Mathematical Questions arising from Vegetation Patterns in Semi-Deserts

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This talk can be downloaded from my web site
`www.ma.hw.ac.uk/~jas`

Outline

Vegetation Pattern Formation

Bushy vegetation in Niger

Mitchell grass in Australia

(Western New South Wales)

- Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia and Mexico
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees

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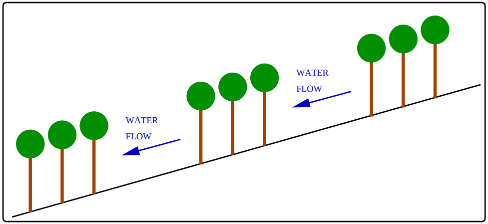

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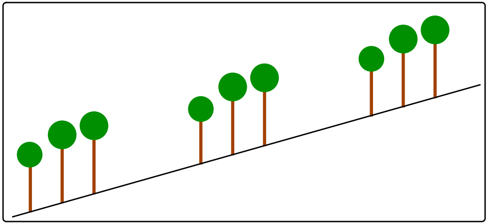

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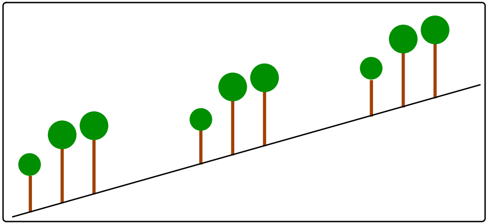

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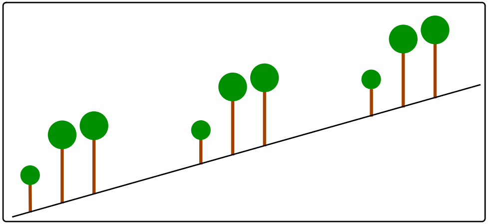

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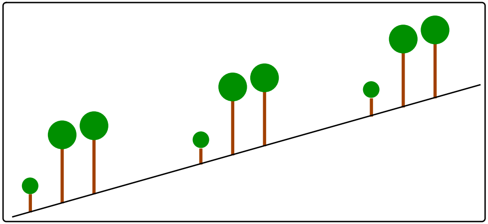

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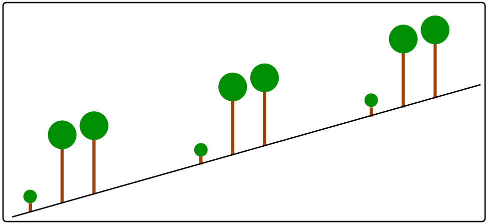

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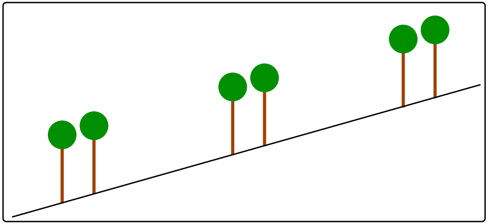

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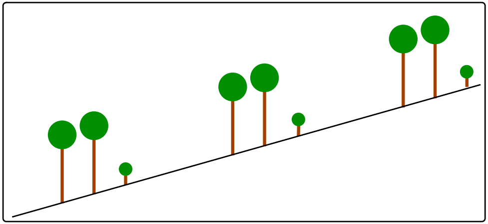

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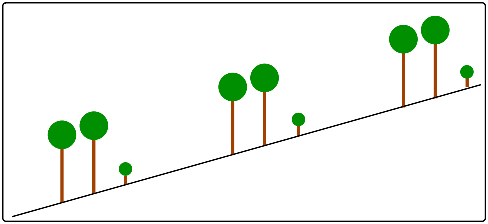

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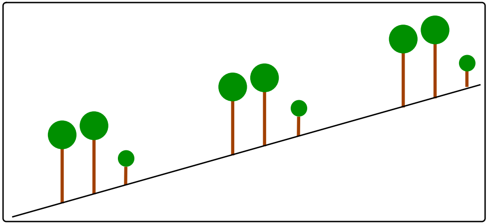

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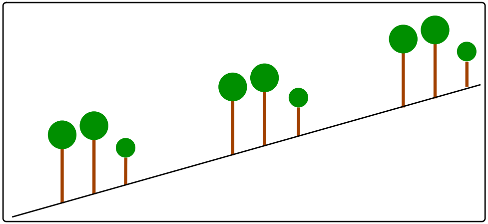

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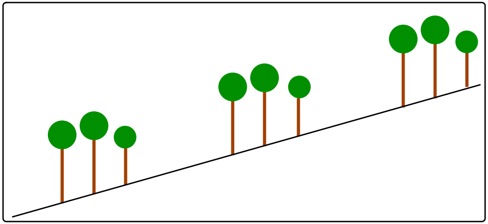

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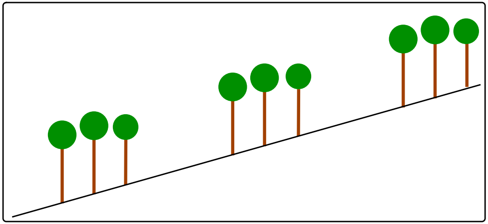

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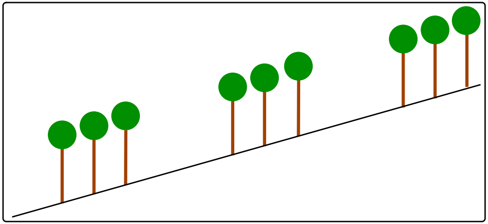

- The stripes move uphill (very slowly)

Two Key Ecological Questions

- How does the spacing of the vegetation bands depend on rainfall, herbivory and slope?
- At what rainfall level is there a transition to desert?

Outline

Mathematical Model of Klausmeier

Rate of change of water = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change plant biomass = Growth, proportional to water uptake – Mortality + Random dispersal

$$\partial w/\partial t = A - w - wu^2 + \nu \partial w/\partial x$$
$$\partial u/\partial t = wu^2 - Bu + \partial^2 u/\partial x^2$$

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The nonlinearity in wu^2 arises because the presence of roots increases water infiltration into the soil.

