{G}_n^m \equiv \mathcal{G}(X_1, X_2, ..., X_n)$ denote the unit ball graph in $\Re^m$ with random vertex set $V = \{X_1, X_2, ..., X_n\}$. (So,in particular, $\mathcal{G}_n = \mathcal{G}_n^2$.) Let $\mathcal{B}_n^m$ denote the event that $\mathcal{G}_n^m$ has a one-point dominating set.

Corollary 2 *For all $m \ge 2$ and for all sufficiently large n,*

$$\Pr\{\mathcal{B}_n^m\} \le \frac{c_m \log^2 n}{n}$$

where $c_m > 0$ is a positive constant which may depend on the dimension m but does not depend on n.

Proof. For each $n > 0$ and for $1 \le i \le n$, let $\mathcal{B}_n(X_i)$ denote the event that X_i is a one-point dominating set of $\mathcal{G}_n^m$. Then we have, for all sufficiently large n,

$$\Pr\{\mathcal{B}_n^m\} = \Pr\{\cup_{i=1}^n \mathcal{B}_n(X_i)\} \le \sum_{i=1}^{n} \Pr\{\mathcal{B}_n(X_i)\} = n \Pr\{\mathcal{B}_n(X_n)\} \le \frac{c_m \log^2 n}{n}$$

since $X_1, X_2, ..., X_n$ are independent and identically distributed. We note that the last inequality follows from the bound obtained for $\Pr\{\mathcal{A}_n\}$ in the proof of Theorem 1. □

3 A Geometric Lemma

In this section and the next, we adopt the following notation. For any $r > 0$, and any $v \in \Re^2$, let $D_r(v)$ be the open unit disk centered at v, and let $\partial D_r(v)$ be the circle of radius r that bounds it. As observed in [6], a unit disk centered at a point o cannot be completely covered with two unit disks having centers at points other than o: $D_1(o) \not\subseteq D_1(u) \bigcup D_1(v)$ for $u \neq o \neq v$. The purpose of this section is to prove Lemma 3, which provides an upper bound for the uncovered region's area (when u and v are suitably situated).

Some notation is needed to state Lemma 3. Throughout this section, $b \geq 3$ will be an unspecified parameter. (When we actually apply Lemma 3 in Section 4, b will correspond to the number of vertices in a random disk graph.) In terms of b, we define $L_b = \lfloor b^{1/3}(\log b)^2 \rfloor, \delta = \delta_b = \frac{1}{\sqrt[3]{b}\log b}$, and $\theta_b = \pi/L_b$. Let $o = (0, 0)$ be the origin in $\Re^2$. We are essentially[1] going to partition $D_{\delta_b}(o)$ into $2L_b$ sectors as follows. For integers i such that $0 \leq i < L_b$, let Q_i be the sector consisting of those points $(x, y) = (r\cos\theta, r\sin\theta)$ whose polar coordinates satisfy $0 < r \leq \delta$ and $(i - \frac{1}{2})\theta_b \leq \theta \leq (i + \frac{1}{2})\theta_b$. Similarly let R_i consist of the points with $0 < r \leq \delta$ and $(i - \frac{1}{2})\theta_b \leq \theta - \pi \leq (i + \frac{1}{2})\theta_b$. Note that the sectors Q_i and R_i are located symmetrically with respect to o. Let $\tilde{q}_i$ and $\tilde{u}_i$ be the extreme points whose polar coordinates are respectively $(\delta, (i - \frac{1}{2})\theta_b)$ and $(\delta, (i + \frac{1}{2})\theta_b + \pi)$. Finally, for any points $u, w \in D_1(o)$, let $X(u, w)$ denote the area of $(D_1(u) \bigcup D_1(w))^c \bigcap D_1(o)$, i.e. the area of the region in $D_1(o)$ that is not covered by $D_1(u) \bigcup D_1(w)$. Our goal in this section is to prove

Lemma 3 *There is a uniform constant $C > 0$ (independent of the parameter b) such that, for all i, and for all $q_i \in Q_i, u_i \in R_i$, we have $X(q_i, u_i) \leq X(\tilde{q}_i, \tilde{u}_i) \leq \frac{C}{b\log^3 b}$.*

We prove four facts which together imply Lemma 3. In the first fact, we observe that for any $v, w \in D_1(o)$ the omitted area $X(v, w)$ increases if we move one (or both) of the two points v and w away from the origin along a radial line.

