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# The mean values of the Weierstrass elliptic function $\wp$ : Theory and application

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ABSTRACT

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# HIGHLIGHTS

- Derivation of differential equations satisfied by the mean values of  $\wp$ .
- Solution of these equations in terms of hypergeometric functions and Legendre functions.
- Numerical computation of the means for both real and complex valued invariants.
- Application of the results to vegetation patterning in semi-arid landscapes.

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### 1. Introduction

The Weierstrass elliptic function  $\wp : \mathbb{C} \to \mathbb{C}$  is defined by

$$\wp(z) = z^{-2} + \sum_{w \in \mathbb{L} \setminus \{0\}} \left[ (z - w^2)^{-1} - w^{-2} \right]$$

where the lattice  $\mathbb{L} = \{2m\omega_1 + 2n\omega_3 | m, n \in \mathbb{Z}\}$  with  $\text{Im}(\omega_3/\omega_1)$ > 0.<sup>1</sup>  $\wp$  is therefore doubly periodic, with  $2\omega_1$  and  $2\omega_3$  both being periods; note that our use of the suffixes 1 and 3 follows convention. An important property of  $\wp$  is that it satisfies the differential

equation

vegetation patterning in semi-arid landscapes.

The Weierstrass elliptic functions can be parameterised using either lattice generators or invariants. Most

presentations adopt the former approach. In this paper the authors give formulae that enable conversion

between the two representations. Using these, they obtain differential equations satisfied by the mean

values of  $\wp$  over its periods; these mean values are considered as functions of the invariants. They show

how to construct exact solutions for the means in terms of both hypergeometric functions and Legendre

functions. These solutions are valid for both real and complex values of the invariants. For the case of real

invariants, the authors prove various monotonicity results for the means with respect to the invariants. They also discuss the numerical computation of the means, and show a number of plots of the means

against both real and complex valued invariants. Finally, they consider an application of their results to

$$(d\wp/dz)^2 = 4\wp^3 - g_2\wp - g_3 \tag{1}$$

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where  $g_2, g_3 \in \mathbb{C}$  are the Weierstrass invariants, defined by

$$g_2 = 60 \sum_{w \in \mathbb{L} \setminus \{0\}} w^{-4} \qquad g_3 = 140 \sum_{w \in \mathbb{L} \setminus \{0\}} w^{-6}.$$
 (2)

For further general background on the Weierstrass elliptic function, see for example the books by Whittaker and Watson [2, Ch. 20], Akhiezer [3, Ch. 3], Walker [4] and Armitage and Eberlein [5].

Most treatments of the Weierstrassian functions consider them to be parameterised by the half-lattice generators  $\omega_1$  and  $\omega_3$ . This is convenient for the development of mathematical theory, and also for many mathematical and some physical applications. The choice of variables has no formal restrictions, so instead of  $\omega_1$  and  $\omega_3$  one may take an arbitrary pair of quantities depending (nondegenerately) on them. For example, some authors use  $\omega = \omega_1$ and  $\tau = \omega_3/\omega_1$  as parameters; these have a natural geometrical interpretation, specifying respectively the size and shape of the







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<sup>&</sup>lt;sup>1</sup> Many presentations of the Weierstrass elliptic functions require only the condition Im $(\omega_3/\omega_1) \neq 0$ . Our assumption follows [1]. It does not result in any loss of generality, and is necessary for our discussion of modular inversion (Sections 3 and 8.2): Klein's function I(.), defined in (32) below, only exists for values of the argument in the upper half of the complex plane.

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"fundamental parallelogram" [6, Section 18.1]. However, in some contexts the invariants  $g_2$  and  $g_3$  are the natural parameters because they correspond directly with physical quantities; usually  $g_2, g_3 \in \mathbb{R}$  in such cases. Examples from Newtonian dynamics include the spherical pendulum, in which the invariants are functions of the radius of the sphere and the initial position and velocity of the bob [7, Section 7.3], and the motion of a gyroscope, for which the invariants depend on the mass, the moments of inertia, and various constants of the motion [7, Section 7.4]. In a recent application to general relativity, Gibbons and Vyska [8] showed that the equation obeyed by a null geodesic in the Schwarzschild metric can be reduced to (1), with  $g_3$  depending on the energy of the light, the angular momentum, and the mass of the black hole. A guite different example comes from the modelling of banded patterns of vegetation in semi-arid environments [9-11], where  $g_3$  determines the migration speed of the patterns; this example will be discussed in more detail in Section 9.

With these considerations in mind, this paper concerns the variation in  $\wp$  and related functions and quantities with the invariants  $g_2$  and  $g_3$ . We will focus in particular on the mean values of  $\wp$  over the lattice generators, which are important quantities in some applications, but which have received almost no discussion in the literature. To study the means, we will use the Weierstrass zeta function, which is defined by

$$\zeta(z) = z^{-1} + \sum_{w \in \mathbb{L} \setminus \{0\}} \left[ (z - w)^{-1} + w^{-1} + zw^{-2} \right]$$

and which satisfies  $d\zeta/dz = -\wp$ . We will also use the Weierstrass eta functions  $\eta_j = \zeta(\omega_j)$  (j = 1, 3).