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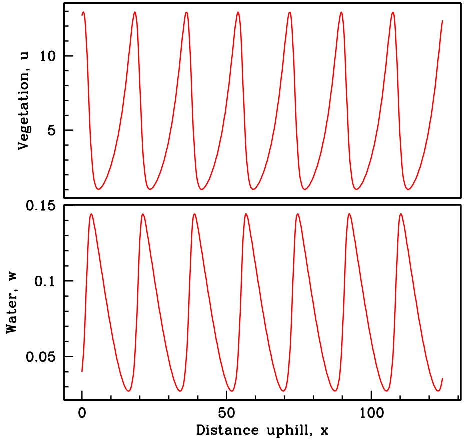
Parameters: A: rainfall B: plant loss ν: slope

Typical Solution of the Model



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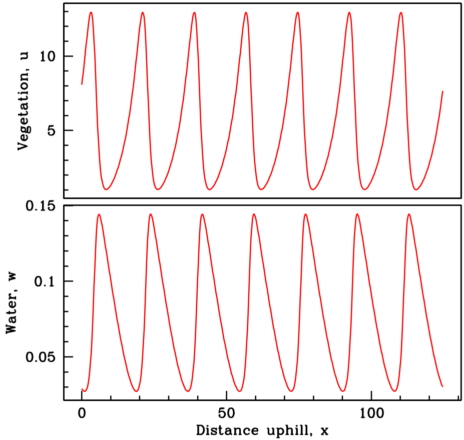

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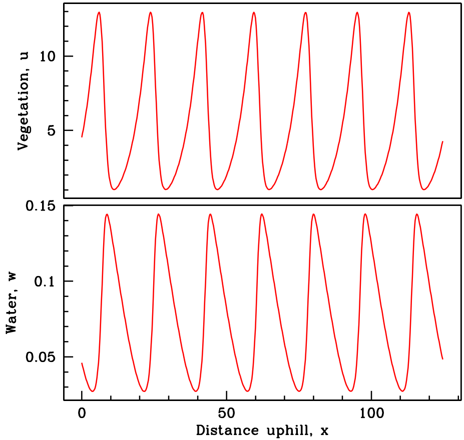

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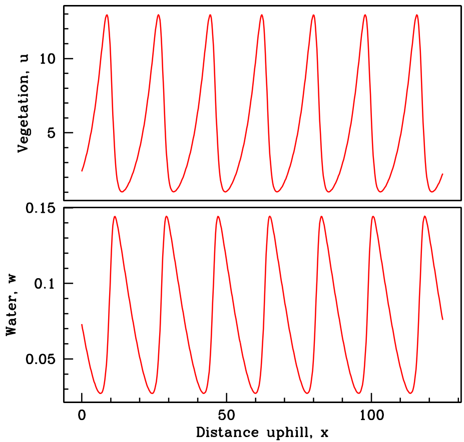

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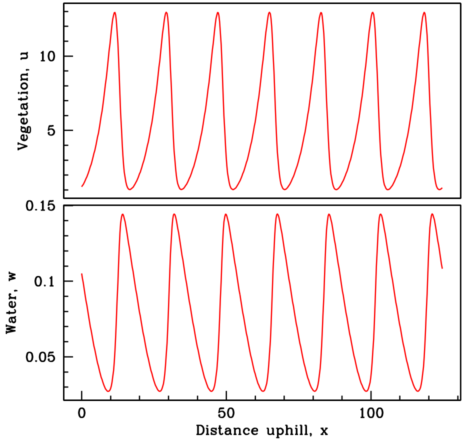

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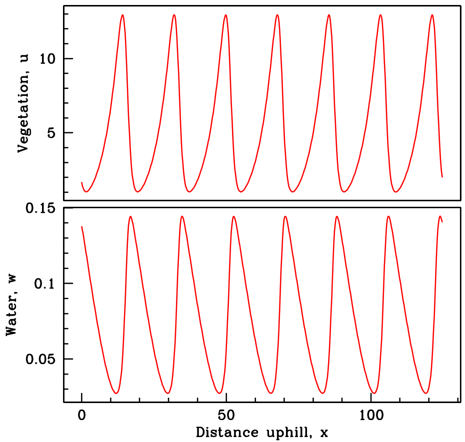

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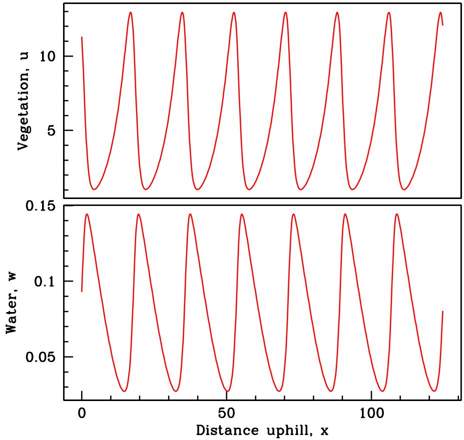

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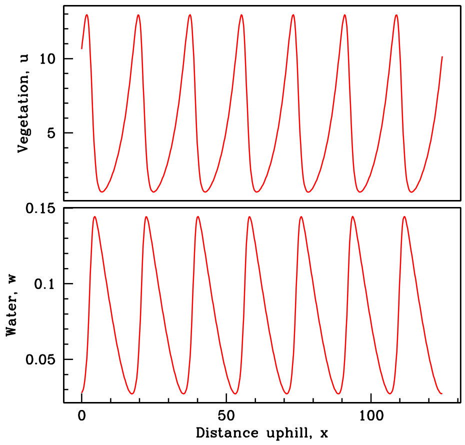

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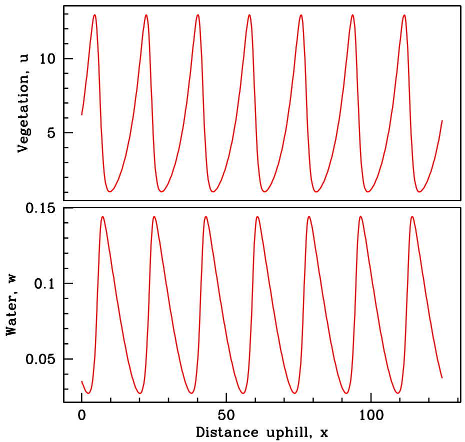

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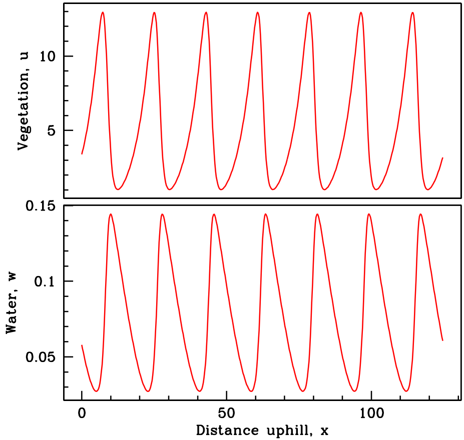