Fact 1 *Let v_1, v_2 and w_1, w_2 be four points in $D_1(o)$ such that v_1 lies on the line segment $\overline{o, v_2}$ and w_1 lies on the line segment $\overline{o, w_2}$. Then $X(v_2, w_2) \geq X(v_1, w_1)$.*

Proof. It suffices to show that $D_1(v_2) \cap D_1(o) \subseteq D_1(v_1) \cap D_1(o)$ and that $D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o)$. Suppose $p \in D_1(v_2) \cap D_1(o)$. Since v_1 lies on the line segment from o to v_2, we have $d(v_1, p) \leq \max(d(o, p), d(v_2, p)) \leq 1$. Hence $p \in D_1(v_1) \cap D_1(o)$. By a similar same argument, $D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o)$.

□

[1] It is not strictly correct to call this a partition of $D_\delta(o)$ since the origin was omitted, the bounding circle was included, and some pairs of sectors have a non-empty intersection (with zero area).

Fact 2 *Let a, b be the two points where the circles $\partial D_1(p), \partial D_1(q)$ intersect. Then, $\overline{a, b} \perp \overline{p, q}$, and the two line segments $\overline{a, b}$ and $\overline{p, q}$ intersect at their midpoints.*

Proof. This follows immediately from the fact that $d(p, a) = d(p, b) = d(q, a) = d(q, b) = 1$. □

Fact 3 *Let o_1, o_2 be two points on the circle $x^2 + y^2 = \delta_b^2$. Then, $X(o_1, o_2)$ is a decreasing function of $\angle o_1 o o_2$.*

Proof. For convenience, we will use polar coordinates. Without loss of generality, let o_1 be the point with polar coordinates $(r_{o_1}, \phi_{o_1}) = (\delta_b, \pi)$. Let o_2 be an arbitrary point on the circle with the polar coordinates (δ_b, ϕ_2). By symmetry, we only need to consider the case when o_2 is in the first or second quadrant; we may, without loss of generality, assume that $0 \leq \phi_2 \leq \pi$. We will show that $X(o_1, o_2)$ is an increasing function of ϕ_2, then the result follows from the fact that $\angle o_1 o o_2 = \pi - \phi_2$.

Let a_1, b_1 be the two points where the circles $\partial D_1(o_1)$ and $\partial D_1(o)$ intersect, with a_1 in the second quadrant and b_1 in the third quadrant.

Let o^* be a point on the circle $x^2 + y^2 = \delta_b^2$ so that $\partial D_1(o^*)$ meets with both $\partial D_1(o)$ and $\partial D_1(o_1)$ at a_1. Let b^*, d^* be the other intersection points of $\partial D_1(o^*)$ with $\partial D_1(o)$ and $\partial D_1(o_1)$, respectively. For convenience, let's denote ϕ_{o^*} by ϕ^*. Figure 1 illustrates the position of $\partial D_1(o_1), \partial D(o)$, and $\partial D_1(o^*)$ and their intersections.

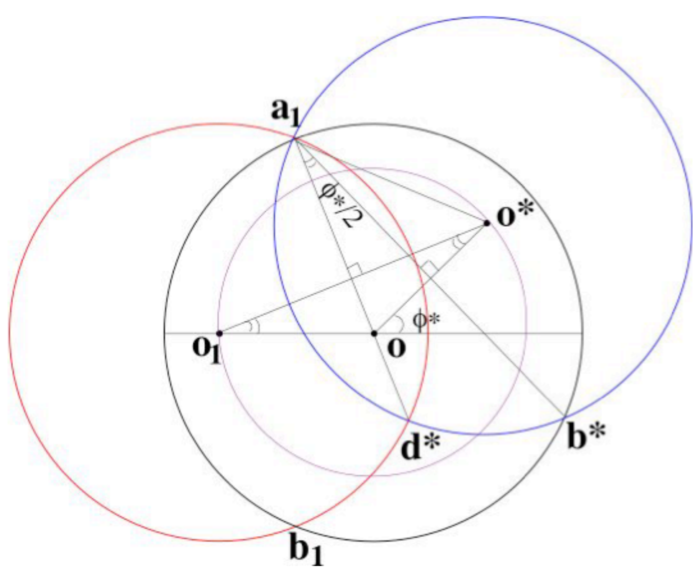

Figure 1: The position of the circle $\partial D_1(o^*)$

As in the proof of Fact 2,we have $\overline{a_1, d^*} \perp \overline{o_1, o^*}$, $\overline{a_1, b^*} \perp \overline{o, o^*}$. Notice also that o is on the line segment $\overline{a_1, d^*}$. So,

$$\angle b^* a_1 o = \angle o o^* o_1 = \angle o^* o_1 o = \frac{\phi^*}{2}. \tag{13}$$

