Separation Distances in Source-Sink Patterns in the Complex Ginzburg-Landau Equation

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This talk can be downloaded from my web site
`www.ma.hw.ac.uk/~jas`

This work is in collaboration with:

Matthew Smith

(Microsoft Research Ltd., Cambridge)

Jens Rademacher

(CWI, Amsterdam)

Outline

Outline

The Complex Ginzburg-Landau Equation

I consider a generic oscillator model, the complex Ginzburg-Landau equation:

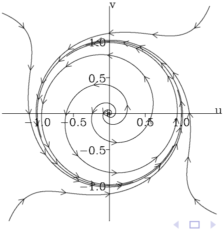
$$A_t = (1 + \mathrm{i}b)A_{xx} + A - (1 + \mathrm{i}c)|A|^2 A.$$

I will look exclusively at $b = 0$. Then writing

$$A(x,t) = e^{-\mathrm{i}at}[u(x,t) + \mathrm{i}v(x,t)]$$

gives a reaction-diffusion system of "λ–ω" type:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1 - r^2)u - (a + cr^2)v \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (a + cr^2)u + (1 - r^2)v \\ \text{where } r &= \sqrt{u^2 + v^2} \end{aligned}$$



Amplitude and Phase Equations

To study these equations, it is helpful to use the variables $r(x,t) = \sqrt{u^2 + v^2}$ and $\theta(x,t) = \tan^{-1}(v/u)$, giving

$$\begin{aligned} r_t &= r_{xx} - r\theta_x^2 + r(1 - r^2) \\ \theta_t &= \theta_{xx} + \frac{2r_x\theta_x}{r} + a - cr^2 \end{aligned}$$

There is a family of wavetrain solutions ($0 < r^* < 1$):

$$\left\{ \begin{array}{l} r = r^* \\ \theta = \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \end{array} \right\}$$

$$\leftrightarrow \left\{ \begin{array}{l} u = r^* \cos \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \\ v = r^* \sin \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \end{array} \right\}$$

Wavetrain Generation by Dirichlet Bndy Conditions

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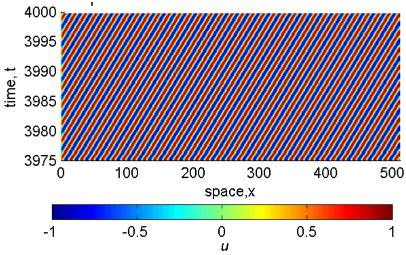
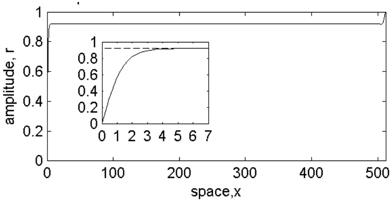




Wavetrain Generation by Dirichlet Bndy Conditions

Conclusion

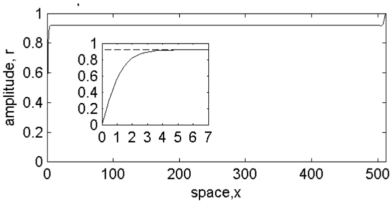
Dirichlet boundary conditions generate a wavetrain

$$R(x) = R^* \tanh\left(x/\sqrt{2}\right)$$

$$\Psi(x) = \Psi^* \tanh\left(x/\sqrt{2}\right)$$

$$R^* = \left\{\tfrac{1}{2}\left[1+\sqrt{1+\tfrac{8}{9}c^2}\right]\right\}^{-1/2}$$

$$\Psi^* = -\mathrm{sign}(c)\left(1-R^{*\,2}\right)^{1/2}$$



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\end{aligned}
$$

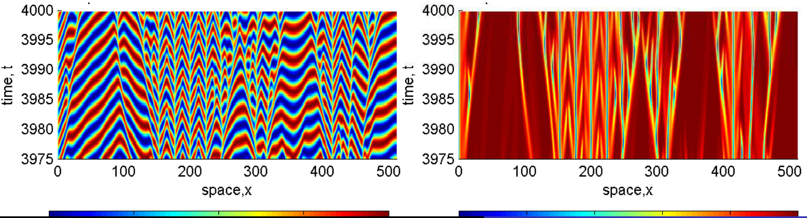
The wavetrain of amplitude R^* is stable $\Leftrightarrow$ $|c| < 1.110468\ldots$

What happens when $|c| > 1.110468\ldots$?

