COMPOUND RANDOM MAPPINGS

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Abstract

In this paper we introduce a compound random mapping model which can be viewed as a generalisation of the basic random mapping model considered by Ross [36] and Jaworski [25]. We investigate a particular example, the Poisson compound random mapping, and compare results for this model with results known for the well-studied uniform random mapping model. We show that although the structure of the components of the random digraph associated with a Poisson compound mapping differs from the structure of the components of the random digraph associated with the uniform model, the limiting distribution of the normalized order statistics for the sizes of the components is the same as in the uniform case, i.e. the limiting distribution is the *Poisson-Dirichlet* (1/2) distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \quad \text{for every} \quad i \geq 1\}$.

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1. Introduction and definitions

The study of random mapping models was initiated independently by several authors (see [6, 14, 15, 23, 30, 38] in the 1950s and the properties of these models have received much attention in the literature. In particular, these models have been useful as models

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for epidemic processes (see [7, 8, 9, 11, 16, 28, 29, 32, 33, 35]) and have provided the basis for tractable heuristic algorithms to solve various combinatorial optimization problems (see [19] and [21]). In this paper we introduce a compound random mapping model, $T_K(\Pi)$, which can be viewed as a generalisation of the random mapping model, $T_K(\pi)$, considered by Ross [36] (see also [2] and [11]) and by Jaworski [25], and as such it provides a richer class of models for applications. Before defining this new model, we review the construction of the basic model $T_K(\pi)$ and some of the known results for the basic model.

Fix $K > 0$ and let $\pi = (p_1, p_2, \ldots, p_K)$ be a fixed probability measure on the set $\{1, 2, \ldots, K\}$, then $T_K(\pi)$ is the random mapping of $\{1, 2, \ldots, K\}$ into itself with distribution given by

$$\Pr\Big\{T_K(\pi) = f\Big\} = \prod_{i=1}^{K} p_{f(i)}$$

for each $f \in \mathcal{M}_K$, where $\mathcal{M}_K$ is the set of all mappings of $\{1, 2, \ldots, K\}$ into itself. The random mapping $T_K(\pi)$ can be represented by a directed random graph $G_K(\pi)$ on vertices labelled $1, 2, \ldots, K$, such that a directed edge from vertex i to vertex j exists in $G_K(\pi)$ if and only if $T_K(\pi)(i) = j$. We note that since each vertex in $G_K(\pi)$ has out-degree 1, the components of $G_K(\pi)$ consist of directed cycles with directed trees attached. Alternatively, $T_K(\pi)$ can also be constructed as follows. Let $X_1, X_2, \ldots, X_K$ be i.i.d. random variables such that $\Pr\{X_i = j\} = p_j$ for all $1 \le i, j \le K$, then $T_K(\pi)$ is the random mapping which satisfies

$$T_K(\pi)(i) = j \quad \text{iff} \quad X_i = j$$

for all $1 \le i, j \le K$. In this construction of $T_K(\pi)$, the variables $X_1, X_2, \ldots X_K$ represent the independent 'choices' of the vertices $1, 2, \ldots, K$ in the random digraph $G_K(\pi)$.

The model which is best understood is the uniform random mapping, $T_K \equiv T_K(\pi)$, where π is the uniform measure on $\{1, 2, \ldots, K\}$. Much is known (see for example the monograph by Kolchin [31]) about the component structure of the random digraph $G_K \equiv G(T_K)$ which represents T_K. Aldous [1] has shown that the joint distribution of the normalized order statistics for the component sizes in G_K converges to the *Poisson-Dirichlet* (1/2) distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \le 1, x_i \ge$

$x_{i+1} \geq 0$ for every $i \geq 1\}$. Also, if M_k denotes the number of components of size k in G_K then the joint distribution of $(M_1, M_2, \ldots, M_b)$ is close, in the sense of total variation, to the joint distribution of a sequence of independent Poisson random variables when $b = o(K/\log K)$ (see Arratia et.al. [3], [4]) and from this result one obtains a functional central limit theorem for the component sizes (see also [17]). The asymptotic distributions of variables such as the number of predecessors and the number of successors of a vertex in G_K are also known (see [28, 29]).

There various ways that the basic random mapping model can be generalized (for an example see Mutafchiev [34] and Jaworski [27]). In this paper we generalize the basic model by introducing another layer of randomness into the model. In particular, let $W_1, W_2, \ldots$ be a sequence of i.i.d. non-negative random variables, let $N = N(K) \equiv \sum_{i=1}^{K} W_i$, and let Π denote the *random* probability measure on $\{1, 2, \ldots, K\}$ given by

$$\Pi = \begin{cases} \frac{1}{N}(W_1, W_2, \ldots, W_K), & \text{if } N = \sum_{i=1}^{K} W_i \neq 0 \\ (\frac{1}{K}, \frac{1}{K}, \ldots, \frac{1}{K}) & \text{otherwise.} \end{cases}$$

The distribution of the compound random mapping $T_K(\Pi)$ on the space $\mathcal{M}_K$ is specified by the distribution of $T_K(\Pi)$ *conditioned* on the random vector $(W_1, W_2, .., W_k)$. In particular, for any $f \in \mathcal{M}_K$ and $(w_1, w_2, .., w_K) \in (R^+)^K \setminus \{\vec{0}\}$, we define

$$\Pr\Big\{T_K(\Pi) = f \,\Big|\, (W_1, W_2, \ldots, W_K) = (w_1, w_2, .., w_K)\Big\} = \prod_{i=1}^{K} \frac{w_{f(i)}}{N} \tag{1.1}$$

and when $N = 0$, we define

$$\Pr\Big\{T_K(\Pi) = f \,\Big|\, N = 0\Big\} = \Pr\Big\{T_K = f\Big\} = \left(\frac{1}{K}\right)^K. \tag{1.2}$$

It follows from (1.1) and (1.2) that the distribution of the compound random mapping $T_K(\Pi)$ on the space $\mathcal{M}_K$ is given by

$$\begin{aligned} &\Pr\Big\{T_K(\Pi) = f\Big\} \\ &= \int_{(R^+)^K} \Pr\Big\{T_K(\Pi) = f \,\Big|\, (W_1, W_2, \ldots, W_K) = (w_1, w_2, .., w_K)\Big\} dF(w_1, .., w_k) \end{aligned}$$

for any $f \in \mathcal{M}_K$, where F is the joint distribution function for $(W_1, W_2, \ldots, W_K)$. The variables $W_1, W_2, \ldots,$ can be viewed as relative 'weights' on the vertices $1, 2, \ldots, K$. Observe that in the case where $W_i \equiv c \geq 0$, we have $T_K(\Pi) \equiv T_K$.

The introduction of an extra layer of randomness in the model $T_K(\Pi)$ complicates the investigation of the structure of $G_K(\Pi)$, the random digraph associated with $T_K(\Pi)$. In particular, if the weight variables $W_1, W_2, \ldots$ are not degenerate, then the distribution of $T_K(\Pi)$ is not uniform on the space of mappings $\mathcal{M}_K$ and we cannot directly use the combinatorial tools which have been useful in the investigation of the structure of uniform random digraph G_K. Nevertheless, some simple observations concerning the structure of $G_K(\Pi)$ are possible. For example, let $V_0(f)$ denote the number of vertices with in-degree 0 in the digraph $G(f)$ which represents the mapping f and suppose that

$$\hat{p} \equiv \Pr\{W_1 = 0\} > e^{-1}.$$

Then as $K \to \infty$, we have

$$E(V_0(T(K))) \sim e^{-1}K$$

whereas

$$E(V_0(T_K(\Pi))) \geq \hat{p}K > e^{-1}K.$$

In other words, in this case the components of $G_K(\Pi)$ are 'leafier' than the components of G_K.