It follows that

$$0 < \phi^*/2 < \pi/2, \text{ and, } \sin\frac{\phi^*}{2} = \frac{\delta_b}{2} \tag{14}$$

Now, for the point o_2 with polar coordinates (δ_b, ϕ_2), let a_2, b_2 denote the two points where $\partial D_1(o_2)$ and $\partial D_1(o)$ intersect, and let c_2, d_2 denote the two points where $\partial D_1(o_2)$ and $\partial D_1(o_1)$ intersect. There are two cases to consider: $\phi_2 \le \phi^*$, and $\phi_2 \ge \phi^*$

Case 1. $\phi_2 \le \phi^*$.

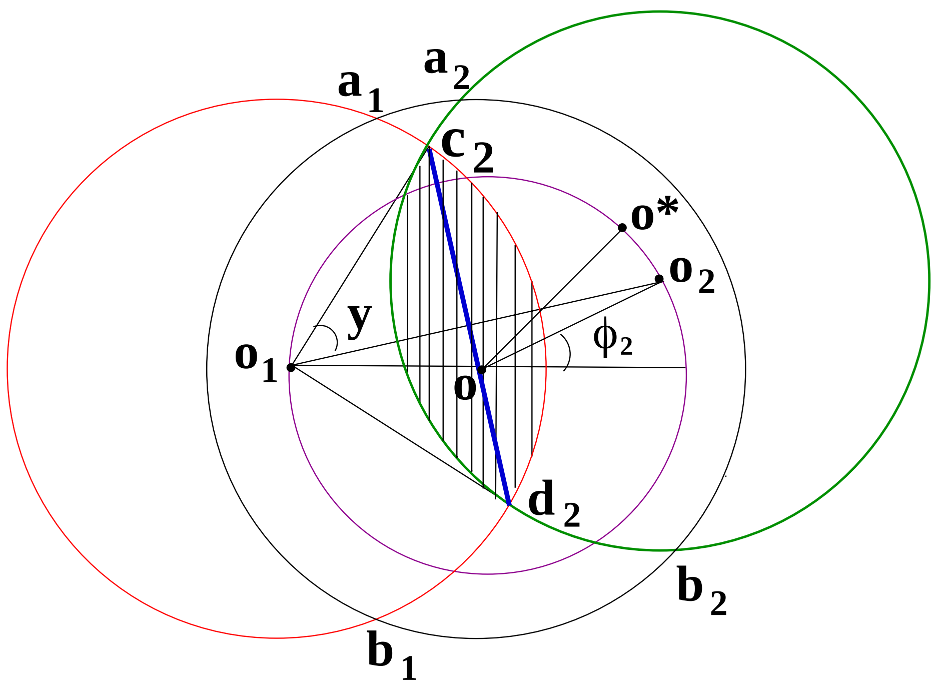

Figure 2: The case when $\phi_2 \le \phi^*$

Notice that a_1, b_1 partitions the circle $\partial D_1(o)$ into two arcs: the right section and the left section. When, $\phi_2 \le \phi^*$, as illustrated in Figure 2, a_2, b_2 are both on the right section of the circle $\partial D_1(o)$ between a_1, b_1. Similarly, c_2, d_2 are both on the right section of the circle $\partial D_1(o_1)$ between a_1, b_1. Clearly,

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3,$$

where

- $B_1 = area(D_1(o_1)^c \cap D_1(o))$

- $B_2 = area(D_1(o) \cap D_1(o_2))$
- $B_3 = area(D_1(o_1) \cap D_1(o_2))$, the shaded area in Figure 2