Outline

Two Types of Solution

- There are two types of solution for $|c| > 1.110468\ldots$



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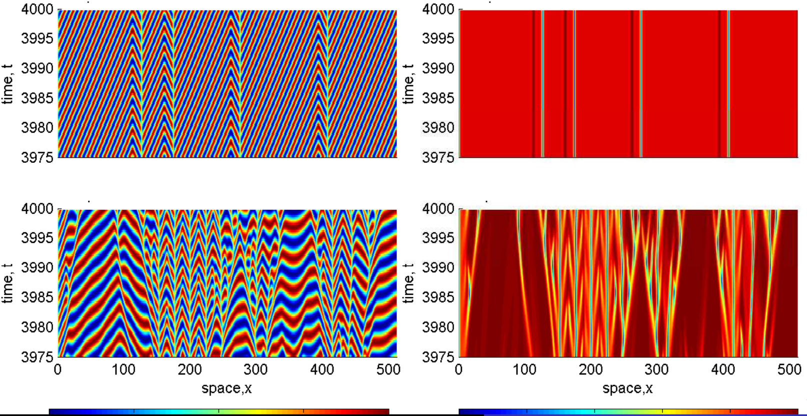

Convective and Absolute Stability

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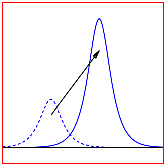
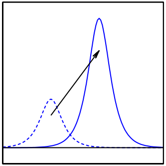
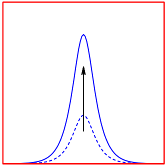


Convective and Absolute Stability

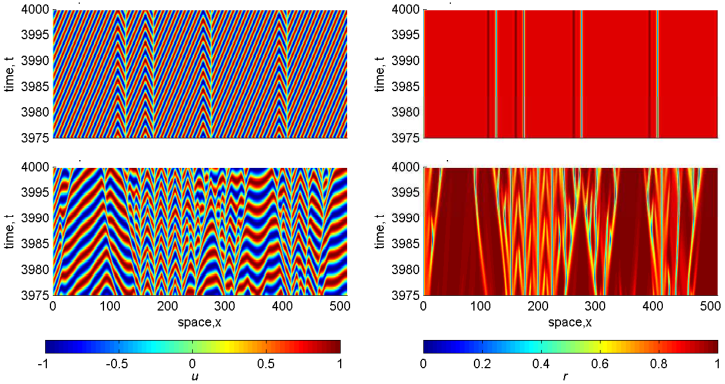
- There are two types of solution for $|c| > 1.110468\ldots$
- The key concept for distinguishing these is "absolute stability".
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".
- Alternatively, a solution can be unstable with perturbations growing without moving. This is "absolute instability".





Generation of Absolutely Stable and Unstable Wavetrains by Dirichlet Boundary Conditions

Numerical simulations show distinct behaviours in the absolutely stable and unstable parameter regimes



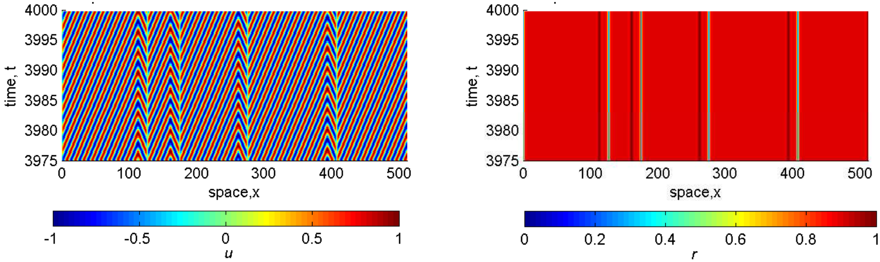
Convectively unstable, absolutely stable

Absolutely unstable

Outline

Sources, Sinks, and Convective Instability

The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.



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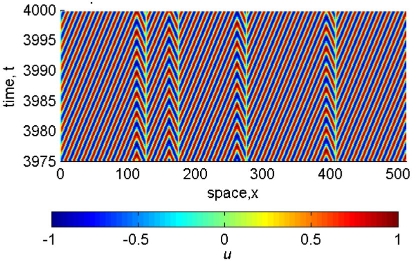

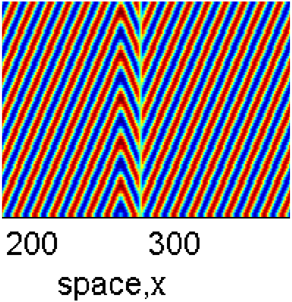

Note: sources and sinks are defined in terms of group velocity.