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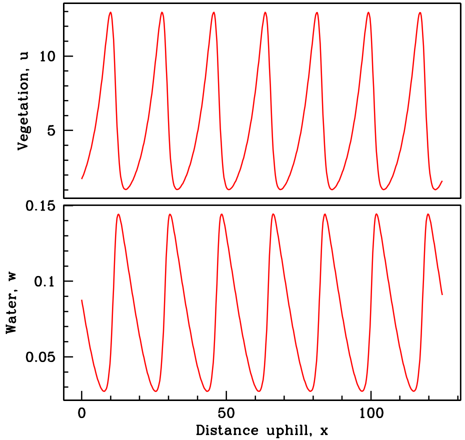

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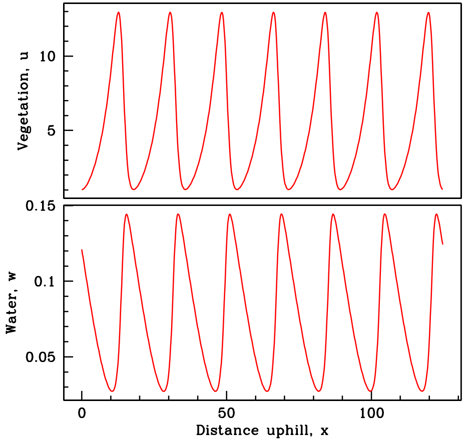

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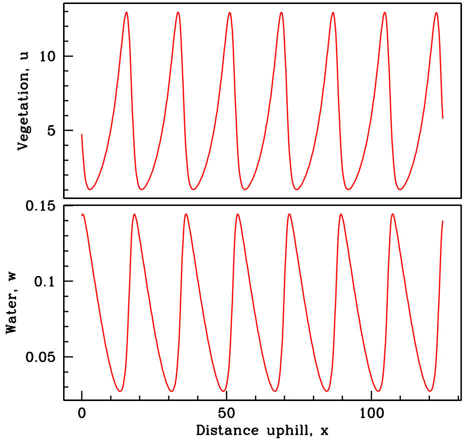

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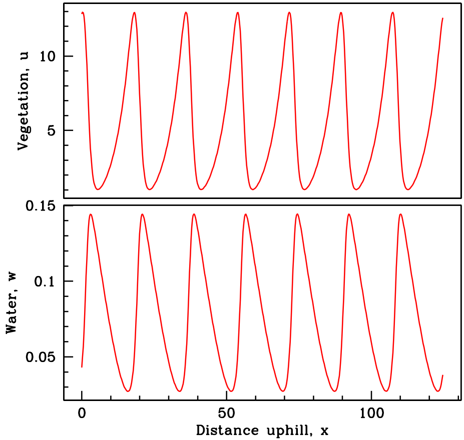

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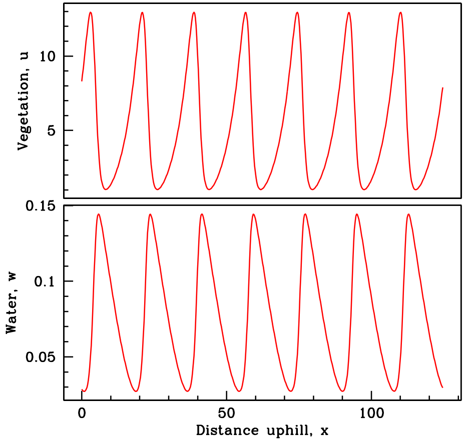

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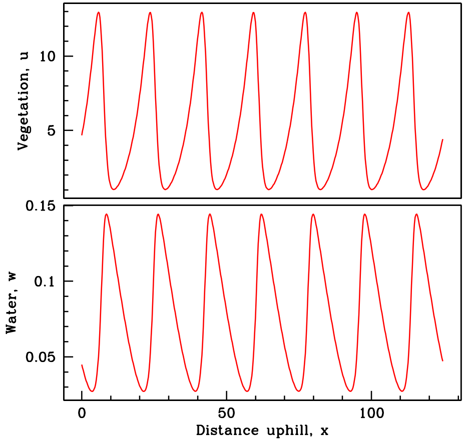

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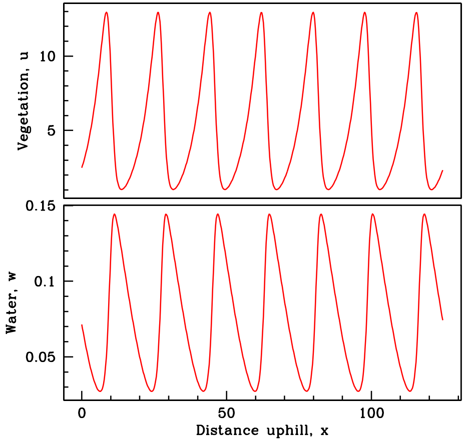

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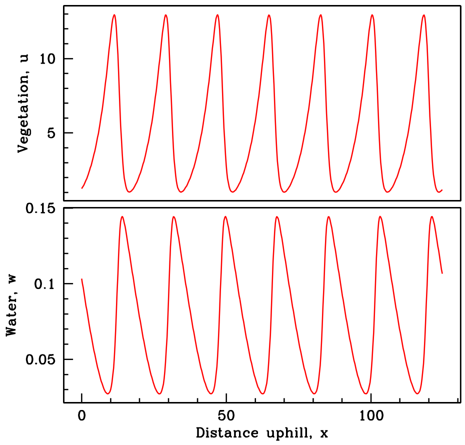

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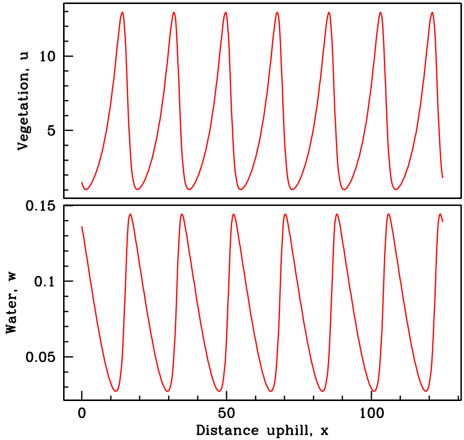

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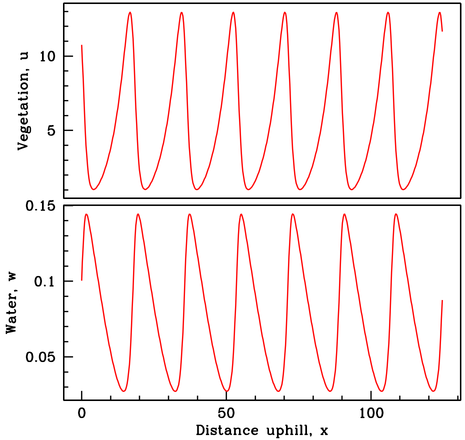

