1 Existence of Travelling Wave Fronts for a Reaction-Diffusion Equation with Quadratic-Type Kinetics

Theorem. *Consider the equation* $u_t = Du_{xx} + f(u)$ *with* $f(0) = f(1) = 0, f(u) > 0$ *on* $0 < u < 1, f'(0) > 0$, *and* $f'(u) < f'(0)$ *on* $0 < u \leq 1$. *There is a positive travelling wave solution for all wave speeds* $\geq 2\sqrt{Df'(0)}$, *and no positive travelling waves for speeds less than this critical value.*

Proof. : The travelling wave solution of speed c is $u(x,t) = U(z), z = x - ct$, satisfying $DU'' + cU' + f(U) = 0$, i.e.

$$\begin{aligned} U' &= V \\ V' &= -\frac{1}{D}\left[cV + f(U)\right], \end{aligned}$$

This has steady states: $(0,0), (1,0)$. Consider the behaviour near $(0,0)$. The stability matrix is:

$$\begin{pmatrix} 0 & 1 \\ -f'(0)/D & -c/D \end{pmatrix}$$

which has eigenvalues:

$$\frac{1}{2D}\left(-c \pm \sqrt{c^2 - 4f'(0)D}\right).$$

Therefore, $(0,0)$ is a stable node if $c \geq 2\sqrt{f'(0)D}$, and a stable focus otherwise. If it is a focus then $U < 0$ at some points, whereas we require $U \geq 0$. Therefore, there are no positive waves if $c < 2\sqrt{f'(0)D}$.

Now consider the behaviour near $(1,0)$. The stability matrix is:

$$\begin{pmatrix} 0 & 1 \\ -f'(1)/D & -c/D \end{pmatrix}$$

which has eigenvalues:

$$\frac{1}{2D}\left(-c \pm \sqrt{c^2 - 4f'(1)D}\right).$$

Therefore, $(1,0)$ is a saddle point. (Note that $f'(1) < 0$). Therefore, there is exactly one trajectory starting at $(1,0)$ with V becoming negative. If there is a travelling wave, it must correspond to this trajectory. Where does the trajectory go? There are three possibilities (see figure 1.1):

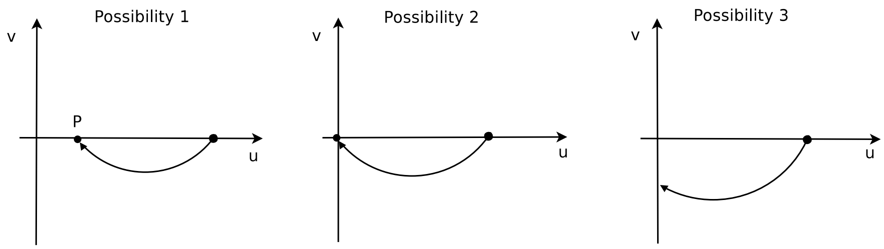

Figure 1.1: The three possible paths of the (unique) trajectory starting at $(1,0)$ with V becoming negative.

- In case 1: At the point P, $f(U) > 0$ and $V = 0 \Rightarrow V' < 0$, while in fact V is changing from negative to positive $\Rightarrow V' > 0$. Therefore, case 1 cannot occur.

- In case 3: Let λ be one of the eigenvalues at $(0,0)$ (real and negative, since $c \geq 2\sqrt{f'(0)D}$). Consider the line $V = \lambda U$ in the phase plane. Then,

$$\begin{aligned} \frac{dV}{dU} = \frac{V'}{U'} &= \frac{-\frac{1}{D}\left[cV + f(U)\right]}{V} \\ &= -\frac{c}{D} + \frac{f(U)}{D(-V)} \\ &< -\frac{c}{D} + \frac{f'(0)U}{D(-V)} \text{ using } f(U) < f'(0)U \\ &= -\frac{c}{D} + \frac{f'(0)}{-\lambda D} \text{ using } V = \lambda U \\ &= \lambda, \end{aligned}$$

  since $D\lambda^2 + c\lambda + f'(0) = 0$ is the eigenvalue equation. At point $Q, dV/dU < \lambda$. But $dV/dU > \lambda$ (see figure 1.2). Hence, case 3 cannot occur.