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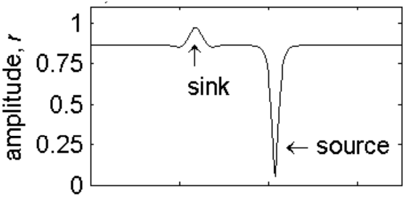

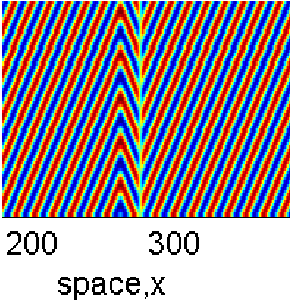

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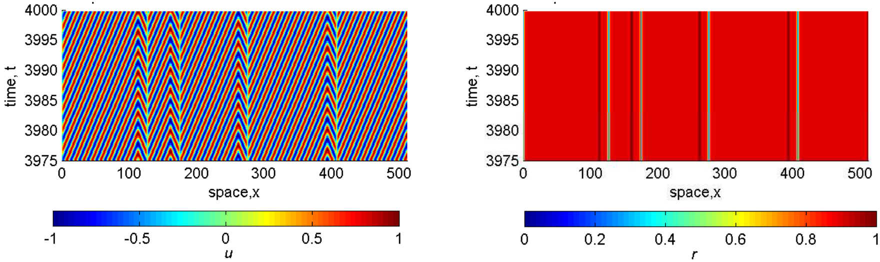

Question: How can an unstable wavetrain persist between the sources and sinks?

Sources, Sinks, and Convective Instability

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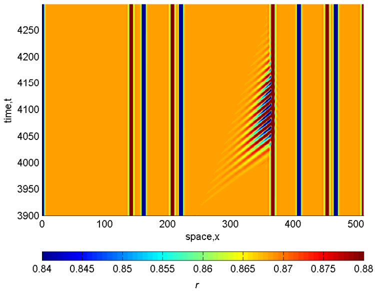
Answer: Any growing perturbations moves, and is absorbed when it reaches a sink.



Previous Mathematical Work on Sources and Sinks

- Sources are “Nozaki–Bekki” holes (Nozaki & Bekki, Phys. Lett. A 110: 133-135, 1985), on which the literature is extensive ($>$ 100 citations).
- Sinks are also well studied, though only numerically.
- But patterns of sources and sinks have received almost no attention.
- One open question is: are there constraints on the distances separating sources and sinks?

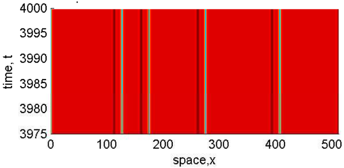
Numerical Study of Source-Sink Separations

Step 1: generate a source-sink pattern via a Dirichlet boundary condition

Step 2: extract a sink-source-sink triple

Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions

Step 4: translate the source and track the subsequent dynamics



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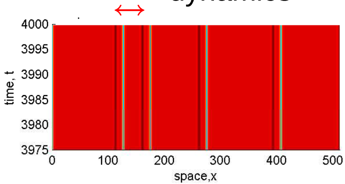

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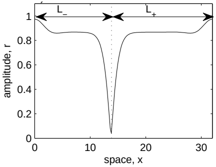

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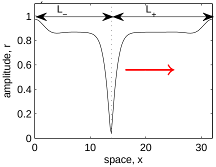

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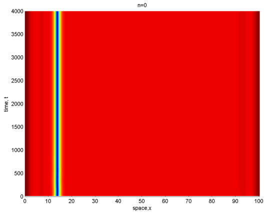

Original solution

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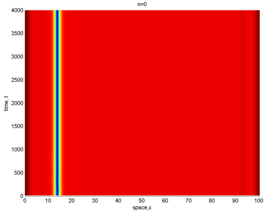

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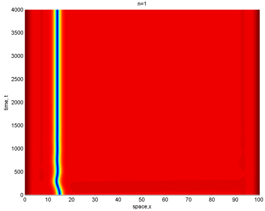

Solution with translated source

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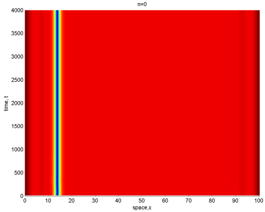