More generally, it can be shown that for some other characteristics the uniform model G_K is an "extremal" case for the compound model $G_K(\Pi)$. For example, consider the probability that $G_K(\Pi)$ is connected. Ross [36] in his paper on the $T_K(\pi)$ model considered the probability that $G_K(\pi)$ is connected in terms of Schur convex functions (where $\pi = (p_1, p_2, \ldots, p_K)$ is a fixed probability measure). It is straightforward to verify ([22], [36]), that this probability is Schur convex function of the vector $(p_1, p_2, \ldots, p_K)$ and therefore it is minimized for $p_i \equiv 1/K$, i.e. for the uniform model. It follows that

$$\begin{aligned}
&\Pr\Big\{G_K(\Pi) \text{ is connected}\Big\} \\
&= \int_{(R^+)^K} \Pr\Big\{G_K(\Pi) \text{ is connected} \,\Big|\, W_i = w_i,\ i = 1, 2, \ldots, K\Big\} dF(w_1, .., w_K) \\
&= \int_{(R^+)^K} \Pr\Big\{G_K(\pi) \text{ is connected}\Big\} dF(w_1, .., w_K) \\
&\geq \int_{(R^+)^K} \Pr\Big\{G_K \text{ is connected}\Big\} dF(w_1, .., w_K) = \Pr\Big\{G_K \text{ is connected}\Big\},
\end{aligned}$$

i.e. probability that $G_K(\Pi)$ is connected is always bounded below by the probability that G_K is connected. Probabilities and expected values for other characteristics of $G_K(\pi)$ can also be shown to be Schur convex functions and in these cases, as in the above calculation, we obtain bounds for the compound model $G_K(\Pi)$ in terms of bounds for the uniform model G_K. However, to obtain more than general bounds for the compound model it is necessary to consider particular examples.

In the remainder of this paper we assume that the weight variables $W_1, W_2, ..$ are i.i.d. Poisson variables with mean $\lambda > 0$. In this case, we say that $T_K(\lambda) \equiv T_K(\Pi)$ is a Poisson compound mapping and $G_K(\lambda)$ denotes the associated random digraph on vertices labelled $1, 2, \ldots, K$. Poisson compound mappings are a tractable class of examples because we can exploit a connection between the component structure of Poisson compound mappings and the component structure of random bipartite mappings. In the Section 2 we prove the key lemma which allows us to translate results for random bipartite mappings into results for the Poisson model and we state our main results. In Sections 3 and 4 we use this lemma to establish our main results.

2. Key lemma and statement of main results

The key to the main results of this paper is the following lemma which establishes the connection between the component structure of Poisson compound mappings and the component structure of random bipartite mappings. A random bipartite mapping $T_{K,L}$ of a finite set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 = \{1, 2, \ldots, K\}$ and $\mathcal{V}_2 = \{K+1, K+2, \ldots, K+L\}$ into itself assigns independently to each $i \in \mathcal{V}_1$ its unique image $j \in \mathcal{V}_2$ with probability $1/L$ and to each $i \in \mathcal{V}_2$ its unique image $j \in \mathcal{V}_1$ with probability $1/K$. The mapping $T_{K,L}$ can be represented by a random bipartite digraph $G(T_{K,L})$ on a set of 'red' labelled vertices corresponding to the set $\mathcal{V}_1$ and a set of 'blue' labelled vertices corresponding to the set $\mathcal{V}_2$. In particular, $G(T_{K,L})$ has a directed edge from red (blue) vertex i to blue (red) vertex j if and only if $T_{K,L}(i) = j$.

Lemma 1. *For any integers $K, L > 0$ and $\lambda > 0$, and any mapping $f \in \mathcal{M}_K$*

$$\Pr\Big\{T_K(\lambda) = f \,\Big|\, N(K) = L\Big\} = \Pr\Big\{T_{K,L}^2 = f\Big\}$$

where $T_{K,L}^2 = T_{K,L} \circ T_{K,L}$ is a random mapping of the vertex set $\mathcal{V}_1 = \{1, 2, \ldots, K\}$

into itself.

Proof. Fix $K, L > 0$, $\lambda > 0$, and the mapping $f \in \mathcal{M}_K$. Let $U_1, U_2, \dots, U_L$ be i.i.d. uniform random variables on the interval $[0, \lambda K)$, and let $X_1, X_2, \dots, X_K$ be i.i.d. discrete random variables such that for each $1 \le j \le K$ and $1 \le i \le L$, $\Pr\{X_j = K + i\} = \frac{1}{L}$. In addition, suppose that the variables $X_1, X_2, \dots, X_K$ are independent of the variables $U_1, U_2, \dots, U_L$. A uniform random bipartite mapping $T_{K,L}$ can be constructed as follows: for any $j \in \mathcal{V}_1, K + i \in \mathcal{V}_2$, $T_{K,L}(j) = K + i$ if and only if $X_j = K + i$ and $T_{K,L}(K + i) = j$ if and only if $U_i \in [\lambda(j - 1), \lambda j)$. Now let $Y_j = |\{i : U_i \in [\lambda(j - 1), \lambda j)\}| = |\{K + i \in V_2 : T_{K,L}(K + i) = j\}|$ for $j = 1, 2, \dots, K$. It is easy to check that for any $(y_1, y_2, \dots, y_K) \in (Z^+)^K$ such that $\sum_{j=1}^{K} y_j = L$, we have

$$\Pr\Big\{T^2_{K,L} = f \,\Big|\, (Y_1, Y_2, \dots, Y_K) = (y_1, y_2, \dots, y_K)\Big\} = \Pr\Big\{T_{\vec{\pi}} = f\Big\} = \prod_{i=1}^{K} \tfrac{y_{f(i)}}{L}$$
$$= \Pr\Big\{T_K(\lambda) = f \,\Big|\, (W_1, W_2, \dots, W_K) = (y_1, y_2, \dots, y_K), N(K) = L\Big\}$$

where $\vec{\pi} = \frac{1}{N}(y_1, y_2, \dots, y_K)$. So it suffices to prove

$$\Pr\Big\{(W_1, W_2, \dots, W_K) = (y_1, y_2, \dots, y_K) \,\Big|\, N(K) = L\Big\}$$
$$= \Pr\Big\{(Y_1, Y_2, \dots, Y_K) = (y_1, y_2, \dots, y_K)\Big\} \tag{2.1}$$

for all $(y_1, y_2, \dots, y_K) \in (Z^+)^K$ such that $\sum_{j=1}^{K} y_j = L$.