Hence, case 2 must occur, i.e. whenever $c \geq 2\sqrt{f'(0)D}$, the trajectory leaving $(1,0)$ ends at $(0,0)$, corresponding to a travelling wave solution. □

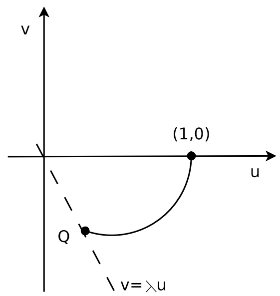

Figure 1.2: An illustration of the intersection that would occur in case 3, between the travelling wave trajectory and the line $V = \lambda U$.

2 Existence of Travelling Wave Fronts for a Reaction-Diffusion Equation with Bistable Kinetics

Theorem. *Consider the equation* $u_t = Du_{xx} + f(u)$ *with* $f(u_1) = f(u_2) = f(u_3) = 0, f'(u_1) < 0, f'(u_2) > 0, f'(u_3) < 0$*. There is a positive travelling wave solution* $u(x,t) = U(x - ct)$ *with* $U(-\infty) = u_1$ *and* $U(+\infty) = u_3$ *for exactly one value of the wave speed* c*.*

Proof. : Write $u(x,t) = U(z), z = x - ct$ (travelling wave solution), and $V(z) = dU/dz$. Without loss of generality assume that f is such that the wave moves from u_1 to u_3. The travelling wave ODEs are:

$$\begin{aligned} U' &= V \\ V' &= -\frac{c}{D}V - \frac{f(U)}{D}. \end{aligned}$$

The eigenvalues at the steady states are:

$$\lambda = \frac{1}{2D}\left[-c \pm \sqrt{c^2 - 4f'(u_i)}\right].$$

Now $f' < 0$ at $(u_1, 0)$ and $(u_3, 0)$ $\Rightarrow$ both are saddle points (two real eigenvalues, one positive, one negative).

A travelling wave corresponds to a trajectory leaving $(u_1, 0)$ and ending at $(u_3, 0)$ with $V > 0$ (U is increasing). Therefore, the trajectory must leave $(u_1, 0)$ along T_1 and enter $(u_3, 0)$ along T_3 (see figure (2.1))

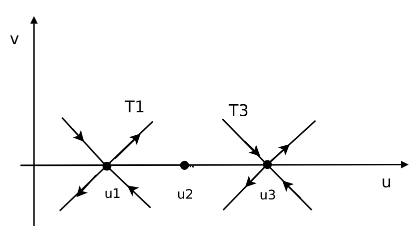

Figure 2.1: An illustration of the trajectories T_1 and T_3.

Therefore, there is a travelling wave $\Leftrightarrow$ T_1 and T_3 are the same trajectory.

Fix $\xi \in (u_1, u_3)$ and let V_1, V_3 be the values of V at which T_1, T_3 hit the line $U = \xi$. Then

$$\frac{dV}{dU} = -\frac{c}{D} - \frac{f(U)}{D}, \text{which decreases as } c \text{ increases.}$$

Hence, V_1 decreases and V_3 increases as c increases (explained below). Moreover

$$\text{as } c \to +\infty,\ dV/dU \to -\infty, \text{ everywhere} \Rightarrow V_3 \to +\infty, V_1 \to -\infty$$

and

$$\text{as } c \to -\infty,\ dV/dU \to +\infty, \text{ everywhere} \Rightarrow V_3 \to -\infty, V_1 \to +\infty.$$

Therefore, V_1 and V_3 are the same for exactly one value of $c \Rightarrow$ there is a travelling wave for this speed only. □

Why does $V_1 \downarrow$ as $c \uparrow$?