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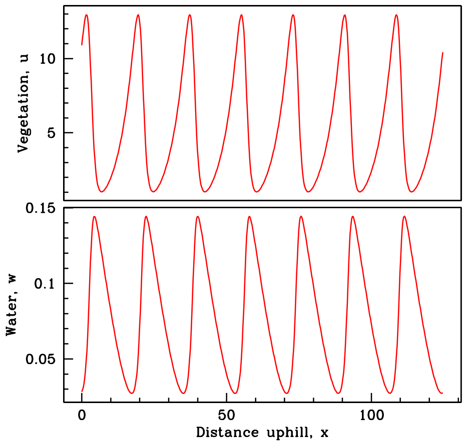

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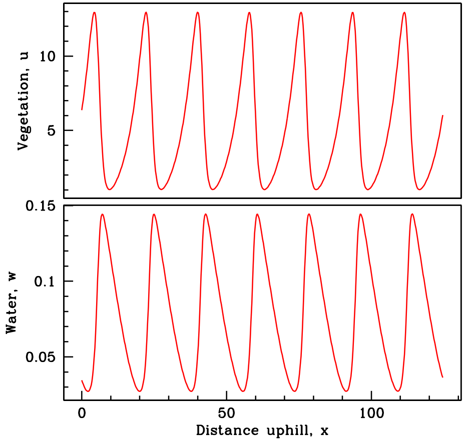

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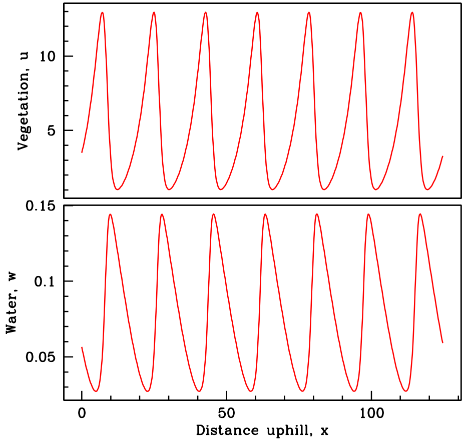

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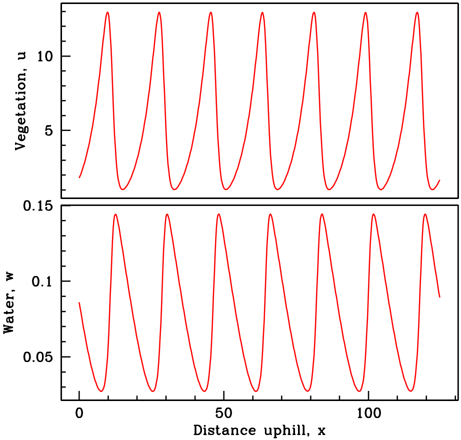

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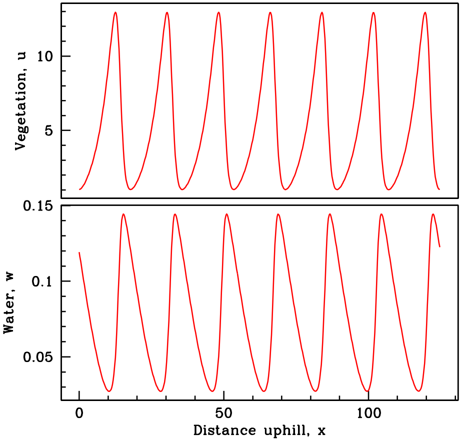

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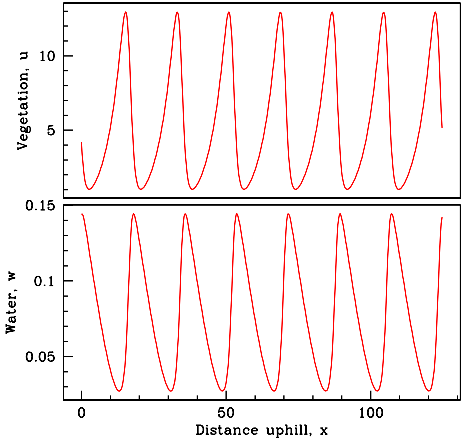

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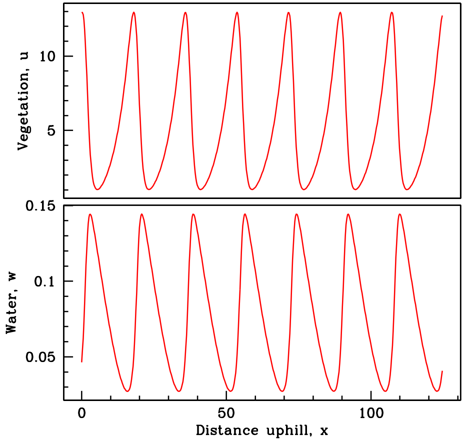

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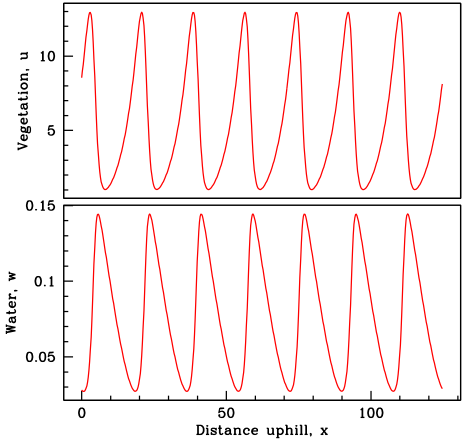

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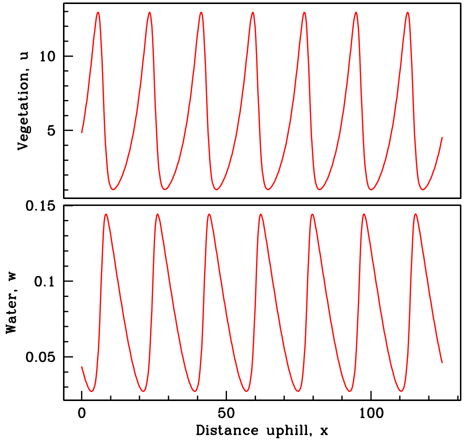