To see that (2.1) holds, recall that if N_t is a homogeneous Poisson process with rate 1, then the random variables $(W_1, W_2, \dots, W_K)$ have the same joint distribution as the variables $(N_\lambda, N_{2\lambda} - N_\lambda, \dots, N_{\lambda K} - N_{\lambda(K-1)})$. It is also well known (see Ross [37], p.67) that

$$\Pr\Big\{(N_\lambda, N_{2\lambda} - N_\lambda, \dots, N_{\lambda K} - N_{\lambda(K-1)}) = (y_1, y_2, \dots, y_K) \,\Big|\, N_{\lambda K} = L\Big\}$$
$$\Pr\Big\{(Y_1, Y_2, \dots, Y_K) = (y_1, y_2, \dots, y_K)\Big\}$$

where the variables Y_j, $1 \le j \le K$ are as defined above. Equation (2.1) now follows and this completes the proof of the lemma.

Since $N(K) \sim Poisson(\lambda K)$, it follows from Lemma 1 that for every $K > 0$, $\lambda > 0$, and $f \in \mathcal{M}_K$,

$$\Pr\Big\{T_K(\lambda) = f\Big\} = \sum_{L=1}^{\infty} \Pr\Big\{T^2_{K,L} = f\Big\} \frac{(\lambda K)^L e^{-\lambda K}}{L!} + \left(\frac{1}{K}\right)^K e^{-\lambda K}. \tag{2.2}$$

Using this relationship, we can translate many known results for bipartite random mappings (see [24], [26]) into results for compound Poisson mappings. For example, for any mapping $f \in \mathcal{M}_K$, we say $v \in \{1, 2, \ldots, K\}$ is a *cyclical vertex* of f if v lies on a cycle in the digraph $G(f)$ which represents f and we define $q(f)$ to be the number of cyclical vertices in $G(f)$. ¿From (2.2) we obtain

$$E_K^\lambda(q) \equiv E(q(T_K(\lambda))) = \sum_{L=1}^{\infty} E_{K,L}(q) \frac{(\lambda K)^L e^{-\lambda K}}{L!} + E_K(q) e^{-\lambda K}, \tag{2.3}$$

where $E_{K,L}(q) \equiv E(q(T^2_{K,L}))$ and $E_K(q) \equiv E(q(T_K))$. Now from [26] we have the explicit expression

$$E_{K,L}(q) = \sum_{i=1}^{\min\{K,L\}} \frac{(K)_i}{K^i} \frac{(L)_i}{L^i} = \sqrt{\frac{\pi}{2} \frac{KL}{K+L}} (1 + \varepsilon_{K,L}), \tag{2.4}$$

where $|\varepsilon_{K,L}| \leq \frac{C}{\min\{K,L\}}$ for some constant $C > 0$. To translate this result into a result for compound Poisson mappings, we use a Chernoff-type bound (see Ross [37])

$$\Pr\left\{|N(K) - \lambda K| > \beta(\lambda K)^\alpha\right\} \leq C_\lambda \exp(-\beta(\lambda K)^{\alpha - \frac{1}{2}}) \tag{2.5}$$

for the Possion distribution, where $\alpha > 1/2$, $\beta > 0$, and $C_\lambda > 0$ is a constant which depends on $\lambda > 0$. It follows from (2.3)-(2.5), that

$$\begin{aligned} E_K^\lambda(q) &= \sum_{|L-\lambda K| < (\lambda K)^{3/4}} E_{K,L}(q) \frac{(\lambda K)^L e^{-\lambda K}}{L!} + O\left(K \exp(-(\lambda K)^{1/4})\right) \\ &= \sqrt{\frac{\pi \lambda K}{2(1+\lambda)}} (1 + o(1)). \end{aligned}$$

On the other hand, $E_K(q) \sim \sqrt{\pi K/2}$ (see [31]), so *on average* $G_K(\lambda)$, the random digraph which represents $T_K(\lambda)$, has *fewer* cyclical vertices than the uniform random digraph G_K. By a similar arguments and using results from [26], we obtain

$$E_K^\lambda(p_v) \equiv E(p_v(T_K(\lambda))) \sim \sqrt{\frac{\pi \lambda K}{2(1+\lambda)}}$$

where, for any $f \in \mathcal{M}_K$ and $v \in \{1, 2, \ldots, K\}$,

$$p_v(f) = \left|\{y : f^l(y) = v \quad \text{for} \quad \text{some} \quad l = 0, 1, 2, ..\}\right|.$$

On the other hand, for uniform random mappings (see [31])

$$E_K(p_v) \equiv E(p_v(T_K)) \sim \sqrt{\frac{\pi K}{2}}.$$

Since $p_v(f)$ can be interpreted as the number of predecessors of vertex v in $G(f)$, we see that for any $\lambda > 0$, a vertex v has *fewer* predecessors on average in $G_K(\lambda)$ than in G_K. These results indicate that the structure of the components of $G_K(\lambda)$ differs from that of the components of G_K. The question arises: how do these differences in component *structure* affect the distribution of the *sizes* of the components of $G_K(\lambda)$? In this paper we show that, perhaps surprisingly, for every $\lambda > 0$ the joint distribution of the normalized order statistics of the components in $G_K(\lambda)$ has the *same* limiting distribution as the joint distribution of the normalized order statistics of the components in G_K. The limiting distribution in both cases is the *Poisson-Dirichlet* (1/2) distribution on the simplex ∇. To prove this result, we first establish the limiting distribution for the size of a component in $G_K(\lambda)$ containing a given vertex and this result may also be of independent interest.

Before stating our main results, we give a convenient characterization of the *Poisson -Dirichlet* (θ) distribution (denoted $\mathcal{PD}(\theta)$) which also yields a useful principle for establishing convergence in distribution to the $\mathcal{PD}(\theta)$ distribution on ∇. Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of i.i.d. random variables such that each Y_i has a *Beta*(θ) distribution (θ >0) with density $h(y) = \theta(1-y)^{\theta-1}$ on the unit interval $(0,1)$. Now define a transformation ϕ of the sequence $(Y_1, Y_2, ..)$ such that $\phi(Y_1, Y_2, \ldots) = (\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, ..)$ where $\tilde{Y}_1 = Y_1$ and $\tilde{Y}_n = Y_n(1-Y_1)(1-Y_2)\cdots(1-Y_{n-1})$ for $n > 1$, and observe that $(\tilde{Y}_1, \tilde{Y}_2, \ldots) \in \tilde{\nabla} = \{\{x_i\} : x_i \geq 0, \sum x_i \leq 1\}$. Finally, define $\psi : \tilde{\nabla} \to \nabla$ such that $(\psi\{x_i\})_k$ is the kth largest term in the sequence $\{x_i\} \in \tilde{\nabla}$; then the random sequence $\psi \circ \phi(Y_1, Y_2, \ldots) = (Q_1, Q_2, Q_3, \ldots) \in \nabla$ has a $\mathcal{PD}(\theta)$ distribution. The following convergence principle is an important consequence of this characterization: suppose that $(Y_1(n), Y_2(n), \ldots)$ is a sequence of random variables such that the joint distribution of $(Y_1(n), Y_2(n), \ldots)$ converges to the joint distribution of the variables $(Y_1, Y_2, \ldots)$, then the joint distribution of the random sequence $\psi \circ \phi(Y_1(n), Y_2(n), \ldots) = (Q_1(n), Q_2(n), \ldots)$ converges to the $\mathcal{PD}(\theta)$ distribution. For further details see Hansen [18] and the references therein.