Notice that B_3 is the only area that depends on ϕ_2. We shall now give an expression for B_3. Let's denote $\angle c_2 o_1 o_2 = y$. Since $\angle o_2 o_1 o = \frac{\phi_2}{2}$, we have

$$0 < y < \frac{\pi}{2}, \text{ and, } \cos y = \delta_b \cos\frac{\phi_2}{2} \tag{15}$$

By symmetry, one can see that the shaded region is partitioned equally by the line $\overline{c_2, d_2}$. So,

$$B_3 = 2(\frac{2y}{2\pi}\pi - \frac{1}{2}(2\sin y)(\cos y)) = 2y - \sin 2y.$$

Here, the first term is the area of the sector $D_1(o_1)$ that extends from c_2 to d_2, and the second term is the area of the triangle(c_2, o_1, d_2).From the above two equations, we have

$$\frac{dX(o_1, o_2)}{d\phi_2} = \frac{dB_3}{d\phi_2} = \frac{dB_3}{dy}\cdot\frac{dy}{d\phi_2} = (1-\cos 2y)\cdot\frac{\delta_b \sin\frac{\phi_2}{2}}{2\sin y} > 0.$$

Here the last inequality follows from the fact that $0 < \frac{\phi_2}{2}, y < \frac{\pi}{2}$. Thus $X(o_1, o_2)$ is an increasing function in ϕ_2.

Case 2. $\phi_2 > \phi^*$.

One can see from Figure 3 that

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3$$

Where B_1, B_2 are defined the same as those in the case 1, but

$$B_3 = area(D_1(o_1) \cap D_1(o_2) \cap D_1(o)), \text{ the shaded area in Figure 3}$$

Again, B_3 is the only area that depends on ϕ_2. We will now give an expression for B_3. We show first that $\angle c_2 o a_1 = \angle a_2 o c_2$ by showing that $\phi_{c_2} - \phi_{a_1} = \phi_{a_2} - \phi_{c_2}$. Then, it follows that the region with area B_3 is split in half by the line segment $\overline{c_2, d_2}$. From Figure 1, one can see that

$$\phi_{a_1} = \phi^* + (\frac{\pi}{2} - \angle b^* a_1 o) = \phi^* + (\frac{\pi}{2} - \frac{\phi^*}{2}) = \frac{\pi}{2} + \frac{\phi^*}{2} \tag{16}$$

To find ϕ_{a_2}, observe that, as in the proof of Fact 2, we have $\overline{a_2, b_2} \perp \overline{o, o_2}$. So, $\sin\angle b_2 a_2 o = \frac{\delta_b}{2}$. Comparing with (14), we see that $\sin\angle b_2 a_2 o = \sin\frac{\phi^*}{2}$. This implies that $\angle b_2 a_2 o = \frac{\phi^*}{2}$. Thus,

$$\phi_{a_2} = \phi_2 + (\frac{\pi}{2} - \angle b_2 a_2 o) = \phi_2 + (\frac{\pi}{2} - \frac{\phi^*}{2}) \tag{17}$$

Lastly, using the fact that $\overline{c_2, o} \perp \overline{o_1, o_2}$, we have

$$\phi_{c_2} = \pi - (\frac{\pi}{2} - \angle o_2 o_1 o) = \pi - (\frac{\pi}{2} - \frac{\phi_2}{2}) = \frac{\pi}{2} + \frac{\phi_2}{2} \tag{18}$$