The eigenvector for T_1 at $(u_1, 0)$ is $\left(1, \frac{1}{2D}\left[-c + \sqrt{c^2 - 4f'(u_1)}\right]\right)$, which becomes shallower as c increases. Therefore, if $V_1(c = c_B) > V_1(c = c_A)$ with $c_B > c_A$, then at P:

$$\left.\frac{dV}{dU}\right|_{c=c_B} > \left.\frac{dV}{dU}\right|_{c=c_A}.$$

But, $dV/dU \downarrow$ as $c \uparrow$. (Similarly, for $V_3 \uparrow$ as $c \downarrow$). □

3 Condition for Stability of Periodic Travelling Waves in $\lambda-\omega$ Equations with $\omega(.)$ constant

Work with the equations for r and θ, which are:

$$r_t = r\lambda(r) + r_{xx} - r\theta_x^2$$

$$\theta_t = \omega_0 + \theta_{xx} + \frac{2}{r} r_x \theta_x$$

where $\omega(.) \equiv \omega_0$. A periodic travelling wave solution is $r = R, \theta = \sqrt{\lambda(R)}x + \omega_0 t$. Consider a small perturbation:

$$r = R + \tilde{r}(x,t),$$

$$\theta = \sqrt{\lambda(R)}x + \omega_0 t + \tilde{\theta}(x,t)\,.$$

Substitute this into the $\lambda - \omega$ PDEs and linearise:

$$\begin{aligned} \tilde{r}_t &= \tilde{r}\lambda(R) + R\lambda'(R)\tilde{r} + \tilde{r}_{xx} - 2R\sqrt{\lambda(R)}\tilde{\theta}_x - \tilde{r}\lambda(R) \\ \tilde{\theta}_t &= \tilde{\theta}_x x + \frac{2}{R}\sqrt{\lambda(R)}\tilde{r}_x\,. \end{aligned}$$

Look for solutions: $\tilde{r} = \bar{r}.e^{\nu t + ikx}, \tilde{\theta} = \bar{\theta}.e^{\nu t + ikx}$, where $\bar{r}, \bar{\theta}$ are constants:

$$\left\{\nu - R\lambda'(R) + k^2\right\}\bar{r} + 2ik.R\sqrt{\lambda(R)}\bar{\theta} = 0$$

$$\frac{2ik}{R}\sqrt{\lambda(R)}\bar{r} - (k^2 + \nu)\bar{\theta} = 0\,.$$

Therefore, for non-trivial solutions we require:

$$\left\{\nu - R\lambda'(R) + k^2\right\}(k^2 + \nu) = 4k^2\lambda(R)$$

$$\text{i.e.} \quad \nu^2 + \left[2k^2 - R\lambda(R)\right]\nu + k^2\left[k^2 - \{4\lambda(R) + R\lambda'(R)\}\right] = 0\,.$$

We have $\lambda'(R) < 0$, so that the coefficient of ν is strictly positive. Therefore there are either:

	two real negative roots for ν	$\Rightarrow$	wave is stable
or	complex conjugate roots with -ve real part for ν	$\Rightarrow$	wave is stable
or	one real +ve and one real -ve root for ν	$\Rightarrow$	wave is unstable.

The condition for the third possibility is $k^2 - \{4\lambda(R) + R\lambda'(R)\} < 0$. This is true for some real $k \iff 4\lambda(R) + R\lambda'(R) > 0$. Therefore,

$$\begin{aligned} \text{the wave is stable} &\iff \text{stable to perturbations with any wavenumber } k \\ &\iff 4\lambda(R) + R\lambda'(R) < 0. \end{aligned}$$

4 Generation of Periodic Waves in $\lambda - \omega$ Systems

Local disturbance of $u = v = 0$ causes travelling fronts in r and θ_x. Ahead of these fronts, u and $v \to 0$, and behind them u and v approach periodic travelling waves (so that r and θ_x approach constant values). The $r - \theta$ PDEs are:

$$\begin{cases} r_t &= r\lambda(r) + r_{xx} - r\theta_x^2 \\ \theta_t &= \omega(r) + \theta_{xx} + 2r_x\theta_x/r\,. \end{cases}$$