Some remarks about our notation are in order. All the Weierstrassian functions are functions of three variables; therefore we use the notation  $\wp(z|\omega_1, \omega_3) = \wp(z; g_2, g_3)$  (and analogously for  $\zeta$  and  $\wp'$ ) when explicit dependence on parameter pairs  $(\omega_1, \omega_3)$  or  $(g_2, g_3)$  is important.

#### 2. Key mathematical formulae

The Weierstrass elliptic functions can be parameterised either by the half-lattice generators  $\omega_1$  and  $\omega_3$ , or by the invariants  $g_2$  and  $g_3$ . Despite the many accounts of the Weierstrass elliptic functions in textbooks and monographs, conversion between  $(\omega_1, \omega_3)$  and  $(g_2, g_3)$  is hardly mentioned. For example, in numbertheoretic considerations computation of the invariants  $g_2$  and  $g_3$  (as functions of the  $\omega_j$ 's) uses Jacobi's theta-functions or the famous Eisenstein series  $E_4$ ,  $E_6$  in  $e^{\pi i \tau}$  multiplied by  $\omega^{-4}$  and  $\omega^{-6}$ respectively [6, Section 18.10]; however this does not furnish a practical means of conversion. Since the standard approach is to regard  $\omega_1$  and  $\omega_3$  as the key parameters, this poses a significant barrier to using the Weierstrassian theory for cases in which the natural parameterisation is via the invariants. Such a situation occurs whenever (1) arises as an important differential equation in a physical application.

We are aware of 3 references discussing conversion between  $(\omega_1, \omega_3)$  and  $(g_2, g_3)$ , all from the 19th Century: the paper of Frobenius and Stickelberger [12, p. 313–316] and the books of Halphen [13, p. 302–307, 319–320] and Forsyth [14, p. 263–265]. The key result is

$$\omega_{1} \frac{\partial F}{\partial \omega_{1}} + \omega_{3} \frac{\partial F}{\partial \omega_{3}} = -4g_{2} \frac{\partial \widetilde{F}}{\partial g_{2}} - 6g_{3} \frac{\partial \widetilde{F}}{\partial g_{3}} = \omega \frac{\partial \overline{F}}{\partial \omega},$$
  

$$\eta_{1} \frac{\partial F}{\partial \omega_{1}} + \eta_{3} \frac{\partial F}{\partial \omega_{3}} = -6g_{3} \frac{\partial \widetilde{F}}{\partial g_{2}} - \frac{1}{3}g_{2}^{2} \frac{\partial \widetilde{F}}{\partial g_{3}}$$
  

$$= \eta_{1} \frac{\partial \overline{F}}{\partial \omega} - \frac{\pi i}{2\omega^{2}} \frac{\partial \overline{F}}{\partial \tau},$$
(3)

where  $F(\omega_1, \omega_3) = \overline{F}(\omega, \tau) = \widetilde{F}(g_2, g_3)$  is any quantity depending on  $\omega_1$  and  $\omega_3$ , or equivalently on  $\omega$  and  $\tau$ , or on  $g_2$  and  $g_3$ . In view of the obscurity of the references, we give a derivation of (3) in Appendix A. The relationship between derivatives with respect to  $(\omega_1, \omega_3)$  and  $(\omega, \tau)$  follows from Legendre's identity  $\omega_3\eta_1 - \omega_1\eta_3 = \frac{1}{2}\pi i$  [1, Section 23.2.14].

Substituting  $F(\omega_1, \omega_3) = \omega_i (j = 1, 3)$  in (3) gives

$$\Delta \frac{\partial \omega_j}{\partial g_2} = -\frac{1}{4}g_2^2 \omega_j + \frac{9}{2}g_3 \eta_j \qquad \Delta \frac{\partial \omega_j}{\partial g_3} = \frac{9}{2}g_3 \omega_j - 3g_2 \eta_j \tag{4}$$

where the discriminant  $\Delta = g_2^3 - 27g_3^2$ . Substituting  $F = \wp$  and  $F = \zeta$  into (3) and using the expressions in [15, Thm. 10.1] for  $\partial \wp / \partial \omega_i$  and  $\partial \zeta / \partial \omega_j$  gives formulae for  $\partial \wp / \partial g_i$  and  $\partial \zeta / \partial g_i$  (i = 2, 3); these are already available in [6, Sections 18.6.19–18.6.22].

Our focus in this paper is on the mean values

$$\mu_{j} = \frac{1}{2\omega_{j}} \int_{z=z_{0}}^{z=z_{0}+2\omega_{j}} \wp(z) \, dz.$$
(5)

Here  $z_0$  can be any point except a lattice point, at each of which  $\wp$  has a non-integrable singularity, and the integration is along any path that does not pass through one of the lattice points. Since  $d\zeta(z)/dz = -\wp(z)$ ,

$$\mu_j = \left(\zeta(z_0) - \zeta(z_0 + 2\omega_j)\right) / (2\omega_j) = -\eta_j / \omega_j,$$

independent of  $z_0$  [1, Section 23.2.11]. Note that for a given lattice, the lattice generators  $2\omega_1$  and  $2\omega_3$  are not unique:  $2l_{11}\omega_1 + 2l_{13}\omega_3$  and  $2l_{31}\omega_1 + 2l_{33}\omega_3$  are also generators for any integers  $l_{ij}$  such that  $l_{11}l_{33} - l_{13}l_{31} = 1$  [1, Section 23.2(i)]. The invariants  $g_2$  and  $g_3$  are determined by the lattice only, not the choice of generators [1, Section 23.3(i)]; however  $\mu_1$  and  $\mu_3$  do depend on the choice of generators.