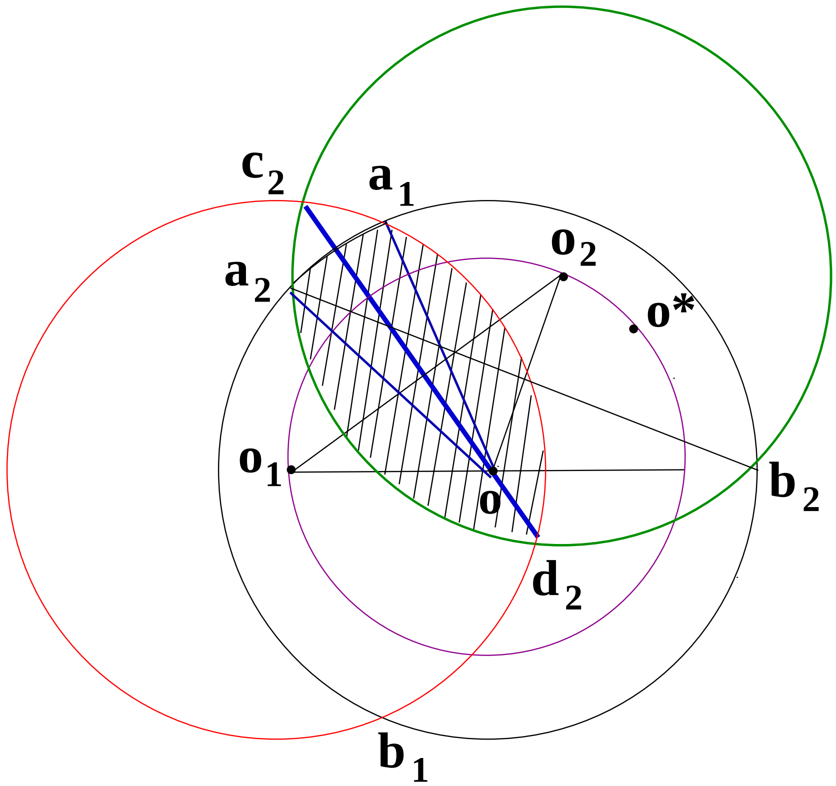

Figure 3: The case when $\phi_2 > \phi^*$

It follows that $\phi_{c_2} - \phi_{a_1} = \phi_{a_2} - \phi_{c_2} = \frac{\phi_2}{2} - \frac{\phi^*}{2}$. Using that the circle $\partial D_1(o_1)$ in the polar system is

$$r = \sqrt{1 - \delta_b^2 \sin^2 \phi} - \delta_b \cos \phi$$

and that

$$\phi_{d_2} = -(\pi - \phi_{c_2}) = -(\frac{\pi}{2} - \frac{\phi_2}{2}) \tag{19}$$

we get

$$\begin{aligned} B_3 &= 2(\int_{-(\frac{\pi}{2}-\frac{\phi_2}{2})}^{\frac{\pi}{2}+\frac{\phi^*}{2}} \int_0^{\sqrt{1-\delta_b^2 \sin^2 \phi}-\delta_b \cos \phi} r\, dr d\phi + \frac{\frac{\phi_2}{2} - \frac{\phi^*}{2}}{2\pi} \cdot \pi) \\ &= \int_{-(\frac{\pi}{2}-\frac{\phi_2}{2})}^{\frac{\pi}{2}+\frac{\phi^*}{2}} 1 - \delta_b^2 \sin^2 \phi + \delta_b^2 \cos^2 \phi - 2\delta_b \cos \phi \sqrt{1 - \delta_b^2 \sin^2 \phi} d\phi + \frac{\phi_2 - \phi^*}{2} \end{aligned}$$

Thus,

$$
\begin{aligned}
\frac{dX(o_1,o_2)}{d\phi_2} = \frac{dB_3}{d\phi_2} &= -\tfrac{1}{2}[1-\delta_b^2\sin^2(-\tfrac{\pi}{2}+\tfrac{\phi_2}{2})+\delta_b^2\cos^2(-\tfrac{\pi}{2}+\tfrac{\phi_2}{2}) \\
&\quad -2\delta_b\cos(-\tfrac{\pi}{2}+\tfrac{\phi_2}{2})\sqrt{1-\delta_b^2\sin^2(-\tfrac{\pi}{2}+\tfrac{\phi_2}{2})}]+\tfrac{1}{2} \\
&= \tfrac{1}{2}[\delta_b^2\cos^2\tfrac{\phi_2}{2}-\delta_b^2\sin^2\tfrac{\phi_2}{2}+2\delta_b\sin\tfrac{\phi_2}{2}\sqrt{1-\delta_b^2\cos^2\tfrac{\phi_2}{2}}] \\
&= \tfrac{1}{2}[-(\delta_b\sin\tfrac{\phi_2}{2}-\sqrt{1-\delta_b^2\cos^2\tfrac{\phi_2}{2}})^2+1] \\
&\geq 0
\end{aligned}
$$

The last inequality follows because $0\leq\delta_b\sin\frac{\phi_2}{2}\leq 1$, $0\leq\sqrt{1-\delta_b^2\cos^2\frac{\phi_2}{2}}\leq 1$, and thus $(\delta_b\sin\frac{\phi_2}{2}-\sqrt{1-\delta_b^2\cos^2\frac{\phi_2}{2}})^2<1$.