Look for solutions of form:

$$\begin{cases} r &= \bar{r}(x - st) \\ \theta_x &= \bar{\psi}(x - st) \Rightarrow \theta = \overbrace{\bar{\Psi}(x - st)}^{\text{integral of } \bar{\psi}} + f(t)\,. \end{cases}$$

As $x \to \infty, r \to 0$ and $\theta_x \to 0 \Rightarrow \theta \to \omega(0)t \Rightarrow$, $f(t) = \omega(0)t+$ constant. Substituting this into the $r - \theta$ PDEs gives:

$$\begin{cases} -s\bar{r}' &= \bar{r}\lambda(\bar{r}) + \bar{r}'' - \bar{r}\bar{\psi}^2 \\ -s\bar{\psi} + \omega(0) &= \omega(\bar{r}) + \psi' + 2\bar{r}'\bar{\psi}/\bar{r}. \end{cases}$$

Now consider behaviour as $x \to -\infty$ (so that $\bar{r} \to r_s$, $\bar{\psi} \to \psi_s$):

$$\begin{cases} 0 &= r_s\lambda(r_s) + 0 - r_s\psi_s^2 \\ -s\psi_s + \omega(0) &= \omega(r_s) + 0 + 0 \end{cases}$$

$$\Rightarrow \begin{cases} \psi_s &= \pm\sqrt{\lambda(r_s)} \\ \underbrace{s^2\lambda(r_s)}_{\text{decreasing function of } r_s} &= \underbrace{[\omega(0) - \omega(r_s)]^2}_{\text{increasing function of } r_s} \end{cases}$$

Therefore, there is a unique solution for r_s, dependent on the front speed s.

The front speed s can be expected to be $2\sqrt{\lambda(0)}$ based on results for scalar equations (linearising ahead of the front gives $r_t = \lambda(0)r + r_{xx}$). A proof of this is currently lacking (but numerical simulations provide strong evidence that it's correct). Hence, a unique periodic wave is selected, with amplitude given by the solution of:

$$4\lambda(0)\lambda(r_s) = [\omega(0) - \omega(r_s)]^2\,.$$

Some examples of wave generation of this type are illustrated on the next page (figure 4.1).