The  $\mu_j \equiv \mu_j(g_2, g_3)$  are potentially important quantities in applications wherein  $g_2$  and  $g_3$  are the physically meaningful parameters. For instance, in the application to vegetation patterning discussed in Section 9,  $\mu_1$  corresponds to the average plant biomass, which is a key variable from environmental, conservation and land management perspectives. However, to the best of our knowledge there is no discussion in the literature of the  $\mu_j$ 's, with the slight exception of a brief comment on page 315 of the 1886 edition of Halphen's book [13].

The formulae for  $\partial \zeta / \partial g_i$  (i = 2, 3), obtained as discussed above, yield

$$\Delta \frac{\partial \eta_j}{\partial g_2} = \frac{1}{4} g_2^2 \eta_j - \frac{3}{8} g_2 g_3 \omega_j \qquad \Delta \frac{\partial \eta_j}{\partial g_3} = -\frac{9}{2} g_3 \eta_j + \frac{1}{4} g_2^2 \omega_j. \tag{6}$$

Combining (4) and (6) gives the system of non-autonomous differential equations

$$\Delta \frac{\partial \mu_j}{\partial g_2} = \frac{3}{8} g_2 g_3 + \frac{1}{2} g_2^2 \mu_j + \frac{9}{2} g_3 \mu_j^2$$

$$\Delta \frac{\partial \mu_j}{\partial g_3} = -9 g_3 \mu_j - 3 g_2 \mu_j^2 - \frac{1}{4} g_2^2.$$
(7)

A remarkable property of Eqs. (7) is that they are self-contained, and do not involve  $\eta_j$  or  $\omega_j$ ; this contrasts with the equations for the  $\eta_j$ 's and  $\omega_j$ 's alone. By construction, the differential equations (7) are compatible, i.e.,  $\partial_{g_2}(\partial_{g_3}\mu_j) = \partial_{g_3}(\partial_{g_2}\mu_j)$ . In Section 5 we will show that (7) is exactly solvable.

It is important to comment that the Weierstrassian theory can be reduced to dependence on a single parameter by renormalisation. In particular, Section 18.2.13 and Section 18.2.14 of [6] yield

$$\mu_j(t^2g_2, t^3g_3) = t\mu_j(g_2, g_3)$$

which holds for any  $t \in \mathbb{C}$ . Substituting  $t = g_2^{-1/2}$  and  $t = g_3^{-1/3}$  gives

$$\mu_j(g_2, g_3) = g_2^{1/2} \mu_j\left(1, g_2^{-3/2} g_3\right) = g_3^{1/3} \mu_j\left(g_3^{-2/3} g_2, 1\right)$$
(8)

(j = 1, 3). In applications it can be convenient to retain pairs of parameters, and therefore we will continue with this description. However we will exploit the reduction to a single parameter in our derivation of exact solutions, in Section 5.

The outline of the paper is as follows. In Section 3 we make some comments on the relationship between the calculation of the  $\mu_j$ 's and the elliptic modular inversion problem. In Section 4 we introduce the two types of lattice (rectangular and rhombic) that arise when  $g_2$ ,  $g_3 \in \mathbb{R}$ . In Section 5 we derive exact solutions for the  $\mu_j$ 's for these two lattice types, and in Section 6 we derive various monotonicity results for the  $\mu_j$ 's as functions of the invariants, again when the latter are real. In Section 7 we discuss numerical calculation of the  $\mu_j$ 's for real  $g_2$  and  $g_3$ . In Section 8 we consider  $g_2$ ,  $g_3 \in \mathbb{C}$ , extending our analytical results to this case and discussing numerical calculation of the  $\mu_j$ 's. Finally in Section 9 we consider an application of our results to vegetation patterning in semi-deserts.

# 3. Remarks concerning modular inversions

The pairs  $(g_2, g_3)$  and  $(\omega_1, \omega_3)$  are mutually dependent. Given  $\omega_1$  and  $\omega_3$ , the values of  $g_2$  and  $g_3$  are implied by (2); the opposite conversion is known as the elliptic modular inversion problem [2, Section 21.73], [16]. Tables of solutions are available (e.g. [6, Table 18.1]) but standard analytical solutions only apply when the Weierstrass elliptic equation  $y^2 = 4x^3 - g_2x - g_3$  has been converted to Jacobian normal form  $Y^2 = (1 - X^2)(1 - k^2X^2)$ . The principal parameter of the theory, the ratio  $\tau = \omega_3/\omega_1$ , is given for this case by the famous Jacobi formula  $\tau = i\mathbf{K}(\sqrt{1-k^2})/\mathbf{K}(k)$ , where  $\mathbf{K}$  is Legendre's complete integral of the first kind [17], [1, Section 19.2.8]. For the Weierstrassian form all the parameters of the theory can, of course, be computed [18, Eqs. 27 and 28], although standard formulae for the solution are not as elegant as in Legendre's case [19, Section 14.6.2], [13, p. 341–348]. However, more compact analytical formulae were obtained recently by Brezhnev [15].