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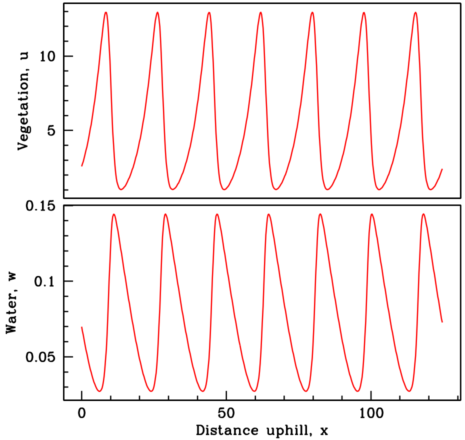

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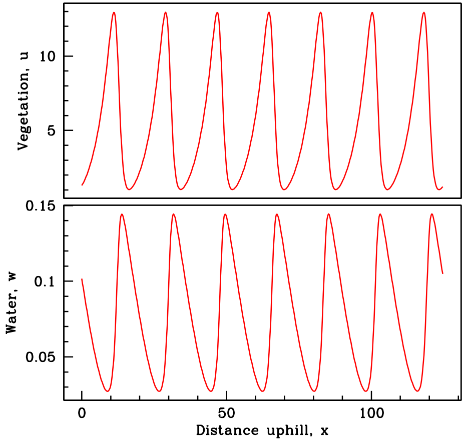

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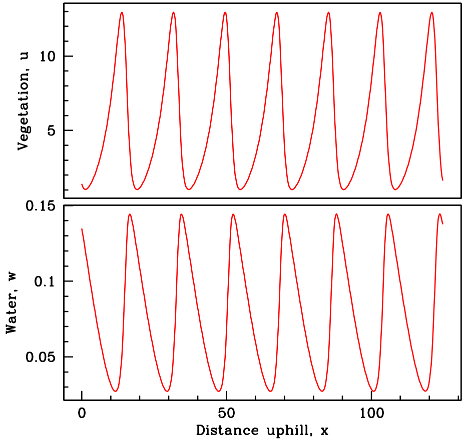

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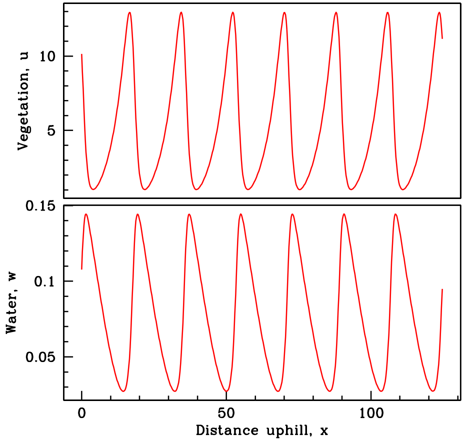

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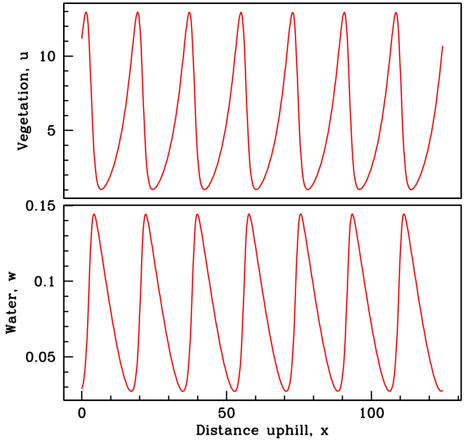

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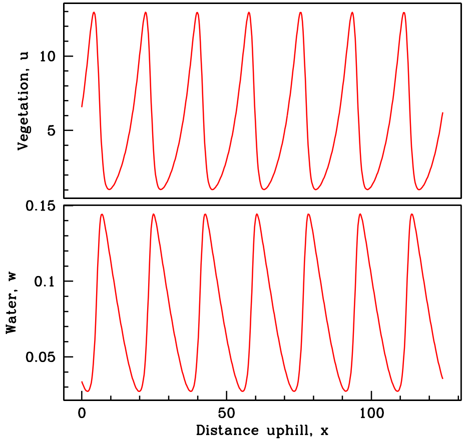

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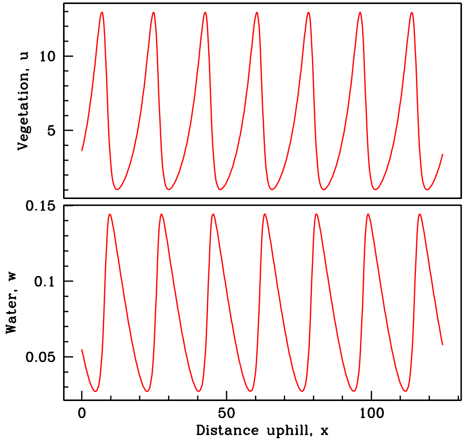

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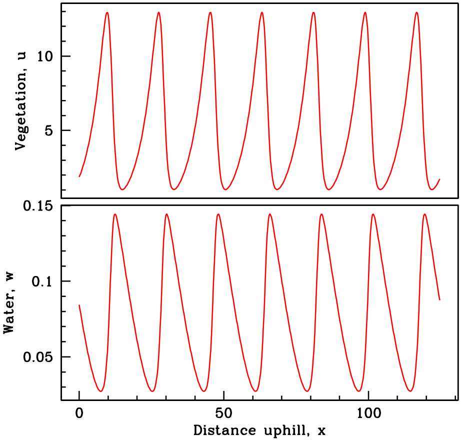

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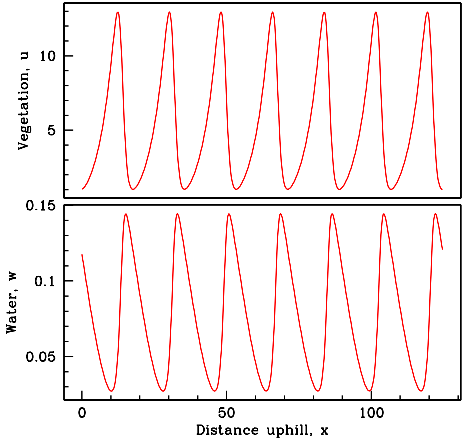

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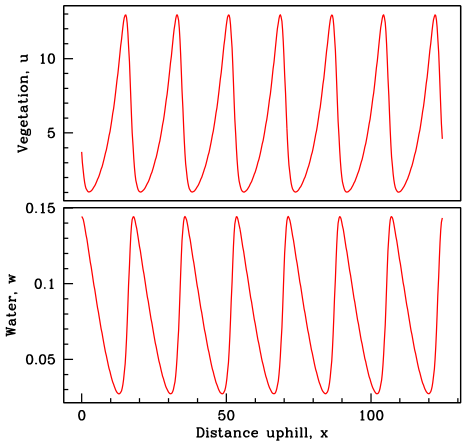