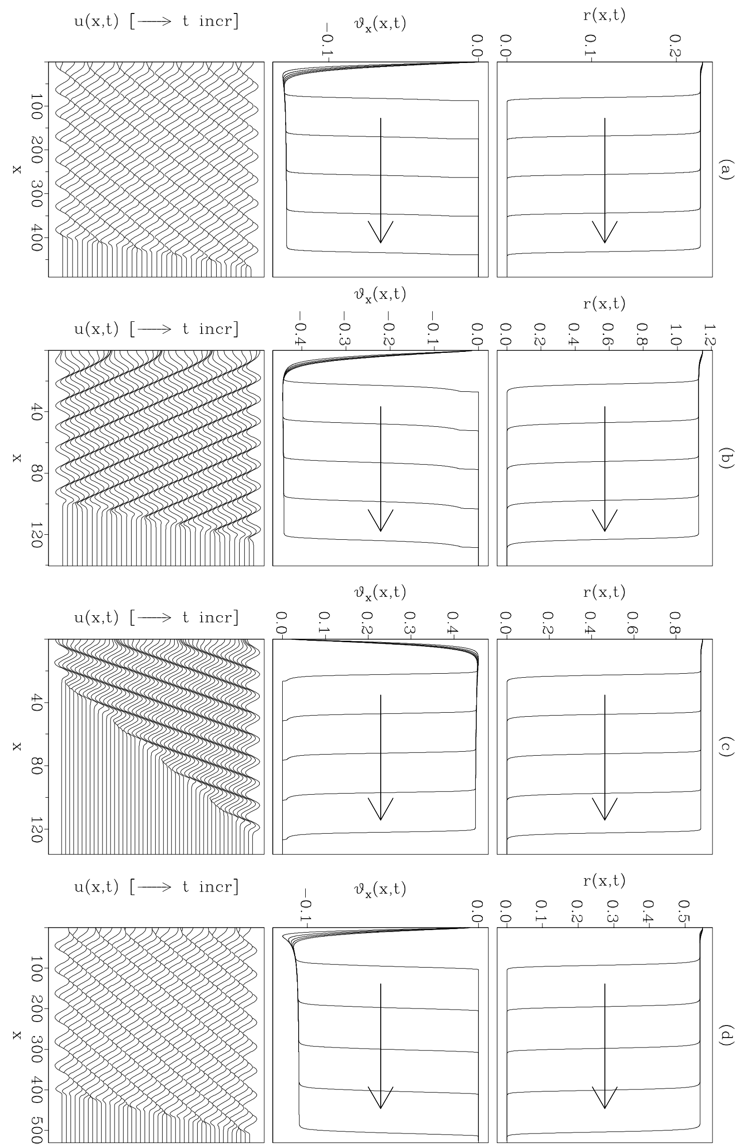

Figure 4.1: Examples of the generation of periodic travelling waves by local disturbance of $u = v = 0$ in λ–ω systems. For details of functional forms and parameter values, see the legend of Figure 2 in the paper J.A. Sherratt: Periodic waves in reaction-diffusion models of oscillatory biological systems. *FORMA* 11: 61-80 (1996). which is available from `www.ma.hw.ac.uk/~jas/publications.html`

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5 Spiral Waves in $\lambda - \omega$ Systems

Look for a solution of the form:

$$\begin{cases} r = r(\rho) & \rho, \phi \equiv \text{polar coordinates in } x-y \text{ plane} \\ \theta = \Omega t + m\phi + \psi(\rho) & r, \theta \equiv \text{polar coordinates in } u-v \text{ plane.} \end{cases}$$

In 2-D, $r - \theta$ PDEs are:

$$\Rightarrow \begin{cases} r_t = r\lambda(r) + \nabla^2 r - r|\nabla\theta|^2 \\ \theta_t = \omega(r) + \nabla^2\theta + \frac{2}{r}\nabla r.\nabla\theta\,. \end{cases}$$

Substitute the solution form into these equations using expressions for ∇ and ∇^2 in polar coordinates:

$$\Rightarrow \begin{cases} r^{''} + \frac{1}{\rho}r^{'} - r\psi^{'2} - \frac{1}{\rho^2}m^2 + r\lambda(r) = 0 \\ \psi^{''} + (\frac{1}{\rho} + \frac{2r^{'}}{r})\psi^{'} = \Omega - \omega(r) \end{cases}$$

We require r and $\psi^{'} \to$ constants as $\rho \to \infty$ ($\leftrightarrow$ periodic travelling wave):

$$\Rightarrow \begin{cases} \lambda(r_\infty) = \psi_\infty^{'2} & \text{(compatible with periodic trav. wave)} \\ 0 = \Omega - \omega(r_\infty) & \text{(this determines } \Omega\text{).} \end{cases}$$

Now consider solutions near $\rho = 0$. Then, to leading order:

$$\begin{cases} r^{''} + \frac{1}{\rho}r^{'} - \frac{m^2}{\rho^2}r = 0 \\ \psi^{''}(0) = \Omega - \omega(0). \end{cases}$$

(Note that $\psi^{'}(0) = 0$ for regularity, but $\psi(0) \neq 0$ in general)

Therefore, the form of the solution near the origin is:

$$\begin{matrix} u \\ v \end{matrix} \sim \rho^m \begin{matrix} \cos \\ \sin \end{matrix} \left(\Omega t + m\phi + \underbrace{\psi(0) + \frac{1}{2}(\Omega - \omega(0))\rho^{2m}}_{\text{Taylor expansion of } \psi \text{ near zero}} \right)$$

The maximum of u occurs when $\Omega t + m\phi + \psi(0) + \frac{1}{2}(\Omega - \omega(0))\rho^{2m} = 0$. For fixed t this implies

$$\phi = \frac{1}{2m}(\omega(0) - \omega(r_\infty))\rho^{2m} + \text{constant.}$$

This is the polar coordinate equation of a spiral.