Eq. (1) implies that modular inversion centres around the calculation of the complete elliptic holomorphic integrals

$$\pm \omega_j = \int_{\infty}^{e_j} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$
(9)

where  $e_1$ ,  $e_2$  and  $e_3$  are the "lattice roots", i.e. the roots of  $y^2 = 4x^3 - g_2x - g_3$ . In comparison we are concerned with an extension of standard modular inversion, in the sense that inversion of (9) is supplemented by an inversion of the meromorphic (second kind elliptic) integral

$$\pm \mu_j(\omega_1, \omega_3) = \frac{1}{2\omega_j} \int_{\infty}^{e_j} \frac{z \, dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$
 (10)

As well as being a very classical subject, this circle of questions attracts much attention in connection with a spread of the theory beyond elliptic curves. For higher genus algebraic dependences, periods of second kind integrals also satisfy differential relations and are related to the theories of integrable systems and theta-constants [20, Section 4: hyperelliptic dependences]. The point here is that choices of dependent/independent variables among transcendental periods and coefficients of algebraic relations allow much freedom. The differential relations above become differential equations and this leads to numerous applications. Extension of these theories to an arbitrary (non-hyperelliptic) case is far from

complete and is a subject of intense recent studies [21]. It may be also mentioned here that results that follow in this work can be directly and effectively applied to some particular cases of [20,21] when abelian integrals are reduced to the elliptic ones. In these cases normalised periods of the second kind abelian integrals are expressed through a set of the *elliptic*  $\mu$ 's whose theory is expounded below. As is known [22], the complete set of periods of both the holomorphic and meromorphic integrals plays a central role in these theories.

# 4. Lattices for $g_2, g_3 \in \mathbb{R}$

The case of  $g_2$ ,  $g_3 \in \mathbb{R}$  is particularly well studied; indeed Abramowitz & Stegun [6] restrict attention to this case. Lattices then fall into one of two categories: "rectangular" and "rhombic", meaning that the "fundamental parallelogram" with vertices at 0,  $2\omega_1$ ,  $2\omega_3$  and  $2\omega_1 + 2\omega_3$  has these shapes. Note that  $g_2$ ,  $g_3 \in \mathbb{R}$ is a necessary and sufficient condition for the lattice to be either rectangular or rhombic [23, Thms. 3.16.2 and 3.16.4].

"Rectangular lattices" occur when  $g_2$ ,  $g_3 \in \mathbb{R}$  and  $\Delta > 0$ . They can always be constructed using half-lattice generators that satisfy  $\omega_1$ ,  $\omega_3/i \in \mathbb{R}^+$  [23, Section 3.16], [1, Section 23.5(i), (ii)] and throughout this paper our use of the phrase "rectangular lattice" will imply this choice of generators. An important special case is the "lemniscatic" lattice,  $\omega_1 \in \mathbb{R}$  and  $\omega_3 = i\omega_1$ . Then  $g_3 = 0$  and Section 23.5(iii) of [1] implies that

$$\mu_1(g_2, 0) = -\mu_3(g_2, 0) = -4\pi^2 g_2^{1/2} / \Gamma(1/4)^4 \quad (g_2 > 0).$$
(11)

"Rhombic lattices" occur when  $g_2$ ,  $g_3 \in \mathbb{R}$  and  $\Delta < 0$ . These can always be constructed using half-lattice generators that satisfy  $\omega_1 \in \mathbb{R}^+$ , Re  $\omega_3 = \frac{1}{2}\omega_1$ , Im  $\omega_3 > 0$  [23, Section 3.16], [1, Section 23.5(i), (ii)] and again our use of the phrase "rhombic lattice" will always imply this choice of generators. There are two important special cases. The "pseudo-lemniscatic" lattice has  $\omega_1 \in \mathbb{R}^+$  and  $\omega_3 = \frac{1}{2}(1 + i)\omega_1$ ; then  $g_2 < 0$ ,  $g_3 = 0$  and

$$\mu_1(g_2, 0) = i\mu_3(g_2, 0) = -4\pi^2(-g_2)^{1/2}/\Gamma(1/4)^4 \quad (g_2 < 0) (12)$$

using Section 18.15 of [6] and (8). The "equianharmonic" lattice has  $\omega_1 \in \mathbb{R}^+$  and  $\omega_3 = e^{\pi i/3}\omega_1$ ; then  $g_2 = 0$ ,  $g_3 > 0$  and

$$u_1(0, g_3) = e^{2\pi i/3} \mu_3(0, g_3)$$
  
=  $-8\pi^3 g_3^{1/3} / \left(3^{1/2} \Gamma(1/3)^6\right) \quad (g_3 > 0)$  (13)

using Section 23.5(v) of [1].

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For the mean values  $\mu_1$  and  $\mu_3$  on these lattices, a key preliminary issue is whether or not they are real.

**Theorem 1.** For a rectangular lattice,  $\mu_1, \mu_3 \in \mathbb{R}$ . For a rhombic lattice,  $\mu_1 \in \mathbb{R}$  and  $\mu_3 \notin \mathbb{R}$ .