To see how the convergence principle can be applied in the context of Poisson compound mappings, we introduce some additional notation. Suppose that T is a random mapping on $\{1, 2, \ldots, K\}$ and let $\mathcal{C}_1 = \mathcal{C}_1(T)$ denote the component in $G(T)$ which contains the vertex labelled 1. If $\mathcal{C}_1 \neq G(T)$, then let $\mathcal{C}_2 = \mathcal{C}_2(T)$

denote the component in $G(T) \setminus \mathcal{C}_1$ which contains the smallest vertex; otherwise, set $\mathcal{C}_2 = \emptyset$. For $k > 2$ we define $\mathcal{C}_t$ iteratively: If $G(T) \setminus (\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{t-1}) \neq \emptyset$, then let $\mathcal{C}_t$ denote the component in $G(T) \setminus (\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{t-1})$ which contains the smallest vertex; otherwise, set $\mathcal{C}_t = \emptyset$. For $t \geq 1$, let $C_t = |\mathcal{C}_t|$ and define the sequence $(Z_1, Z_2, \ldots) = (Z_1(T), Z_2(T), \ldots)$ by

$$Z_1 = \frac{C_1}{K},\ Z_2 = \frac{C_2}{K - C_1}, \ldots,\ Z_t = \frac{C_t}{K - C_1 - C_2 - \ldots - C_{t-1}}, \ldots$$

where $Z_t = 0$ if $K - C_1 - C_2 - \ldots - C_{t-1} = 0$. In Section 3 we show that for each $t \geq 1$ and $0 < a_i < b_i < 1,\ i = 1, 2, \ldots, t$

$$\lim_{K \to \infty} \Pr\Big\{ a_i < Z_i(\lambda, K) \leq b_i,\ i = 1, 2, .., t \Big\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} . \tag{2.6}$$

where $Z_i(\lambda, K) \equiv Z_i(T_K(\lambda))$. We establish (2.6) by an inductive argument, the first step of which is established in Section 3, where we prove

Theorem 1. *Suppose that $\lambda > 0$ is fixed, then for every $0 < a < b < 1$*

$$\Pr\Big\{ aK < C_1(\lambda, K) \leq bK \Big\} \to \int_a^b \frac{du}{2\sqrt{1-u}} \qquad \text{as} \quad K \to \infty ,$$

where $C_1(\lambda, K) = C_1(T_K(\lambda))$.

To describe Theorem 2 below, let $D_1(\lambda, K)$ denote the size of the largest connected component in $G_K(\lambda)$, let $D_2(\lambda, K)$ denote the size of the second largest component and so on. It is easy to check that

$$\psi \circ \phi(Z_1(\lambda, K), Z_2(\lambda, K), \ldots) = \left(\frac{D_1(\lambda, K)}{K}, \frac{D_2(\lambda, K)}{K}, \ldots \right) ,$$

so using the convergence principle for the Poisson-Dirichlet distribution, we obtain from (2.6)

Theorem 2. *For any fixed $\lambda > 0$,*

$$\left(\frac{D_1(\lambda, K)}{K}, \frac{D_2(\lambda, K)}{K}, \ldots \right) \quad \xrightarrow{d} \quad \mathcal{PD}(1/2) \quad \text{as} \quad K \to \infty ,$$

where $D_1(\lambda, K), D_2(\lambda, K), \ldots$ are as defined above, and $\mathcal{PD}(1/2)$ denotes the Poisson -Dirichlet(1/2) distribution on the simplex

$$\nabla = \Big\{ \{x_i\} : \sum x_i \leq 1, \quad x_i \geq x_{i+1} \geq 0 \quad \text{for every} \quad i \geq 1 \Big\} .$$

3. The size of a connected component

In this section we prove Theorem 1. ¿From Lemma 1 we obtain the identity

$$\begin{aligned}
&\Pr\{aK < C_1(\lambda, K) \le bK\} \\
&= \sum_{L>0} \Pr\{aK < C_1(T^2_{K,L}) \le bK\} \Pr\{N(K) = L\} \\
&+ \Pr\{aK < C_1(T(K)) \le bK\} \Pr\{N(K) = 0\}.
\end{aligned} \tag{3.1}$$

Now for values of L which are neither too big nor too small, $\Pr\{aK < C_1(T^2_{K,L}) \le bK\}$ is 'close' to $\int_a^b \frac{dx}{2\sqrt{1-x}}$. More precisely, we have

Lemma 2. *Fix $0 < \xi < \eta$, then for all $K, L > 0$ such that $\xi K \le L \le \eta K$ and for every $0 < a < b < 1$, there is a constant $C(a, b, \xi, \eta)$ which only depends on a, b, ξ and η, such that*

$$\left| \Pr\Big\{aK < C_1(T^2_{K,L}) \le bK\Big\} - \int_a^b \frac{dx}{2\sqrt{1-x}} \right| \le \frac{C(a,b,\xi,\eta)}{K^{1/16}}.$$

Proof. Fix $0 < \xi < \eta$ and $0 < a < b < 1$, then there exists $K(a, b, \xi, \eta) > 0$ such that $(\eta/\xi)K^{-3/8} \le \frac{1}{2}\min\{a, 1-a, b, 1-b\}$ whenever $K > K(a, b, \xi, \eta)$. Throughout the proof $C(a, b, \xi, \eta)$ will denote any constant which may depend on a, b, ξ and η but which does not depend on K and which statisfies $C(a, b, \xi, \eta) \ge 2K(a, b, \xi, \eta)$. Now fix $K > K(a, b, \xi, \eta)$ and L such that $\xi K \le L \le \eta K$, and suppose m is such that $aK < m \le bK$. Let $x = m/K$ and $\alpha = \frac{L}{K}$ (so $x \in (a, b]$ and $\alpha \in [\xi, \eta]$), then

$$\Pr\{\, C_1(T^2_{K,L}) = m\} = \Pr\{R_1 = m\} = \sum_{-\lceil Lx\rceil < j \le L - \lceil Lx\rceil} \Pr\{R_1 = m, B_1 = \lceil Lx\rceil + j\}$$

where R_1 is the number of red vertices and B_1 is the number of blue vertices in the connected component which contains the vertex 1 in $G(T_{K,L})$. We split the above sum into two sums:

$$\text{(i)} \quad \sum_{|j| \le \tau\sqrt{\alpha K}} \qquad\qquad \text{(ii)} \quad \sum_{|j| > \tau\sqrt{\alpha K}} \tag{3.2}$$

where $\tau = K^{1/8}$ and consider each sum separately. Approximations of the terms in each sum depend on the following lemma which we state without proof.