□

Fact 4 *Uniformly for all i, we have $X(\tilde{q}_i,\tilde{u}_i)=O(\frac{1}{b\log^3 b})$.*

Proof. Without loss of generality, let $i=0$ and $v=(0,0)$. To simplify notation, define $x_b=\delta_b\cos(-\frac{1}{2}\theta_b)$, $y_b=\delta_b\sin(-\frac{1}{2}\theta_b)$. Let (ξ,η) be the point in the first quadrant where the circles $x^2+y^2=1$ and $(x-x_b)^2+(y-y_b)^2=1$ meet. Then

$$X(\tilde{q}_0,\tilde{u}_0)\leq 4\int_0^{\xi}\sqrt{1-x^2}-(y_b+\sqrt{1-(x-x_b)^2})dx$$

$$=-4y_b\xi+4\int_0^{\xi}\frac{-2xx_b+x_b^2}{\sqrt{1-x^2}+\sqrt{1-(x-x_b)^2}}dx$$

Hence we have

$$X(\tilde{q}_0,\tilde{u}_0)=O(\xi y_b)+O(x_b\xi^2)+O(x_b^2\xi). \tag{20}$$

Note that $x_b^2+y_b^2=\delta_b^2=\frac{1}{b^{2/3}\log^2 b}$, that $\xi^2+\eta^2=1$, that $(\xi-x_b)^2+(\eta-y_b)^2=1$, that $x_b=\delta_b(1+O(\theta_b^2))$, and that $y_b=\frac{-\delta_b\theta_b}{2}(1+O(\theta_b^2))$. Combining these equations, we get $\xi=O(\delta_b)$. Putting this estimate back into (20), we get

$$X(\tilde{q}_0,\tilde{u}_0)=O(\frac{1}{b\log^3 b}). \tag{21}$$

□

4 Two Point Dominating Sets

Let n be an integer such that $n\geq 3$, and let $L_n=\lfloor n^{1/3}(\log n)^2\rfloor$ and $\delta_n=\frac{1}{n^{1/3}\log n}$. Select n points $X_1,X_2,...,X_n$ independently and uniform randomly from the unit disk $D_1(o)$ and form the unit disk graph $\mathcal{G}_n(\equiv\mathcal{G}_n^2)$ by putting an edge between two of the n points iff the distance between them is less than 1. Our goal in this section is to prove that, with high probability, $\mathcal{G}_n$ contains a dominating set consisting of two vertices of $\mathcal{G}_n$ that are adjacent to each other.

For $0 \leq i < L_n$, let Q_i, R_i denote the sectors of $D_{\delta_n}(o)$ as defined in the previous section and let $N(Q_i), N(R_i)$ respectively be the number of vertices of $\mathcal{G}_n$ that lie in Q_i and R_i. Let $\tau_n = \sum_{i=0}^{L_n-1} I_i$ where, in this section only, the indicator variable $I_i = 1$ if and only if $N(R_i) = N(Q_i) = 1$ (and otherwise $I_i = 0$.)

Lemma 4 $\Pr\left(\tau_n < \frac{n^{1/3}}{16\log^6 n}\right) = O(\frac{\log^6 n}{n^{1/3}})$

Proof. Let

$$p = \frac{\text{Area}(Q_i)}{\text{Area}(D_1(0))} = \pi\delta_n^2/\pi 2L_n = \frac{1}{2n\log^4 n}\left(1 + O\left(\frac{1}{n^{1/3}\log^2 n}\right)\right). \quad (22)$$

Then

$$E(I_i) = n(n-1)p^2(1-2p)^{n-2}, \quad (23)$$

and

$$E(\tau_n) = L_n n(n-1)p^2(1-2p)^{n-2} = \frac{n^{1/3}}{4(\log n)^6}\Big(1 + O(\frac{1}{n^{1/3}(\log n)^2}). \quad (24)$$

Similary, for $i \neq j$

$$E(I_iI_j) = n(n-1)(n-2)(n-3)p^4(1-4p)^{n-4}. \quad (25)$$

Since $\tau_n = \sum_{i=0}^{L_n-1} I_i$, and the I_i's are identically distributed, we have

$$Var(\tau_n) = L_n(L_n-1)E(I_1I_2) + L_nE(I_1) - (E(\tau))^2.$$

Combining this identity with the expression for $E(I_i)$ in (23), the expression for $E(I_iI_j)$ in (25), and the definitions for L_n, δ_n and p, we get

$$Var(\tau_n) = E(\tau_n)\left(1 + O\left(\frac{1}{(\log n)^8}\right)\right). \quad (26)$$

The lemma now follows by Chebyshev's inequality. □

Theorem 5 *There is a constant $c > 0$ such that, with probability greater than $1 - \frac{c}{(\log n)^3}$, the random graph $\mathcal{G}_n$ has a connected dominating set that consists of two vertices in $D_{\delta_n}(o)$.*

Proof.