**Proof.** For the rectangular lattice case,  $\wp(z) \in \mathbb{R}$  when  $\operatorname{Im} z = \omega_3/i$ . Therefore taking  $z_0 = \omega_3$  and j = 1 in (5) shows that  $\mu_1 \in \mathbb{R}$ . For  $\mu_3$ , homogeneity relations imply that  $\wp(z; g_2, g_3) = -\wp(iz; g_2, -g_3)$  and  $\zeta(z; g_2, g_3) = i\zeta(iz; g_2, -g_3)$  (e.g. substitute t = i into Section 18.2.13 and Section 18.2.14 of [6]). Therefore for a rectangular lattice

$$\mu_3(g_2, g_3) = -\mu_1(g_2, -g_3) \tag{14}$$

and thus  $\mu_3 \in \mathbb{R}$  also. Note that the validity of (14) does depend on the lattice being rectangular.

For rhombic lattices, (7) implies that if Im  $\mu_j = 0$  for some values of  $g_2$  and  $g_3$ , then  $\partial \operatorname{Im} \mu_j / \partial g_2 = \partial \operatorname{Im} \mu_j / \partial g_3 = 0$  and thus Im  $\mu_j \equiv 0$ . Now  $\mu_1 \in \mathbb{R}$  and  $\mu_3 \notin \mathbb{R}$  when  $g_3 = 0$  and  $g_2 < 0$ . Therefore  $\mu_1 \in \mathbb{R}$  and  $\mu_3 \notin \mathbb{R}$  for all  $g_2$  and  $g_3$ .

For  $\mu_1$ , one can alternatively use the infinite series Section 23.8.5 of [1], which shows immediately that  $\eta_1$  and thus  $\mu_1$  is real-valued for both lattice types.  $\Box$ 

# 5. Analytical solutions to (7)

In this section we present exact solutions for  $\mu_i(g_2, g_3)$  (j =1, 3). We present these in the context of  $g_2, g_3 \in \mathbb{R}$  but our derivations and solutions extend easily to  $g_2, g_3 \in \mathbb{C}$ , and this will be discussed in Section 8. As a prelude, it is helpful to sketch out why Eqs. (7) are exactly solvable. The  $\mu_i$ 's contain two objects:  $\eta_i$  and the half-period  $\omega_i$ . The former are expressible in terms of Legendre's complete elliptic integrals (see for example [6, Section 18.9.13]), while  $\omega_i$  is given by integrals of the form (9), which can also be written in terms of Legendre's complete elliptic integrals [6, Sections 18.9.7-18.9.8]. In principle, these combine to give exact solutions for the  $\mu_i$ 's, but the formulae are very cumbersome and inconvenient to use. It should also be noted that they require knowing the lattice roots  $e_1$ ,  $e_2$ , and  $e_3$  in terms of  $g_2$ ,  $g_3$ . This further complicates the result, to say nothing of analysis of signs in numerous radicals of branch-points (see for example the tables in Section 13.5 of [19]). Therefore we follow a different approach, inspired by the old work of Bruns [18] and Innes [24] on the representation of periods of holomorphic and meromorphic elliptic integrals in terms of hypergeometric functions. The latter satisfy a linear differential equation of second order and the relationship between the periods of integrals and such equations has an extensive theory. It is known nowadays as the Picard-Fuchs theory, and Bruns [18] was the first (1875) to construct the closed 'hypergeometric' theory for 'elliptic' periods. See [25] for history and classical references and [26,27] for some modern applications.

# 5.1. Solution in terms of hypergeometric functions

From (8) we have

$$\mu_j(g_2, g_3) = g_3^{1/3} \mu_j(g_3^{-2/3}g_2, 1) \equiv g_3^{1/3} \mu(g_3^{-2/3}g_2)$$
(15)

where for notational simplicity we omit the subscript *j* from  $\mu$ . Here and throughout this section we choose the real cube root when  $g_3 < 0$ .

Substituting (15) into either of the equations in (7) and writing  $z = g_3^{-2/3}g_2$  gives

$$(z^3 - 27)\frac{d\mu}{dz} = \frac{9}{2}\mu^2 + \frac{1}{2}z^2\mu + \frac{3}{8}z.$$
 (16)

This has the form of a first order Riccati equation, and the substitution  $\widetilde{\mu} = \mu + \frac{1}{18}z^2$  gives the canonical form

$$\frac{d\tilde{\mu}}{dz} = \frac{9}{2} \frac{\tilde{\mu}^2}{z^3 - 27} + \frac{7}{72} z.$$
 (17)

Using a standard trick for the Riccati equation [28, Section 1.2.1], we write

$$\widetilde{\mu} = \frac{2}{9}(27 - z^3)\frac{\psi_z}{\psi} \Rightarrow (z^3 - 27)\psi_{zz} + 3z^2\psi_z + \frac{7}{16}z\psi = 0.$$

Intuitively, this linear ODE arises because the differential equations in (7) may be viewed as two separate ODE s, each having the Riccati equation form. The appearance of the self-similarity variable z is not surprising because it is directly related to a quantity uniquely characterising the Weierstrass equation (1), namely Klein's absolute invariant

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{z^3}{z^3 - 27}.$$

This suggests the variable change  $s = \frac{1}{27}z^3$ . The equation for  $\psi(z) = \Psi(s)$  then becomes

$$s(s-1)\Psi'' + \frac{1}{3}(5s-2)\Psi' + \frac{7}{144}\Psi = 0,$$
(18)

which is a standard hypergeometric equation [6, Section 15.5], [1, Section 15.10]. Eq. (18) and all the ensuing equations of hypergeometric or Legendrian type provide examples of the Picard–Fuchs equations in the context mentioned above. The singularity in (18) at s = 1 is expected, since this point corresponds to  $\Delta = 0$ . There are a variety of different exact forms for the general solution of the hypergeometric equation: Kummer famously constructed 24 different solutions [29], [1, Section 15.10(ii)]. We will use different forms for the cases |s| < 1 and |s| > 1; note that  $|s| < 1 \Rightarrow \Delta < 0$ , while |s| > 1 allows both  $\Delta > 0$  and  $\Delta < 0$ .