Lemma 3. ([20]) *For* $k = 0, 1, \ldots, K-1$ *and* $l = 1, \ldots, L$ *we have*

$$\begin{aligned}
&\Pr\Big\{R_1 = k+1,\ B_1 = l\Big\} \\
&= \binom{K-1}{k}\binom{L}{l}\left(\frac{k+1}{K}\right)^{l-1}\left(1-\frac{k+1}{K}\right)^{L-l}\left(\frac{l}{L}\right)^{k}\left(1-\frac{l}{L}\right)^{K-1-k} \\
&\quad \times \frac{1}{KL}\sum_{j=1}^{min\{l,k+1\}} \frac{(l)_j}{l^j}\frac{(k+1)_j}{(k+1)^j}\,(k+l+1-j)\,.
\end{aligned}$$

We note from Lemma 3 that the expression for $\Pr\{R_1 = m,\ B_1 = l\}$ where $l = \lceil Lx \rceil + j$, can be split into two factors. The first factor

$$\binom{K-1}{m-1}\left(\frac{l}{L}\right)^{m-1}\left(1-\frac{l}{L}\right)^{K-m}\binom{L}{l}\left(\frac{m}{K}\right)^{l}\left(1-\frac{m}{K}\right)^{L-l}$$

is the product of binomial probabilities and, provided $|j| \leq \tau\sqrt{\alpha K}$, an approximation for this expression with an appropriate error bound can be obtained by following the proof of the de Moivre-Laplace Theorem (see Feller [13], p.182). In particular,

$$\begin{aligned}
&\binom{K-1}{m-1}\left(\frac{l}{L}\right)^{m-1}\left(1-\frac{l}{L}\right)^{K-m}\binom{L}{l}\left(\frac{m}{K}\right)^{l}\left(1-\frac{m}{K}\right)^{L-l} \\
&= \frac{1}{2\pi x(1-x)\sqrt{KL}}\cdot \exp\left(\frac{-y^2}{2}\left(\frac{1+\alpha}{\alpha x(1-x)}\right)\right)\cdot(1+\tilde{\rho}_j(x))
\end{aligned} \tag{3.3}$$

where $y = j/\sqrt{\alpha K}$, $x = m/K$, and $|\tilde{\rho}_j(x)| \leq C(a,b,\xi,\eta)K^{-1/16}$. We note that to obtain the bound for $|\tilde{\rho}_j(x)|$, we use the inequality $K > K(a,b,\xi,\eta)$.

Next, for $k+1 = m$ and $l = \lceil Lx \rceil + j$ with $|j| \leq \tau\sqrt{\alpha K}$, one can show, as in the proof of Theorem 7 in [26], that

$$\frac{1}{mL}\sum_{i=1}^{\min\{m,l\}} \frac{(l)_i (m)_i}{(l)^i (m)^i}(m+l-i) = \sqrt{\frac{\pi x(K+L)}{2KL}}\cdot(1+\hat{\varepsilon}(x,j)) \tag{3.4}$$

where

$$\hat{\varepsilon}(x,j) \leq \frac{C(a,b,\xi,\eta)}{K^{1/8}}\,.$$

¿From (3.3) and (3.4) we obtain for $a < x = \frac{m}{K} \leq b$, and $|j| \leq \tau\sqrt{\alpha K}$

$$\begin{aligned}
&\Pr\big\{R_1 = m, B_1 = \lceil Lx \rceil + j\big\} = \Pr\big\{R_1 = m, B_1 = \lceil Lx \rceil + y\sqrt{\alpha K}\big\} \\
&= \frac{1}{K}\cdot\frac{1}{2\sqrt{1-x}}\frac{1}{\sqrt{\alpha K}}\cdot\sqrt{\frac{1+\alpha}{2\pi\alpha x(1-x)}}\exp\left(\frac{-y^2}{2}\left(\frac{1+\alpha}{\alpha x(1-x)}\right)\right)\cdot(1+\rho_j(x))
\end{aligned}$$

where $|\rho_j(x)| \leq C(a,b,\xi,\eta)K^{-1/16}$ and $y = j/\sqrt{\alpha K}$. It follows that for $a < x = \frac{m}{K} \leq b$

$$
\begin{aligned}
&\sum_{|j|\le\tau\sqrt{\alpha K}} \Pr\bigl\{R_1=m, B_1=\lceil Lx\rceil+j\bigr\}\\
&=\frac{1}{K}\cdot\frac{1}{2\sqrt{1-x}}\sum_{|j|\le\tau\sqrt{\alpha K}}\frac{1}{\sqrt{\alpha K}}\cdot\sqrt{\frac{1+\alpha}{2\pi\alpha x(1-x)}}\exp\left(\frac{-y^2(1+\alpha)}{2\alpha x(1-x)}\right)\cdot(1+\rho_j(x))\\
&=\frac{1}{K}\cdot\frac{1}{2\sqrt{1-x}}\cdot(1+\delta_x) \qquad (3.5)
\end{aligned}
$$

where $|\delta_x|\le C(a,b,\xi,\eta)\cdot K^{-1/16}$.

It remains to determine a bound for the second sum in (3.2). Since this is a 'two-sided' sum, we consider one side of the sum; the other case follows by similar calculations. The first step is to note (see [20], p.324) that for all $k=0,1,\dots,K-1$ and $l=1,\dots,L$

$$
\Pr\bigl\{R_1=m, B_1=l\bigr\}\le\binom{L}{l}\left(\frac{m}{K}\right)^l\left(1-\frac{m}{K}\right)^{L-l}=\binom{L}{l}x^l(1-x)^{L-l}.
$$

It follows that

$$
\begin{aligned}
\sum_{j\ge\tau\sqrt{\alpha K}}\Pr\bigl\{R_1=m,\ B_1=\lceil Lx\rceil+j\bigr\} &\le \sum_{l\ge\lceil Lx\rceil+\tau\sqrt{\alpha K}}\binom{L}{l}x^l(1-x)^{L-l}\\
&\le \Pr\Bigl\{\frac{X-Lx}{\sqrt{Lx(1-x)}}\ge\frac{\tau}{\sqrt{x(1-x)}}\Bigr\}\\
&\le C(a,b,\xi,\eta)\exp\left(\frac{-\tau}{\sqrt{x(1-x)}}\right)\\
&\le C(a,b,\xi,\eta)\exp(-K^{1/16})
\end{aligned}
\qquad (3.6)
$$

where $X\sim Bin(L,x)$ and $\tau=K^{1/8}$. The third inequality follows from Chernoff-type bounds for tail probabilities of the binomial distribution. Similarly,

$$
\sum_{j\le-\tau\sqrt{\alpha K}}\Pr\bigl\{R_1=m,\ B_1=\lceil Lx\rceil+j\bigr\}\le C(a,b,\xi,\eta)\exp(-K^{1/16}). \qquad (3.7)
$$

Combining the bounds (3.6) and (3.7) and approximation (3.5), we obtain

$$
\Pr\bigl\{C_1(T^2_{K,L})=m\bigr\}=\frac{1}{K}\cdot\frac{1}{2\sqrt{1-m/K}}\cdot(1+\delta_x)+\gamma_m
$$

where

$$
|\delta_x|\le C(a,b,\xi,\eta)\cdot K^{-1/16}
$$

and

$$\gamma_m = \sum_{|j|\geq\tau\sqrt{\alpha K}} \Pr\big\{R_1 = m, B_1 = \lceil Lx\rceil + j\big\} \leq C(a,b,\xi,\eta)\exp(-K^{1/16})\,.$$

Hence

$$\Pr\big\{aK < C_1(T^2_{K,L}) \leq bK\big\} = \sum_{aK<m\leq bK} \frac{1}{K}\cdot\frac{1}{2\sqrt{1-m/K}}\cdot(1+\delta_x) + \sum_{aK<m\leq bK}\gamma_m$$

and it follows that

$$\Big|\Pr\big\{aK < C_1(T^2_{K,L}) \leq bK\big\} - \int_a^b \frac{dx}{2\sqrt{1-x}}\Big| \leq \frac{C(a,b,\xi,\eta)}{K^{1/16}}$$

in the case $K > K(a,b,\xi,\eta)$. The result holds trivially in the case $K \leq K(a,b,\xi,\eta)$ since $C(a,b,\xi,\eta) \geq 2K(a,b,\xi,\eta)$.