Let $\mathcal{T}_n \subseteq \{0, 1, 2, 3, \ldots, L_n - 1\}$ be the random subset of indices such that $i \in \mathcal{T}_n$ iff $N(Q_i) = N(R_i) = 1$. If $\mathcal{T}_n \neq \emptyset$, define $Y = \min \mathcal{T}_n$ to be the smallest of the indices in $\mathcal{T}_n$; otherwise, if $\mathcal{T}_n = \emptyset$, set $Y = -1$. Define the indicator random variable X_n as follows: If $\tau_n = |\mathcal{T}_n| = 0$ then $X_n = 0$; otherwise, if $\mathcal{T}_n = \{i_1, i_2, \ldots i_{\tau_n}\}$ and $i_1 < i_2 < \ldots < i_{\tau_n}$, then $X_n = 1$ iff $Q_{i_1} \cup R_{i_1}$ contains a two-point connected dominating set for $\mathcal{G}_n$.

Let $\mathcal{V} = \{v_1, v_2, \ldots, b_n\}$ be the set of vertices of $\mathcal{G}_n$, selected independently and uniform randomly from $D_1(o)$. Define $\mathcal{Z} = \mathcal{V} \bigcap D_{\delta_n}(o)$ to be set of vertices that lie near the origin o, and let $Z = |\mathcal{Z}|$ be the number of these points. Then

$$\Pr(X_n = 0) \leq \Pr\left(X_n = 0, \tau_n \neq 0, Z \leq \frac{2n^{1/3}}{(\log n)^2}\right) + \Pr(\tau_n = 0)$$
$$+\Pr\left(Z > \frac{2n^{1/3}}{(\log n)^2}\right). \tag{27}$$

Note that Z has a binomial distribution: $Z \overset{d}{=} Bin(n, \delta_n^2)$. If $\beta = \frac{2n^{1/3}}{(\log n)^2}$, then by Chernoff's inequality,

$$\Pr_(Z \geq \beta) \leq \exp(-n^{1/3}/4(\log n)^2). \tag{28}$$

By Lemma 4, $\Pr(\tau_n = 0) = O(\frac{\log^6 n}{n^{1/3}})$. Therefore

$$\Pr(X_n = 0) \leq \Pr(X_n = 0, \tau_n \neq 0, Z \leq \beta) + O(\frac{\log^6 n}{n^{1/3}}). \tag{29}$$

Now we decompose the first term on the right side of (29) according to the value of Y.

$$\Pr(X_n = 0, \tau_n \neq 0, Z \leq \beta) = \sum_{k=0}^{L_n - 1} \Pr(X_n = 0 | Y = k, Z \leq \beta)\Pr(Y = k, Z \leq \beta). \tag{30}$$

(The redundant condition $\tau_n \neq 0$ need not be included on the right side of (30) because it a consequence of the condition $Y \geq 0$.) We have

$$\Pr(X_n = 0 | Y = k, Z \leq \beta) = \sum_S \Pr(X_n = 0 | \mathcal{Z} = S, Y = k)\Pr(\mathcal{Z} = S \big| Y = k, Z \leq \beta) \tag{31}$$

where the sum is over subsets $S \subseteq [n]$ such that $2 \leq |S| \leq \beta$.

$$\Pr(X_n = 0 | \mathcal{Z} = S, Y = k) = 1 - \Pr(X_n = 1 | \mathcal{Z} = S, Y = k), \tag{32}$$

so it is enough to find a lower bound for $\Pr(X_n = 1 | \mathcal{Z} = S, Y = k)$.

To simplify notation, let $\gamma = X(\tilde{q}_0, \tilde{u}_0)$, and recall that $\gamma = O(\frac{1}{n \log^3 n})$.