 $|\mathbf{s}| < \mathbf{1}$ . For this case the general solution of (18) that we use is

$$\Psi = A \cdot {}_{2}F_{1}\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}\middle|s\right) + B \cdot s^{1/3} \cdot {}_{2}F_{1}\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}\middle|s\right).$$
(19)

Here  $_2F_1$  denotes the hypergeometric function, which is given simply by the single-valued hypergeometric series since |s| < 1. We comment that the ability to represent the solution in terms of hypergeometric functions is expected, since all of the elliptic integrals mentioned at the start of this section can be represented in terms of such functions [6, Sections 17.3.9–12].

We now perform the inverse transformations, using the identity

$$\frac{d}{ds}{}_{2}F_{1}(a, b; c|s) = \frac{ab}{c} \cdot {}_{2}F_{1}(a+1, b+1; c+1|s)$$

[1, Section 15.5.1]. After some algebraic manipulation, one obtains

$$\mu_{j}(g_{2}, g_{3}) = \frac{27g_{3}^{2} - g_{2}^{2}}{6^{5}g_{3}^{3}} \frac{14Cg_{2}^{2}g_{3}^{2/3}G_{3} - 55g_{2}^{3}G_{4} - 12^{3}g_{3}^{2}G_{2}}{Cg_{3}^{2/3}G_{1} - g_{2}G_{2}} - \frac{1}{18}\frac{g_{2}^{2}}{g_{3}}$$
(20)

where

$$G_{1} = {}_{2}F_{1}\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}\middle|s\right), \quad G_{2} = {}_{2}F_{1}\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}\middle|s\right),$$

$$G_{3} = {}_{2}F_{1}\left(\frac{19}{12}, \frac{13}{12}; \frac{5}{3}\middle|s\right), \quad G_{4} = {}_{2}F_{1}\left(\frac{23}{12}, \frac{17}{12}; \frac{7}{3}\middle|s\right)$$
(21)

with  $s = g_2^3/(27g_3^2)$ . Here C = -3A/B is a constant of integration, which depends on *j*. For the equianharmonic case  $g_2 = 0$ ,  $g_3 \in \mathbb{R}^+$ , (20) implies  $\mu(0, g_3) = -6g_3^{1/3}/C$ . Comparing this with (13) shows that for  $\mu_1$ 

$$-\frac{6}{C}\sqrt[3]{g_3} = -\frac{8\pi^3\sqrt[3]{g_3}}{\sqrt{3}\Gamma^6(\frac{1}{3})} \Longrightarrow C = \frac{3\sqrt{3}}{4\pi^3}\Gamma^6\left(\frac{1}{3}\right) \approx 15.486339.$$

Similarly for  $\mu_3$ ,

$$C = \frac{3\sqrt{3}}{4\pi^3} \Gamma^6\left(\frac{1}{3}\right) e^{2\pi i/3} \approx -7.743169 + 13.411563 \,i.$$

 $|\mathbf{s}| > \mathbf{1}$ . For this case we use the general solution of (18) given by

$$\Psi = \widetilde{A}s^{-1/12} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}\middle|s^{-1}\right) + \widetilde{B}s^{-7/12} \cdot {}_2F_1\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}\middle|s^{-1}\right).$$

This generates a new solution for  $\mu_i$ :

$$\mu_{j}(g_{2}, g_{3}) = \frac{g_{2}^{3} - 27g_{3}^{2}}{36g_{2}^{4}g_{3}} \times \frac{2\widetilde{C}g_{2}^{9/2}\widetilde{G}_{1} + 45\widetilde{C}g_{2}^{3/2}g_{3}^{2}\widetilde{G}_{3} + 14g_{2}^{3}g_{3}\widetilde{G}_{2} + 231g_{3}^{3}\widetilde{G}_{4}}{\widetilde{C}g_{2}^{3/2}\widetilde{G}_{1} + g_{3}\widetilde{G}_{2}} - \frac{1}{18}\frac{g_{2}^{2}}{g_{3}},$$
(22)

where 
$$C = A/(B\sqrt{27})$$
 and  
 $\widetilde{G}_1 = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| s^{-1}\right), \quad \widetilde{G}_2 = {}_2F_1\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2} \middle| s^{-1}\right),$ 
 $\widetilde{G}_3 = {}_2F_1\left(\frac{17}{12}, \frac{13}{12}; \frac{3}{2} \middle| s^{-1}\right), \quad \widetilde{G}_4 = {}_2F_1\left(\frac{23}{12}, \frac{19}{12}; \frac{5}{2} \middle| s^{-1}\right)$ 
(23)