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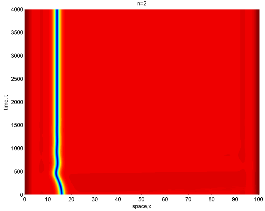

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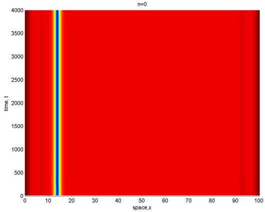

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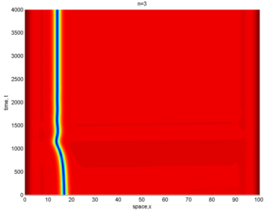

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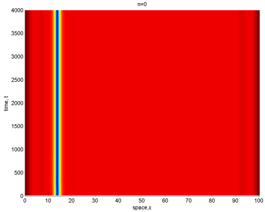

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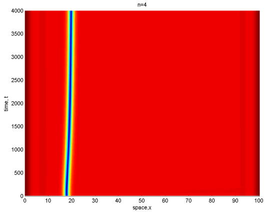


Solution with translated source

Numerical Study of Source-Sink Separations

Conclusion: source-sink separations appear to be constrained to a discrete set of possible values.

Next Step: analytical investigation of the separations.

Outline

Travelling Waves of Amplitude

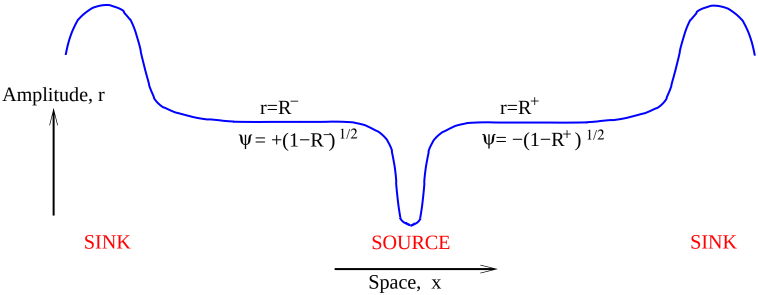
For stationary source-sink patterns, substitute $r(x,t) = \widehat{r}(x)$, $\theta_x(x,t) = \widehat{\psi}(x)$

$$\Longrightarrow \quad d^2\widehat{r}/dx^2 + \widehat{r}\left(1 - \widehat{r}^{\,2} - \widehat{\psi}^{\,2}\right) = 0$$

$$d\widehat{\psi}/dx + K - c\widehat{r}^{\,2} + 2\widehat{\psi}\,(d\widehat{r}/dx)/\widehat{r} = 0$$

(K is a constant of integration).

Solution Structure



Based on source-sink patterns seen in numerical simulations, we consider large separations.

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\widehat{r}/dx^2 + \widehat{r}\left(1 - \widehat{r}^{\,2} - \widehat{\psi}^{\,2}\right) &= 0 \\ d\widehat{\psi}/dx + K - c\widehat{r}^{\,2} + 2\widehat{\psi}\,(d\widehat{r}/dx)/\widehat{r} &= 0. \end{aligned}$$

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Linearise about the wavetrain

⇒ isolated sources decay to the wavetrain at rate $\sqrt{2}$
& isolated sinks decay to the wavetrain at rate $1/\sqrt{2} \pm i\delta/4$

$(\delta = \sqrt{11 - 12R^{*\,2}} \in \mathbb{R})$

⇒ in patterns, the effect of sinks on sources dominates
the effect of sources on sinks,for large separations

⇒ we can just consider the correction to an isolated source:

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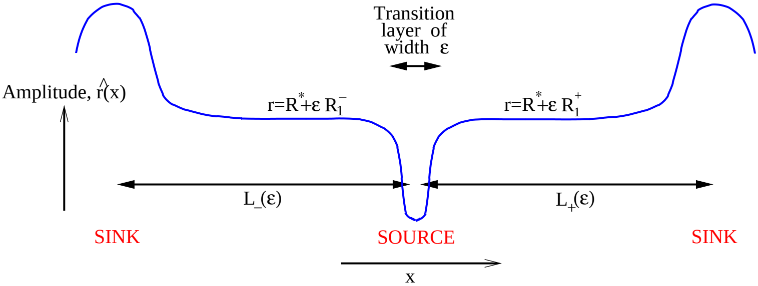
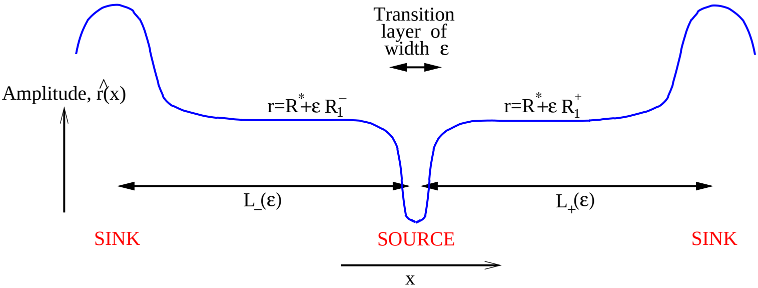
$r = R^* |\tanh(x/\sqrt{2})|$