Proof of Theorem 1. Fix $K > 0$, then it follows from Lemma 1 that

$$\begin{aligned}\sum_{\frac{\lambda K}{2}\leq L}^{\lfloor 3\lambda K/2\rfloor} &\Pr\big\{aK < C_1(T^2_{K,L}) \leq bK\big\}\Pr\big\{N(K)=L\big\} \leq \Pr\big\{aK < C_1(\lambda,K)\leq bK\big\} \\ &\leq \sum_{\frac{\lambda K}{2}\leq L}^{\lfloor 3\lambda K/2\rfloor} \Pr\big\{aK < C_1(T^2_{K,L}) \leq bK\big\}\Pr\big\{N(K)=L\big\} \\ &\qquad + \Pr\big\{|N(K)-\lambda K| > \frac{\lambda K}{2}\big\}.\end{aligned} \tag{3.8}$$

¿From Lemma 2, with $\xi = \frac{\lambda}{2}$ and $\eta = \frac{3\lambda}{2}$, we obtain

$$\begin{aligned}&\left(\int_a^b \frac{dx}{2\sqrt{1-x}} - \frac{C(a,b,\frac{\lambda}{2},\frac{3\lambda}{2})}{K^{1/16}}\right)\left(1-\Pr\Big\{|N(K)-\lambda K| > \frac{\lambda K}{2}\Big\}\right) \\ &\leq \sum_{\frac{\lambda K}{2}\leq L}^{\lfloor 3\lambda K/2\rfloor} \Pr\Big\{aK < C_1(T^2_{K,L}) \leq bK\Big\}\Pr\Big\{N(K)=L\Big\} \\ &\leq \int_a^b \frac{dx}{2\sqrt{1-x}} + \frac{C(a,b,\frac{\lambda}{2},\frac{3\lambda}{2})}{K^{1/16}}.\end{aligned} \tag{3.9}$$

Finally, from (2.5) we obtain

$$\Pr\Big\{|N(K)-\lambda K| > \frac{\lambda K}{2}\Big\} \leq C_\lambda \exp(-\sqrt{\lambda K}/2) \tag{3.10}$$

and the theorem follows from inequalities (3.8)-(3.10).

4. Order statistics for component sizes

In this section we prove Theorem 2. By the convergence principle outlined in Section 1, it is enough to show that for any integer $t \geq 1$ and any $0 < a_i < b_i < 1$, $i = 1, 2, \ldots, t$,

$$\lim_{K\to\infty} \Pr\Big\{a_i < Z_i(\lambda, K) \leq b_i, \quad i = 1, 2, \ldots, t\Big\} = \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{\sqrt{1-u}}. \tag{4.1}$$

To establish (4.1), we condition on the value of $N(K)$ and appeal to the following lemma.

Lemma 4. *Suppose $\lambda > 0$, $t \in Z^+$, and $0 < a_i < b_i < 1$, $1 \leq i \leq t$ are fixed. Then for every $K > 0$ and $\frac{\lambda}{2}K < L < \frac{3\lambda}{2}K$, there is a constant C which does not depend on K (but which may depend on λ, t, and $a_1, a_2, \ldots, a_t$, $b_1, b_2, \ldots, b_t$), such that*

$$\Big| \Pr\Big\{a_i < Z_i(T^2_{K,L}) \leq b_i : \ \ i = 1, 2, \ldots, t\Big\} - \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{\sqrt{1-u}} \Big| \leq \frac{C}{K^{1/16}}.$$

Before proving this result, we need to extend our notation. For any random bipartite mapping $T_{K,L}$ let $\mathcal{C}_1(T_{K,L})$ denote the component in $G(T_{K,L})$ which contains the vertex labelled 1. If $\mathcal{C}_1(T_{K,L}) \neq G(T_{K,L})$, then let $\mathcal{C}_2(T_{K,L})$ denote the component in $G(T_{K,L}) \setminus \mathcal{C}_1(T_{K,L})$ which contains the smallest vertex; otherwise, set $\mathcal{C}_2(T_{K,L}) = \emptyset$. For $k > 2$ we define $\mathcal{C}_t(T_{K,L})$ iteratively: If $G(T_{K,L}) \setminus (\mathcal{C}_1(T_{K,L}) \cup \ldots \cup \mathcal{C}_{t-1}(T_{K,L})) \neq \emptyset$, let $\mathcal{C}_t(T_K, L)$ denote the component in $G(T_{K,L}) \setminus (\mathcal{C}_1(T_{K,L}) \cup \ldots \cup \mathcal{C}_{t-1}(T_{K,L}))$ which contains the smallest vertex; otherwise, set $\mathcal{C}_t(T_{K,L}) = \emptyset$. In addition, let R_i denotes the number of red vertices and B_i denotes the number of blue vertices in $\mathcal{C}_i(T_{K,L})$ and note that $R_i = C_i(T^2_{K,L})$ for $i \geq 1$. $\ldots$. Finally, let $K_1 = K$, $L_1 = L$, and for $i \geq 2$,

$$K_i = K_{i-1} - R_{i-1}; \qquad L_i = L_{i-1} - B_{i-1};$$

and note that for $i \geq 2$, K_i and L_i are random variables. With this notation we have $Z_i(T^2_{K,L}) = C_i(T^2_{K,L})/K_i = R_i/K_i$ whenever $K_i > 0$.

Proof of Lemma 4. Fix $K > 0$ and $L > 0$ such that $\frac{\lambda}{2}K \leq L \leq \frac{3\lambda}{2}K$. For conciseness, we introduce

$$\mathcal{A}_j = \big\{a_i < Z_i(T^2_{K,L}) \leq b_i : \quad i = 1, 2, \ldots, j\big\} \quad \text{for} \quad j = 1, 2, \ldots, t\,,$$

and we write

$$\Pr\{a_i < Z_i(T^2_{K,L}) \le b_i : 1 \le i \le t\} = \Pr\{\mathcal{A}_t\} = \Pr\{\mathcal{B}_t \cap \mathcal{A}_t\} + \Pr\{\mathcal{B}^c_t \cap \mathcal{A}_t\} \quad (4.2)$$

where

$$\mathcal{B}_1 = \left\{ \frac{\lambda}{2} K_1 < L_1 < \frac{3\lambda}{2} K_1 \right\},$$

and for $j = 2, \ldots, t$,

$$\mathcal{B}_j = \left\{ \frac{\lambda}{2} K_1 < L_1 < \frac{3\lambda}{2} K_1, \quad \frac{\lambda}{2^{i+1}} \le \frac{L_{i+1}}{K_{i+1}}, \quad i = 1, 2, \ldots, j-1 \right\}.$$

Observe that

$$\begin{aligned}\Pr\{\mathcal{B}_t \cap \mathcal{A}_t\} = &\prod_{j=1}^{t-1} \Pr\Big\{ \frac{\lambda}{2^{j+1}} \le \frac{L_{j+1}}{K_{j+1}} \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j \Big\} \\ &\times \prod_{i=1}^{t} \Pr\Big\{ a_i < Z_i(T^2_{K,L}) \le b_i \,\Big|\, \mathcal{B}_i \cap \mathcal{A}_{i-1} \Big\},\end{aligned} \quad (4.3)$$

where $\mathcal{B}_1 \cap \mathcal{A}_0 := \mathcal{B}_1$ and $\Pr\{\mathcal{B}_1\} = 1$. The first step is to show that for $i = 1, 2, \ldots, t$

$$\left| \Pr\Big\{ a_i < Z_i(T^2_{K,L}) \le b_i \mid \mathcal{B}_i \cap \mathcal{A}_{i-1} \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right| \le \frac{C(i)}{K^{1/16}} \quad (4.4)$$

where $C(i)$ is a constant which may depend on λ, i and $a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_i$ but which does not depend on K. Since $\mathcal{B}_1 \cap \mathcal{A}_0 := \mathcal{B}_1 = \{\frac{\lambda}{2} K_1 < L_1 < \frac{3\lambda}{2} K_1\}$, inequality (4.4) follows from Lemma 2, by the choice of K and L, when $i = 1$.