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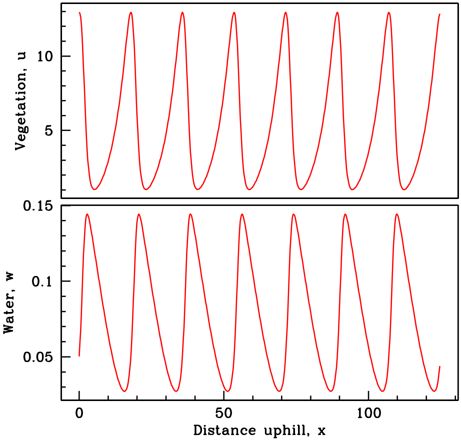

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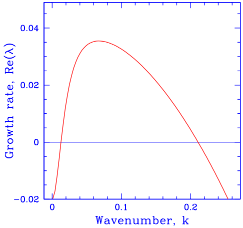
- For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$
- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations
- Patterns develop when the other steady state (u_s, w_s) is unstable to inhomogeneous perturbations

Approximate Conditions for Patterning

Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0) \exp\{ikx + \lambda t\}$



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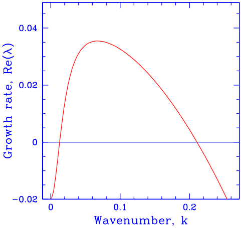

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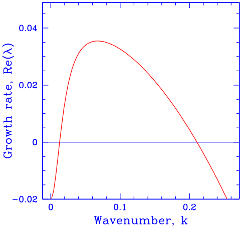


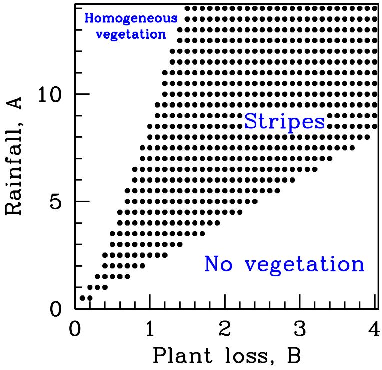
The dispersion relation $\mathrm{Re}[\lambda(k)]$ is algebraically complicated

An approximate condition for pattern formation is

$$2B < A < \nu^{1/2} B^{5/4} / 8^{1/4}$$

One can niavely assume that existence of (u_s, w_s) gives a second condition

An Illustration of Conditions for Patterning



The dots show parameters for which there are growing linear modes.

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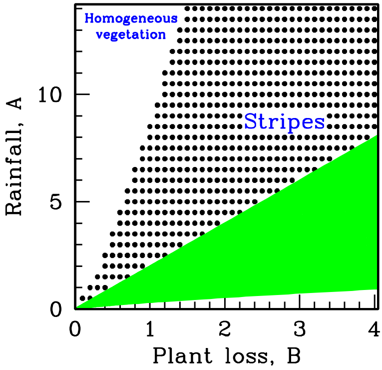
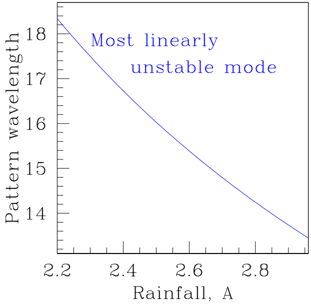


Numerical simulations show patterns in both the dotted and green regions of parameter space.

Predicting Pattern Wavelength

Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography. Wavelength can be predicted from the linear analysis.



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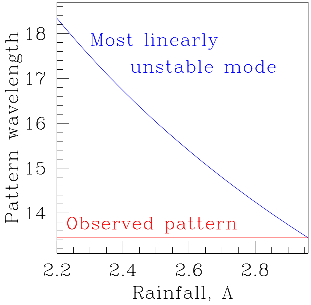


However this prediction doesn't fit the patterns seen in numerical simulations.

Shortcomings of Linear Stability Analysis

Linear stability analysis fails in two ways:

- It significantly over-estimates the minimum rainfall level for patterns.
- Close to the maximum rainfall level for patterns, it incorrectly predicts a variation in pattern wavelength with rainfall.

Outline

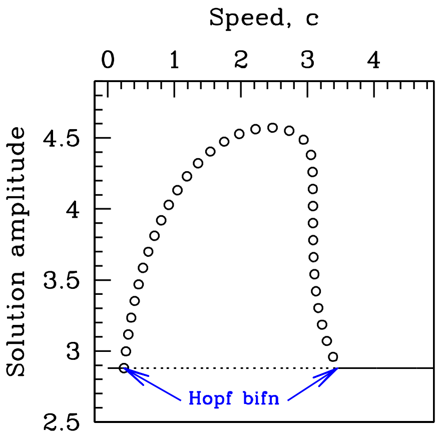
Travelling Wave Equations

The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z)$, $w(x,t) = W(z)$, $z = x - ct$

$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these equations

Bifurcation Diagram for Travelling Wave Equations



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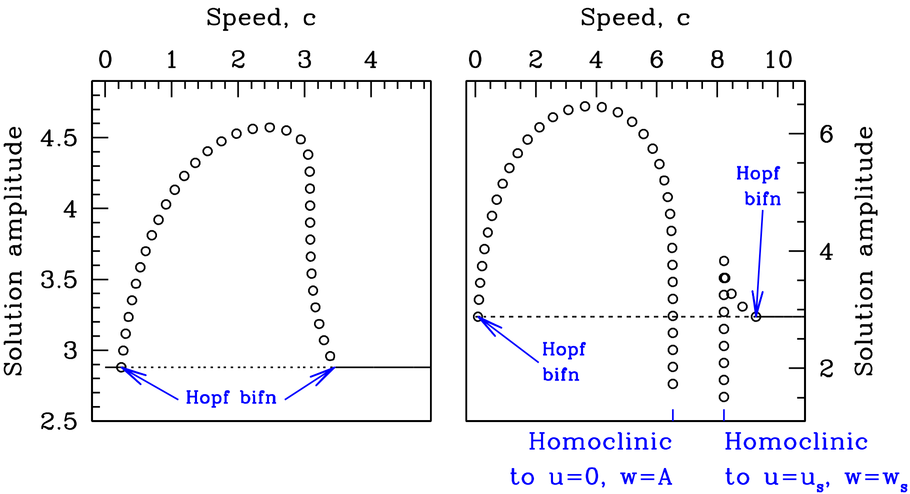
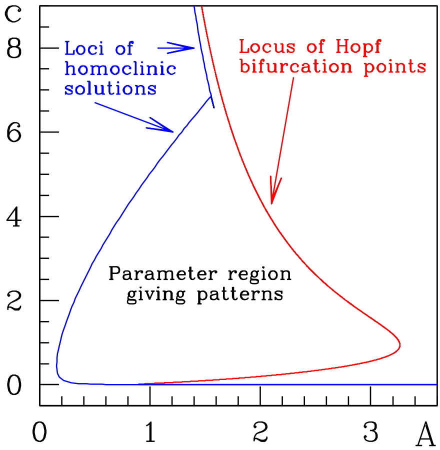
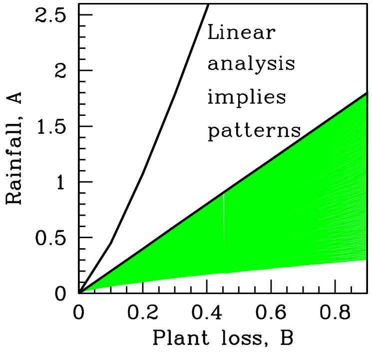