(recall that  $s = g_2^3/(27g_3^2)$ ). The constant  $\widetilde{C}$  depends on the lattice and on *j*. For a rectangular lattice (s > 1), the  $\mu_j$ 's corresponding to  $s = \infty$  are given by (11), while (22) reduces to  $\mu_j = g_2^{1/2}/(3\widetilde{C})$  when  $g_3 = 0$ . Therefore

$$\widetilde{C} = -\Gamma^4 \left(\frac{1}{4}\right) / (12\pi^2) \text{ for } j = 1 \text{ and}$$
  
 $\widetilde{C} = \Gamma^4 \left(\frac{1}{4}\right) / (12\pi^2) \text{ for } j = 3.$ 

 $\sim \sim \sim$ 

For a rhombic lattice (s < -1), we use the values of the  $\mu_j$ 's corresponding to  $s = -\infty$ , which are given by (12); these imply

$$\widetilde{C} = i \Gamma^4 \left(\frac{1}{4}\right) / (12\pi^2) \quad \text{for } j = 1 \quad \text{and}$$

$$\widetilde{C} = -\Gamma^4 \left(\frac{1}{4}\right) / (12\pi^2) \quad \text{for } j = 3.$$
Note that  $\Gamma^4 \left(\frac{1}{4}\right) / (12\pi^2) \approx 1.4589597.$ 

**Remark.** There are old results on the representation of the  $\omega_j$ 's and  $\eta_j$ 's in terms of hypergeometric functions, due to Bruns [18]. His formulae (25), (28) (on pages 241 and 243 of the 1886 reprint) can be used to derive a formula for  $\mu_j$  that is equivalent to (22). Note that Bruns's formulae were rewritten by Innes [24] to facilitate numerical evaluation.

*Continuity of* (20) and (22). It is important to note that (20) and (22) are continuous for all  $g_2, g_3 \in \mathbb{R}$ . This continuity includes values of  $g_2$  and  $g_3$  for which  $\Delta = 0$ . To see this, one uses the known behaviour of the hypergeometric function when the argument is close to unity (e.g. [1, Sections 15.4.20–23]). This implies that for (20) with  $s \in \mathbb{R}, \mu_j \rightarrow -\frac{1}{2}g_3^{1/3}$  as  $s \rightarrow 1^-$ , while for (22)  $\mu_j \rightarrow -\frac{1}{2}g_3^{1/3}$  as  $s \rightarrow 1^+$ ; these apply irrespective of the values of the constants *C* and  $\widetilde{C}$ .

#### 5.2. Solution in terms of Legendre functions

We now present the derivation of an alternative solution form for the  $\mu_j$ 's, in terms of Legendre functions. Summaries of the theory of these special functions are given, for example, in [6, Ch. 8] and [1, Ch. 14].

Eq. (18) is a hypergeometric equation with parameters  $\frac{1}{12}$ ,  $\frac{7}{12}$  and  $\frac{2}{3}$ . Notably, the first two of these differ by  $\frac{1}{2}$ . This condition implies that (18) can be reduced to a Legendre equation by a quadratic transformation (see Section 3.2 of [17]). Moreover the fact that the sum of the first two parameters is equal to the third  $(\frac{1}{12} + \frac{7}{12} = \frac{2}{3})$  implies that the Legendre equation is of the simpler non-associated type. Specifically, we substitute  $s = \xi^{-2}$  and  $\Psi = s^{-1/3}Y$ , arbitrarily fixing  $\xi > 0$  if s > 0 and  $\text{Im } \xi > 0$  if s < 0. This gives

$$(1 - \xi^2)Y'' - 2\xi Y' - \frac{5}{36}Y = 0, \tag{24}$$

which is in the standard form of Legendre's non-associated equation. Note that different transformations, such as  $\hat{s} = 1 - \xi^{-2}$ , would reduce (18) to a complete Legendre equation, i. e. of associated type. We consider separately the three cases 0 < s < 1, s > 1 and s < 0.

**0** < **s** < **1**. In this case  $\Delta$  < 0 so that the lattice is rhombic. We have  $\xi \in (1, \infty)$ , so that (24) has the general solution  $Y(\xi) = \widehat{AP}_{-1/6}(\xi) + \widehat{BQ}_{-1/6}(\xi)$  [1, Section 14.2(i)]. Derivatives of Legendrian functions generate Legendrian functions with different indices [6, Section 8.5]. Thus inverting the various transformations in Section 5.1 gives a solution for  $\mu$  in terms of  $P_{-1/6}(\xi)$ ,  $Q_{-1/6}(\xi)$ ,  $P_{5/6}(\xi)$  and  $Q_{5/6}(\xi)$ . We omit the details and just give the final result:

$$\mu_j(g_2, g_3) = -\frac{5}{6}\sqrt{3}g_2^{1/2}\frac{\widehat{C}P_{5/6}(\xi) + Q_{5/6}(\xi)}{\widehat{C}P_{-1/6}(\xi) + Q_{-1/6}(\xi)} + 6\frac{g_3}{g_2},$$
 (25)

where  $\xi = 3\sqrt{3}|g_3|g_2^{-3/2}$ ; note that  $g_2 > 0$  in this parameter regime, and we take  $g_2^{-3/2} > 0$ . The integration constant  $\widehat{C} = \widehat{A}/\widehat{B}$  again depends on *j*.