In this section of the paper, define $|D_{\delta_n}(o)| = \frac{\pi}{n^{2/3}(\log n)^2}$ to be the area of the disk $D_{\delta_n}(o)$, and $|D_1(o)| = \pi$ = area of the unit disk centered at o. An important observation is that, once we have specified $n - |S|$ = the number of points that fall *outside* $D_{\delta_n}(o)$, the locations in $D_{\delta_n}(o)^c$ of these $n-|S|$ points are independent of the locations of the $|S|$ points *in* $D_{\delta_n}(o)$. Hence

$$\Pr(X_n = 1|\mathcal{Z} = S, Y = k) \geq \frac{\left(1 - \frac{|D_{\delta_n}(o)|}{|D_1(o)|} - \frac{\gamma}{|D_1(o)|}\right)^{n-|S|}}{\left(1 - \frac{|D_{\delta_n}(o)|}{|D_1(o)|}\right)^{n-|S|}} \tag{33}$$

$$\geq \left(1 - \frac{C}{n(\log n)^3}\right)^{n-|S|} \geq 1 - \frac{C'}{(\log n)^3} \tag{34}$$

for some constants C and C' which are independent of $\mathcal{Z}, Y$. Hence

$$\Pr(X_n = 0) \leq \frac{c}{(\log n)^3} \tag{35}$$

for some positive constant c that does not depend on n.

□

We note that the result obtained in Theorem 5 depends on a delicate trade-off: We must choose δ_n small enough and L_n large enough to guarantee that for any $q \in Q_i$ and any $u \in R_i$, where (Q_i, R_i) is a pair of opposite sectors of $D_{\delta_n}(o)$, there is high probability that none of the points $X_1, X_2, ..., X_n$ lie in the 'uncovered' region $(D_1(q) \bigcup D_1(u))^c \bigcap D_1(o)$. On the other hand, δ_n must not be so small or L_n so large that we cannot find (with high probability) some pair of opposite sectors (Q_i, R_i) such that there is some $X_j \in Q_i$ and $X_k \in R_i$. The necessity for this 'trade-off' stems from the fact that a unit disk centered at a point o cannot be completely covered with two unit disks having centers at points other than o, i.e. $D_1(o) \nsubseteq D_1(u) \bigcup D_1(v)$ for $u \neq o \neq v$.

5 Final Comments

The original question posed in the introduction concerned the typical size of a minimum connected dominating set in the random disk graph $\mathcal{G}_n^2$. Theorem 5 establishes that, with asymptotic probability one, $\mathcal{G}_n^2$ has a two-point connected dominating set. By Theorem 1, this two-point dominating set is also a minimum connected dominating set (with asymptotic probability one).

Theorem 5 was difficult because a unit disk, centered at a point o, cannot be completely covered with two unit disks having centers at points other than o. In contrast, one can easily find *three* points $u, v, w \in D_1(o) \setminus \{o\}$ such that $D_1(o) \subseteq D_1(u) \bigcup D_1(v) \bigcup D_1(w)$. Using this fact, the authors show in [4] that there is some α, with $0 < \alpha < 1$, such that for every $k \geq 3$ the probability that there does *not* exist a k-point connected dominating set in $\mathcal{G}_n^2$ is less than $3\alpha^n$. This exponential probability bound was used to analyze the performance of the Rule k local algorithm for constructing a connected dominating set in a wireless network model when $k \geq 3$. By comparing the exponential $O(\alpha^n)$ probability bound for $k \geq 3$ with the $O(\frac{1}{\log^3 n})$ bound for $k = 2$, we gain some insight into the empirical observation that the Rule k algorithm does not perform as well for $k = 2$ as it does for $k \geq 3$.

Finally, we have not determined the typical size of the minimum connected dominating set for dimensions $m > 2$. The case $m = 2$ was already challenging, and we did not we did not see how to extend our methods to the general case. We did prove in [3] that, when $m = 3$, the probability that there does *not* exist a 4-point CDS is exponentially small. Therefore, with with high probability the smallest CDS in $\mathcal{G}_n^3$ consists of either 2 or 3 vertices. It is reasonable to conjecture that an analogous statement holds for all $m \geq 2$: with asymptotic probability 1, $\mathcal{G}_n^m$ has an m point CDS.

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