For $2 \le i \le t$ we exploit the identity

$$\Pr\Big\{ a_i < Z_i(T^2_{K,L}) \le b_i \,\Big|\, K_i = k,\, L_i = l,\, \mathcal{B}_{i-1} \cap \mathcal{A}_{i-1} \Big\} = \Pr\Big\{ a_i < \frac{C_1(T^2_{k,l})}{k} \le b_i \Big\} \quad (4.5)$$

where $C_1(T^2_{k,l})$ is the size of the component which contains a given red vertex in the random mapping $T^2_{k,l} = T_{k,l} \circ T_{k,l}$, where $T_{k,l}$ is a random bipartite mapping on k red vertices and l blue vertices. This identity is a straightforward consequence of the independence and uniformity which is built into the bipartite model, namely, that each vertex is assigned independently, according to the uniform distribution, to a vertex in the other set.

Now suppose that $K \prod_{s=1}^{i-1}(1 - b_s) \le k < K \prod_{s=1}^{i-1}(1 - a_s)$ and $\frac{\lambda k}{2^i} \le l < L$, then since $L \le \frac{3\lambda}{2} K$, we have

$$\xi(i) k \le l \le \eta(i) k$$

where

$$\xi(i) = \frac{\lambda}{2^i} \quad \text{and} \quad \eta(i) = \frac{3\lambda}{2\prod_{s=1}^{i-1}(1-b_s)}.$$

So by Lemma 2 and identity (4.5), we have

$$\left| \Pr\Big\{ a_i < Z_i(T^2_{K,L}) \le b_i \,\Big|\, K_i = k,\, L_i = l,\, \mathcal{B}_{i-1} \cap \mathcal{A}_{i-1} \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right|$$
$$= \left| \Pr\Big\{ a_i < \frac{C_1(T^2_{k,l})}{k} \le b_i \Big\} - \int_{a_i}^{b_i} \frac{du}{2\sqrt{1-u}} \right| \le \frac{C(a_i, b_i, \xi(i), \eta(i))}{k^{1/16}} \le \frac{C(i)}{K^{1/16}}. \tag{4.6}$$

Since $\{\mathcal{B}_i \cap \mathcal{A}_{i-1}\} = \bigcup_{k,l}\{K_i = k,\, L_i = l,\, \mathcal{B}_{i-1} \cap \mathcal{A}_{i-1}\}$, where $\frac{\lambda k}{2^i} \le l < L$ and $K\prod_{s=1}^{i-1}(1-b_s) \le k < K\prod_{s=1}^{i-1}(1-a_s)$, it follows from (4.6) that (4.4) holds for $2 \le i \le t$, and hence

$$\left| \prod_{i=1}^{t} \Pr\Big\{ a_i < Z_i(T^2_{K,L}) \le b_i \Big| \mathcal{B}_i \cap \mathcal{A}_{i-1} \Big\} - \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{\sqrt{1-u}} \right| \le \frac{\sum_{i=1}^{t} C(i)}{K^{1/16}}. \tag{4.7}$$

Next we show that for $1 \le j \le t-1$

$$\left| \Pr\Big\{ \frac{\lambda}{2^{j+1}} \le \frac{L_{j+1}}{K_{j+1}} \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j \Big\} - 1 \right| \le \frac{\hat{C}(j)}{K} \tag{4.8}$$

where $\hat{C}(j)$ is a constant which may depend on λ, j and $a_1, a_2, \ldots, a_j$, $b_1, b_2, \ldots, b_j$ but which does not depend on K. Now given $\mathcal{B}_j \cap \mathcal{A}_j$, we have for $1 \le j \le t-1$

$$\frac{L_{j+1}}{K_{j+1}} = \frac{L_j - (L_j Z_j(T^2_{K,L}) - D_j)}{K_j - K_j Z_j(T^2_{K,L})} = \frac{L_j}{K_j} \cdot \left(1 + \frac{D_j}{L_j(1 - Z_j(T^2_{K,L}))} \right)$$
$$\ge \frac{\lambda}{2^j} \cdot \left(1 + \frac{D_j}{L_j(1 - Z_j(T^2_{K,L}))} \right)$$

where $D_j = L_j Z_j(T^2_{K,L}) - B_j$. Hence

$$\Pr\Big\{ \frac{1}{2} \le 1 + \frac{D_j}{L_j(1 - Z_j(T^2_{K,L}))} \Big| \mathcal{B}_j \cap \mathcal{A}_j \Big\} \le \Pr\Big\{ \frac{\lambda}{2^{j+1}} \le \frac{L_{j+1}}{K_{j+1}} \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j \Big\}. \tag{4.9}$$

Given the event $\mathcal{B}_j \cap \mathcal{A}_j$, we have $\frac{\lambda}{2^j} K \prod_{s=1}^{j-1}(1-b_s) \le \lambda K_j / 2^j \le L_j$ (with the convention that for $j = 1$ the product in this inequalitity is equal to 1). So *given* the event $\mathcal{B}_j \cap \mathcal{A}_j$, if $|D_j| \le (L_j)^{2/3}$, then

$$\frac{|D_j|}{L_j(1 - Z_j(T^2_{K,L}))} \le \frac{\tilde{C}(j)}{K^{1/3}} < \frac{1}{2}$$

for all $K > (2\tilde{C}(j))^3$, where $\tilde{C}(j)$ is a constant which depends on λ, j, $b_1, b_2, \ldots, b_j$. Therefore,

$$\Pr\Big\{|D_j| \le (L_j)^{2/3} \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j\Big\} \le \Pr\Big\{\frac{1}{2} \le 1 + \frac{D_j}{L_j(1 - Z_j(T^2_{K,L}))} \Big| \mathcal{B}_j \cap \mathcal{A}_j\Big\} \quad (4.10)$$

for all $1 \le j \le t-1$ and $K > \max\{(2\tilde{C}(j))^3 : 1 \le j \le t-1\}$.