Perturbation Theory Calculation



We study the problem using perturbation theory.

Perturbation Theory Calculation



Key result (phase-locking condition):

$$\arg\left[\exp\left(-L_-(1+i\delta)/\sqrt{2}\right)+\exp\left(-L_+(1+i\delta)/\sqrt{2}\right)\right]=\text{constant}\,.$$

The constant is determined explicitly.

Illustration of the Locking Condition

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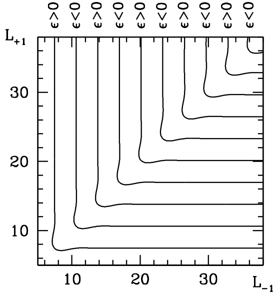
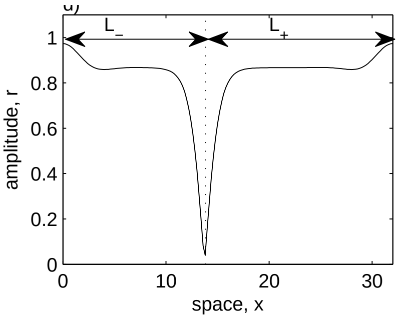


Numerical Verification of the Analysis



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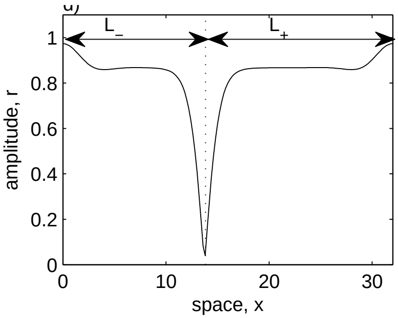

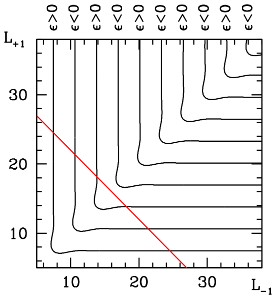

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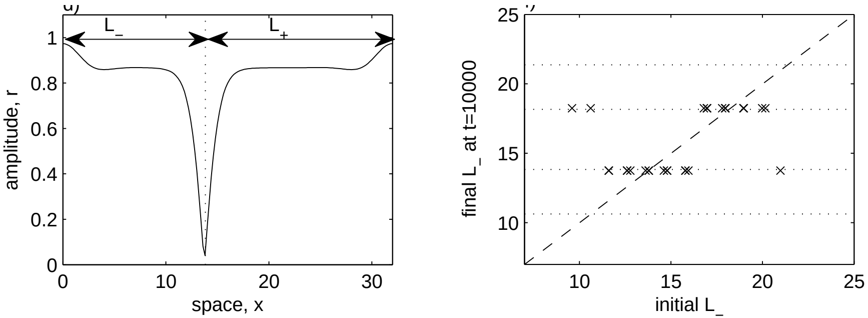


Outline

Summary of Main Results

Main Results:

- For behaviour induced by Dirichlet boundary conditions, the transition from a wavetrain to spatiotemporal disorder occurs via source-sink patterns.
- The separations between a source and its neighbouring sinks, L_- and L_+, are constrained to lie on one of a discrete infinite sequence of curves in the L_-–L_+ plane (to leading order as velocity $\to 0$ and separations $\to \infty$).

Implications for Real Systems

Implications for Real Systems:

Physics

- Experiments are sufficiently precise that the prediction of discrete spacings are testable.

Ecology

- Empirical testing is not feasible.
- In the convectively unstable parameter regime, wavetrains will only be detected in field data if the spatial scale of the data is small compared to source-sink separations.