When do Patterns Form?

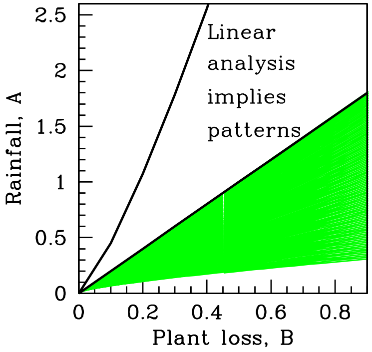


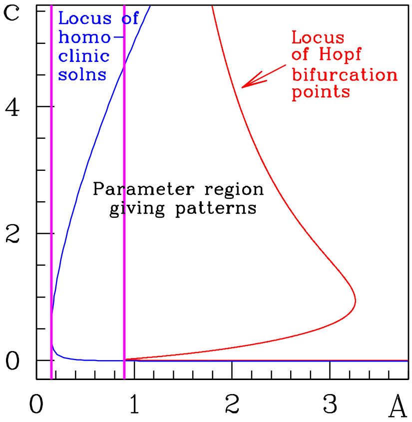
Pattern Formation for Low Rainfall



Patterns are also seen for parameters in the green region.

Pattern Formation for Low Rainfall





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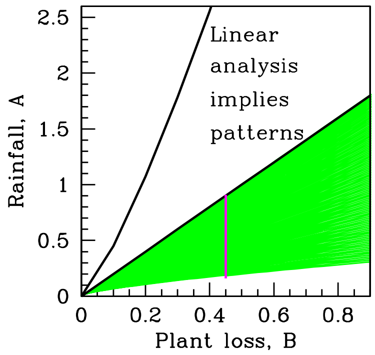
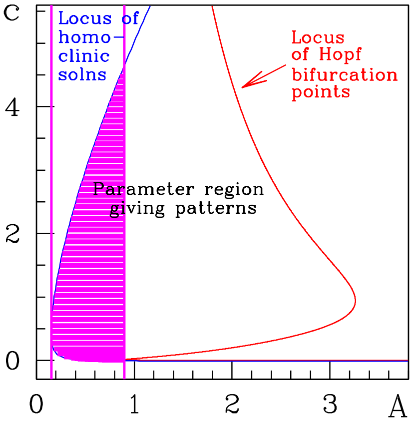
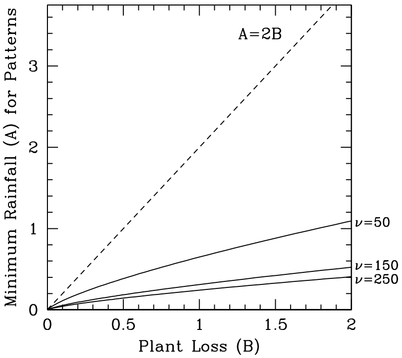




Minimum Rainfall for Patterns



Outline

The Eigenvalue Problem

PDE model:
$$u_t = u_{zz} + cu_z + f(u,w)$$
$$w_t = \nu w_z + cv_z + g(u,w)$$

Periodic wave satisfies:
$$0 = U_{zz} + cU_z + f(U,W)$$
$$0 = \nu W_z + cW_z + g(U,W)$$

Consider
$$u(z,t) = U(z) + e^{\lambda t}\overline{u}(z) \quad \text{with } |\overline{u}| \ll |U|$$
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$\Rightarrow$ Eigenfunction eqn:
$$\lambda\overline{u} = \overline{u}_{zz} + c\overline{u}_z + f_u(U,W)\overline{u} + f_w(U,W)\overline{w}$$
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Boundary conditions:
$$\overline{u}(0) = \overline{u}(L)e^{i\gamma} \quad (0 \leq \gamma < 2\pi)$$
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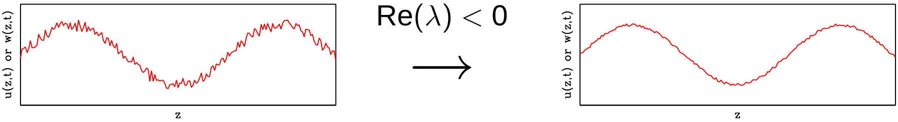

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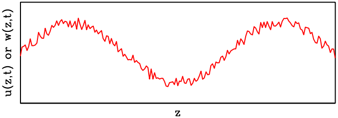
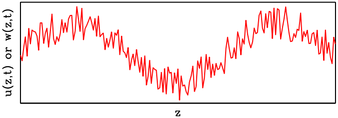


$\mathrm{Re}(\lambda) > 0$

$\longrightarrow$



Numerical Calculation of Eigenvalue Spectrum

(based on Jens Rademacher, Bjorn Sandstede, Arnd Scheel Physica D 229 166-183, 2007)

1. solve numerically for the periodic wave by continuation in c from a Hopf bifn point in the travelling wave eqns

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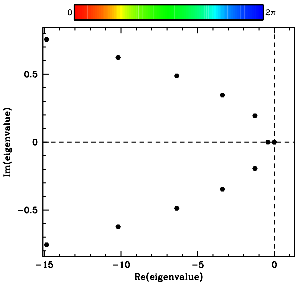

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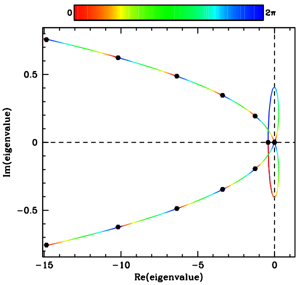