Next, for $j \ge 2$

$$\begin{aligned} &\Pr\{\, R_j = r, B_j = b \,|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1} \,\} \\ &\qquad = \Pr\{\, R_1(k,l) = r,\, B_1(k,l) = b \,\} \end{aligned} \quad (4.11)$$

where $R_1(k,l)$ is the number of red vertices and $B_1(k,l)$ is the number of blue vertices in the connected component $\mathcal{C}_1(T_{k,l})$. So for k, l and m chosen such that $\lambda k/2^j \le l < L$, $K\prod_{s=1}^{j}(1-b_s) \le k < K\prod_{s=1}^{j}(1-a_s)$, and $a_j k < m \le b_j k$, we have

$$\begin{aligned} &\Pr\Big\{|D_j| > l^{2/3},\, R_j = m \,\Big|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\} \\ &\le \sum_{|i| > l^{2/3}-1} \Pr\Big\{R_1(k,l) = m, B_1(k,l) = \lceil lx \rceil + i\Big\} \\ &\le \hat{C}(j)\exp(-l^{1/6}) \;\le\; \hat{C}(j)\exp(-\hat{C}(j)K^{1/6}) \end{aligned} \quad (4.12)$$

where $x = \frac{m}{k}$ and $\hat{C}(j)$ is a constant which may depend on λ, j and $a_1, a_2, \ldots, a_j$, $b_1, b_2, \ldots, b_j$ but which does not depend on K. We note that the second inequality in (4.12) follows from arguments similar to those which established inequalities (3.6) and (3.7) and the last inequality follows from the inequality $\frac{\lambda}{2^j}K\prod_{s=1}^{j}(1-b_s) \le l$. Since these bounds are uniform over all k, l, and m satisfying $\lambda k/2^j \le l < L$, $K\prod_{s=1}^{j}(1-b_s) \le k < K\prod_{s=1}^{j}(1-a_s)$, and $a_j k < m \le b_j k$, we have

$$\begin{aligned} &\Pr\Big\{|D_j| > l^{2/3},\, a_j k < R_j \le b_j k \;|\; K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\} \\ &= \sum_{m > a_j k}^{b_j k} \Pr\Big\{|D_j| > l^{2/3},\, R_j = m \;|\; K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\} \\ &\le K\hat{C}(j)\exp(-\hat{C}(j)K^{1/6}) \end{aligned} \quad (4.13)$$

It follows from (4.13) and identity (4.11) that for $1 \le j \le t-1$ and all large K,

$$\begin{aligned}
&\Pr\Big\{|D_j| > l^{2/3} \,\Big|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_j\Big\} \\
&= \frac{\Pr\Big\{|D_j| > l^{2/3},\, a_j k < R_j \le b_j k \,\Big|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\}}{\Pr\Big\{a_j k < R_j \le b_j k \,\Big|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\}} \\
&= \frac{\Pr\Big\{|D_j| > l^{2/3},\, a_j k < R_j \le b_j k \,\Big|\, K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_{j-1}\Big\}}{\Pr\Big\{a_j k < C_1(T_{k,l}^2) \le b_j k\Big\}} \\
&\le \frac{\hat{C}(j)}{K}
\end{aligned}$$

provided $K\prod_{s=1}^{j-1}(1-b_s) \le k < K\prod_{s=1}^{j}(1-a_s)$, and $\lambda k/2^{j-1} \le l < L$. Since $\{\mathcal{B}_j \cap \mathcal{A}_j\}$ $= \bigcup_{k,l}\{K_j = k,\, L_j = l,\, \mathcal{B}_{j-1} \cap \mathcal{A}_j\}$ where $K\prod_{s=1}^{j-1}(1-b_s) \le k < K\prod_{s=1}^{j}(1-a_s)$ and $\lambda k/2^{j-1} \le l < L$, it follows that

$$\Pr\Big\{|D_j| \le L^{2/3} \Big| \mathcal{B}_j \cap \mathcal{A}_j\Big\} \ge 1 - \frac{\hat{C}(j)}{K} \tag{4.14}$$

for $1 \le j \le t-1$. Inequality (4.8) now follows from (4.9), (4.10) and (4.14) and so for all large K and $1 \le j \le t-1$,

$$\left| \prod_{j=1}^{t-1} \Pr\Big\{\frac{1}{2^{j+1}} \le \frac{L_{j+1}}{K_{j+1}},\ \Big| \mathcal{B}_j \cap \mathcal{A}_j\Big\} - 1 \right| \le \sum_{j=1}^{t-1} \frac{\hat{C}(j)}{K}. \tag{4.15}$$

Finally, it follows from (4.3), (4.7), and (4.15) that

$$\left| \Pr\Big\{\mathcal{B}_t \cap \mathcal{A}_t\Big\} - \prod_{i=1}^{t} \int_{a_i}^{b_i} \frac{du}{\sqrt{1-u}} \right| \le \frac{\sum_{i=1}^{t} C(i)}{K^{1/16}} + \frac{\sum_{j=1}^{t-1} \hat{C}(j)}{K}. \tag{4.16}$$

To complete the proof of the lemma, observe that

$$\Pr\Big\{\mathcal{B}_t^c \cap \mathcal{A}_t\Big\} \le \sum_{j=1}^{t-1} \Pr\Big\{\frac{L_{j+1}}{K_{j+1}} < \frac{\lambda}{2^j} \,\Big|\, \mathcal{B}_j \cap \mathcal{A}_j\Big\} \Pr\Big\{\mathcal{B}_j \cap \mathcal{A}_j\Big\} \le \frac{\sum_{j=1}^{t-1} \hat{C}(j)}{K}. \tag{4.17}$$

The result now follows from (4.2), (4.16), and (4.17).

Proof of Theorem 2. It suffices to note that equation (4.1) now follows immediately from Lemma 4 and proof of Theorem 1.

5. Final remarks

The results above give some indication of which limit results for the uniform random mapping model T_K are 'robust' under the introduction of extra randomness into the random mapping model. In particular, we have seen that the limiting distributions for the number of cyclic vertices and the number of predecessors of a vertex in $G_K(\lambda)$ *cannot* be the same as for the G_K, whereas the limiting distribution for the normalized order statistics of the component sizes in both $G_K(\lambda)$ and G_K is $\mathcal{PD}(1/2)$.

It would be interesting to determine which other limit results for the uniform mapping T_K remain the same for the Poisson compound mapping $T_K(\lambda)$. For example, it is not difficult to show, using the methods of this paper, that a central limit theorem for $S_K(\lambda)$, the total number of components in $G_K(\lambda)$, follows from Theorem 7 in [26]. In particular, we have

$$\frac{S_K(\lambda) - \frac{1}{2}\log K}{\sqrt{\frac{1}{2}\log K}} \quad \xrightarrow{d} \quad \mathcal{N}(0,1)$$

as $K \to \infty$, where the normalizing constants above are the same as in the case of the uniform random mapping. We also conjecture that the joint distribution of $(M_1, M_2, \ldots, M_b)$ is close, in the sense of total variation, to the joint distribution of a sequence of independent Poisson random variables when $b = o(K/\log K)$ where M_k denotes the number of components of size k in $G_K(\lambda)$. This result, however, does not follow from the results of Arratia, et al. (see [5]) as a compound random mapping is not a logarithmic combinatorial assembly.

We conclude by noting that our results can also be interpreted as a Bayesian approach to random mappings, and as such, they are in the spirit of recent work by Diaconis and Holmes [12] on Bayesian versions of the classic birthday problem, coupon collector's problem and matching problem. In this light, it would also be interesting to determine whether there are other tractable (and non-trivial) models $T_K(\Pi)$ which differ in their their component structure from that of either the uniform model T_K or the Poisson compound mapping model $T_K(\lambda)$.

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