Future Work and Publications

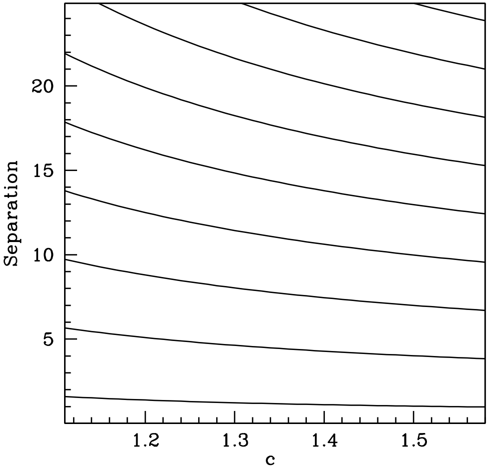
- Selection of source-sink separations from the discrete family by initial and boundary conditions
- Stability of source-sink patterns
- Higher order terms (sink-sink coupling)
- Extension to $b \neq 0$

M.J. Smith, J.D.M. Rademacher, J.A. Sherratt:
Absolute stability of wavetrains can explain spatiotemporal dynamics in reaction-diffusion systems of lambda-omega type.
SIAM J. Appl. Dyn. Systems 8, 1136-1159 (2009).

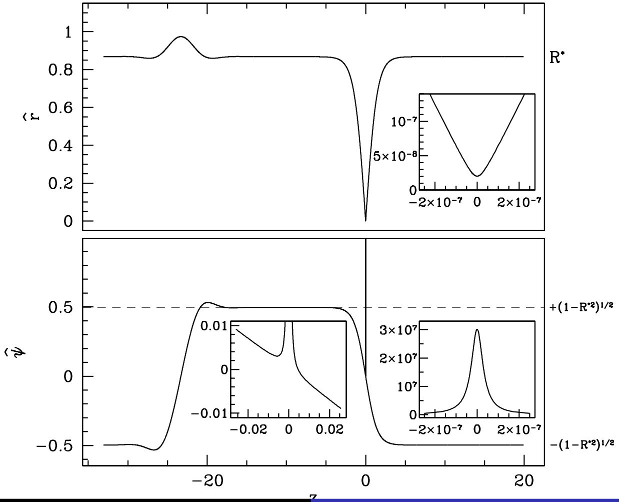
J.A. Sherratt, M.J. Smith, J.D.M. Rademacher:
Patterns of sources and sinks in the complex Ginzburg-Landau equation with zero linear dispersion.
SIAM J. Appl. Dyn. Systems 9, 883-918 (2010).

List of Frames

Dependence of Source-Sink Separations on c



Detailed form of a Source-Sink Pair



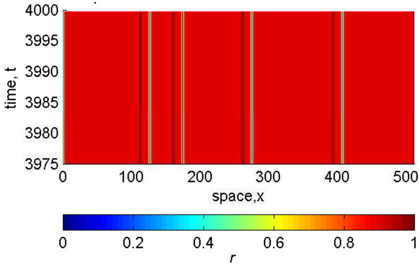
Experimental Observation of Sources and Sinks

Experimental systems in which sources and sinks have been observed include:

- chemical reactions
- electrochemical systems
- heated wire convection
- binary fluid convection
- convection waves generated by heating at a boundary
- the “printer’s instability”, in which the thin gap between two rotating acentric cylinders is filled with oil.

Movement of Sources and Sinks

These sources and sinks appear to be stationary........



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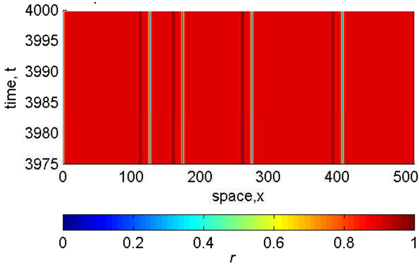
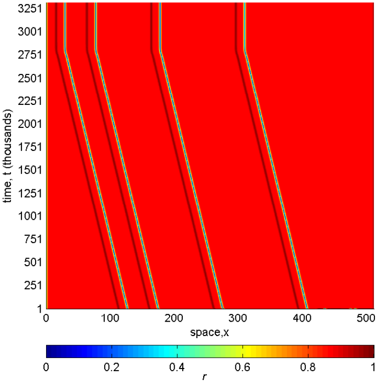
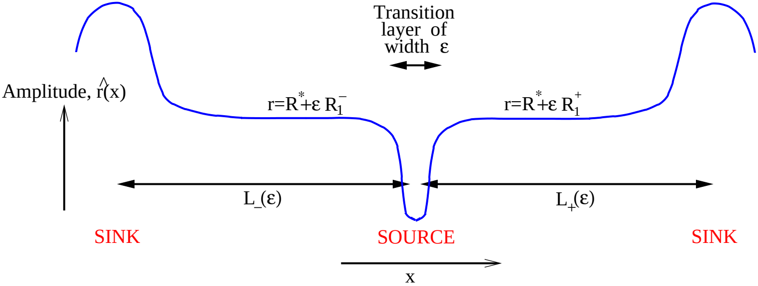


..........but very long simulations show that they move.