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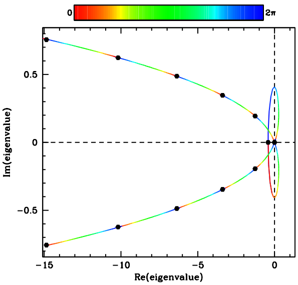

This gives the eigenvalue spectrum, and hence (in)stability

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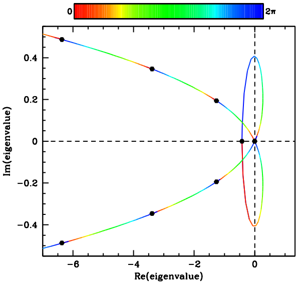
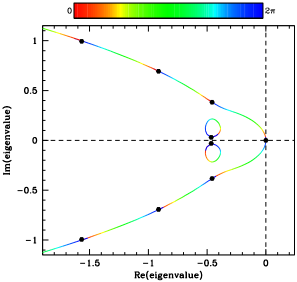


UNSTABLE

Eckhaus instability



STABLE

This gives the eigenvalue spectrum, and hence (in)stability

Stability in a Parameter Plane

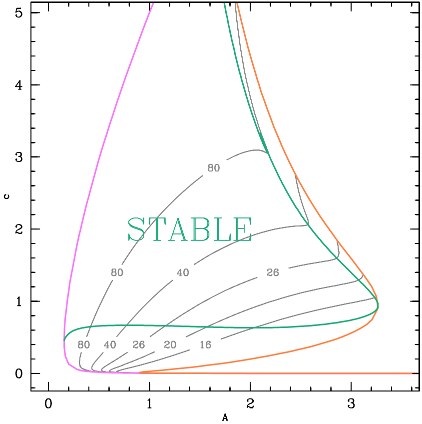
By following this procedure at each point on a grid in parameter space, regions of stability/instability can be determined.

In fact, stable/unstable boundaries can be computed accurately by numerical continuation of the point at which

$$\mathrm{Re}\lambda = \mathrm{Im}\lambda = \gamma = \partial^2 \mathrm{Re}\lambda / \partial\gamma^2 = 0$$

(Eckhaus instability point)

Stability in a Parameter Plane



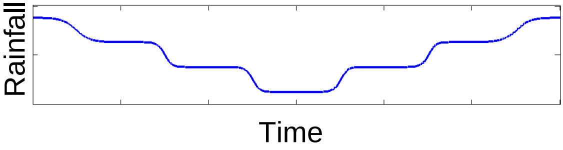
Pattern Stability: The Key Result

Key Result

Many of the possible patterns are unstable and thus will never be seen.

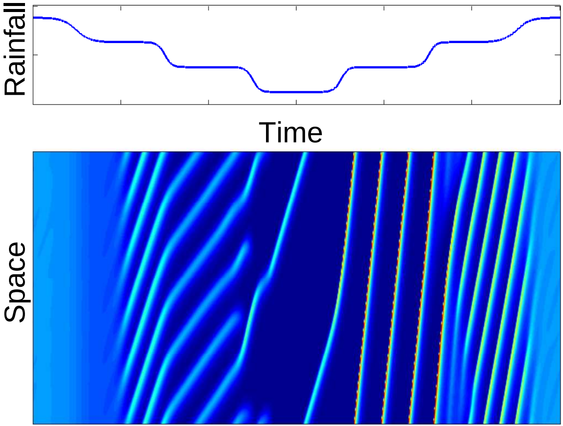
However, for a wide range of rainfall levels, there are multiple stable patterns.

Hysteresis



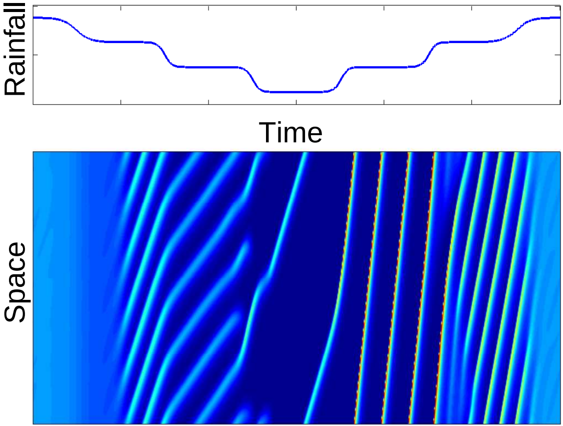
- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A
- Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year

Hysteresis



<< Mode 5 >> <<<<< Mode 1 >>>>> < Mode 3 >

Hysteresis



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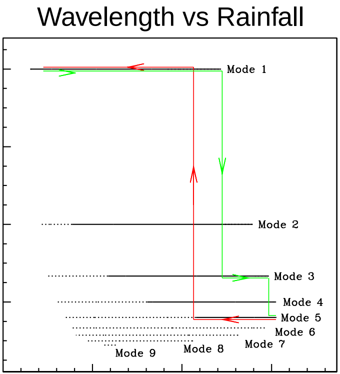
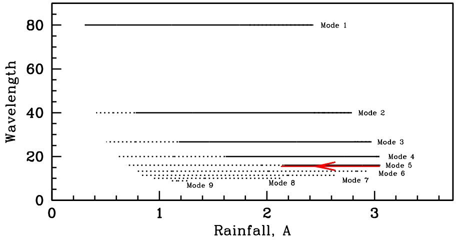


Outline

1. Ecological Background
2. The Mathematical Model
3. Travelling Wave Equations
4. Pattern Stability
5. Conclusions

Predictions of Pattern Wavelength

- In general, pattern wavelength depends on initial conditions
- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.



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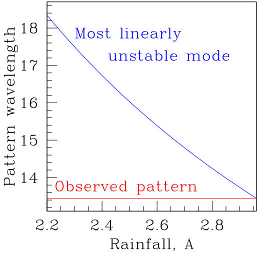
$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



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J.A. Sherratt: An analysis of vegetation stripe formation in semi-arid landscapes. *J. Math. Biol.* 51, 183-197 (2005).

J.A. Sherratt, G.J. Lord: Nonlinear dynamics and pattern bifurcations in a model for vegetation stripes in semi-arid environments. *Theor. Pop. Biol.* 71, 1-11 (2007).

J.A. Sherratt: Pattern solutions of the Klausmeier model for banded vegetation in semi-arid environments I. *Nonlinearity* 23, 2657-2675 (2010).

J.A. Sherratt: Pattern solutions of the Klausmeier model for banded vegetation in semi-arid environments II. Patterns with the largest possible propagation speeds. Submitted.

List of Frames