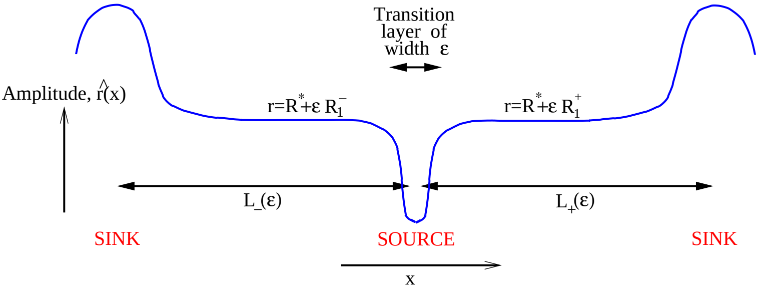


Perturbation Theory Calculation



We study the problem using perturbation theory.

Perturbation Theory Calculation

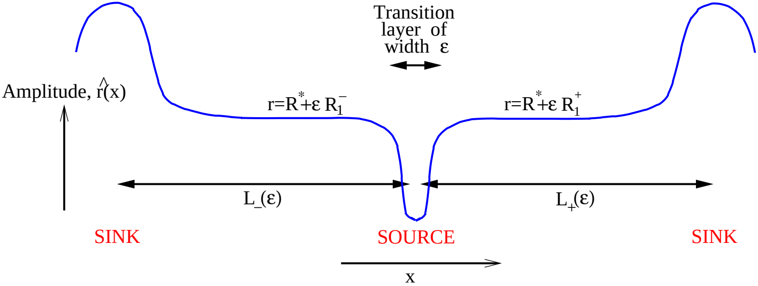


We study the problem using perturbation theory.

For $\epsilon = 0$:

$$\begin{aligned} K &= (9 - \sqrt{81 + 72c^2})/(4c) \\ \hat{r} &= R^* |\tanh(x/\sqrt{2})| \\ \hat{\psi} &= -(1 - R^{*2})^{1/2} \tanh(x/\sqrt{2}) \end{aligned}$$

Perturbation Theory Calculation

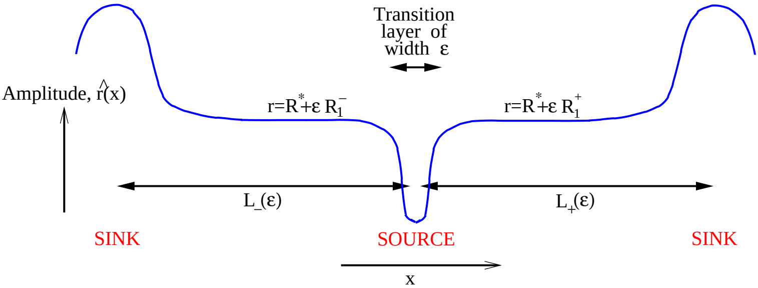


We study the problem using perturbation theory.

For $\epsilon \neq 0$:

$$\begin{aligned} K &= (9-\sqrt{81+72c^2})/(4c) + \epsilon K_1 + O(\epsilon^2) \\ \widehat{r} &= R^*|\tanh(x/\sqrt{2})| + \epsilon\widehat{r}_1(x) + O(\epsilon^2) \\ \widehat{\psi} &= -(1-R^{*2})^{1/2}\tanh(x/\sqrt{2}) + \epsilon\widehat{\psi}_1(x) + O(\epsilon^2) \end{aligned}$$

Perturbation Theory Calculation



Key result (phase-locking condition):

$$\arg\left[\exp\left(-L_-(1+i\delta)/\sqrt{2}\right)+\exp\left(-L_+(1+i\delta)/\sqrt{2}\right)\right]=\text{constant}\,.$$

The constant is determined explicitly.