**s** > **1**. In this case  $\Delta$  > 0: a rectangular lattice. Then  $Y(\xi) = \widehat{A}P_{-1/6}(\xi) + \widehat{B}Q_{-1/6}(\xi)$  where  $P_{-1/6}$  and  $Q_{-1/6}$  are Ferrer's functions [17, Section 3.4], [1, Section 14.23]. The derivation above proceeds in exactly the same way with these functions replacing the Legendre functions, giving

$$\mu_{j}(g_{2},g_{3}) = -\frac{5}{6}\sqrt{3}g_{2}^{1/2}\frac{\widehat{\mathsf{CP}}_{5/6}(\xi) + \mathsf{Q}_{5/6}(\xi)}{\widehat{\mathsf{CP}}_{-1/6}(\xi) + \mathsf{Q}_{-1/6}(\xi)} + 6\frac{g_{3}}{g_{2}}.$$
 (26)

Again  $g_2 > 0$  and  $\xi = 3\sqrt{3}|g_3|g_2^{-3/2}$  with  $g_2^{-3/2} > 0$ . It is straightforward to show that (25) and (26) are continuous at s = 1.

**s** < **0**. This is the case of  $g_2$  < 0. Then the lattice is necessarily rhombic, and  $\xi$  is pure imaginary. We assume principal branches of the Legendre functions, with a cut along  $(-\infty, 1]$ . The fact that the Legendre functions are multi-valued is reflected in the multi-valuedness of the  $\mu_j$ 's for  $g_2, g_3 \in \mathbb{C}$ , and this is discussed in detail in Section 8.

The solution (25) applies in this case. The Legendre functions are complex-valued on the imaginary axis, and thus the  $\mu_j$  given by (25) is in general complex. We have shown (Theorem 1) that  $\mu_1 \in \mathbb{R}$  for a rhombic lattice, and yet it is far from obvious that  $\widehat{C}$  can be chosen in order that the solution (25) is real. However we show in Appendix B that for  $g_2 < 0$  and under our assumption  $g_2^{1/2}/i > 0$ , (25) is always real provided that  $\widehat{C}$  satisfies the constraint  $|\widehat{C}| = 2\pi \cos[\arg(\widehat{C}) - \pi/6]$ .

Note that 0 < s < 1 (i.e.  $0 < g_2 < (27g_3^2)^{1/3}$ ) and s < 0 (i.e.  $g_2 < 0$ ) both correspond to a rhombic lattice, but different values of  $\widehat{C}$  are required in these two cases. The explanation for this is that (25) is not continuous at  $g_2 = 0$ , which corresponds to  $\xi = \infty$ : this follows from standard results on the behaviour of  $P_{\nu}(\xi)$  as  $\xi \to \infty$  (e.g. [1, Section 14.8.12]). Therefore to give a continuous solution for  $\mu_j$  one requires different values of  $\widehat{C}$  for  $g_2 > 0$  and  $g_2 < 0$ .

As in Section 5.1, the values of  $\widehat{C}$  and  $\widehat{C}$  corresponding to  $\mu_1$  and  $\mu_3$  for rectangular and rhombic lattices can be determined using the known values for  $g_2 = 0$  and  $g_3 = 0$ .

### 6. Monotonicity results for $g_2, g_3 \in \mathbb{R}$

In applications it can be important to consider whether or not the  $\mu_j$ 's are monotonic as functions of the  $g_i$ 's. In this section we prove the following two theorems pertaining to this issue.

**Theorem 2.** For a rectangular lattice,  $\mu_1$  and  $\mu_3$  satisfy

(i) 
$$\partial \mu_1 / \partial g_2 < 0$$
 (ii)  $\partial \mu_1 / \partial g_3 < 0$   
(iii)  $\partial \mu_3 / \partial g_2 > 0$  (iv)  $\partial \mu_3 / \partial g_3 < 0$ .

Moreover, the ranges of  $\mu_1$  and  $\mu_3$  as one of  $g_2$  and  $g_3$  are varied with the other fixed are specified by

(v) 
$$\mu_1 = -c$$
,  $\mu_3 = -c$  on  $g_2 = 12c^2$ ,  $g_3 = 8c^3$   
( $\Rightarrow \Delta = 0$ ) with  $c \in \mathbb{R}$ 

(vi) 
$$\mu_1 \rightarrow -\infty, \mu_3 \rightarrow +\infty$$
 as  $g_2 \rightarrow \infty$  with  $g_3$  fixed.

Theorem 3. For a rhombic lattice

- (i) for  $g_2 > 0$ :  $\partial \mu_1 / \partial g_2 < 0$  for  $g_3 > 0$  and  $\partial \mu_1 / \partial g_2 > 0$  for  $g_3 < 0$ ;
- (ii) for  $g_2 < 0$  and  $g_3 > 0$ :  $\mu_1$  is non-monotonic as a function of both  $g_2$  and  $g_3 > 0$ . Specifically, for any  $g_2 < 0$  there is a  $g_3 > 0$  at which  $\partial \mu_1 / \partial g_3 = 0$  and  $\partial^2 \mu_1 / \partial g_3^2 < 0$ , and for any  $g_3 > 0$  there is a  $g_2 < 0$  at which  $\partial \mu_1 / \partial g_2 = 0$  and  $\partial^2 \mu_1 / \partial g_2^2 < 0$ .

Although these monotonicities must follow from the exact solutions derived in Section 5, we have found it easier to prove them directly from the governing differential equation (7).

**Proof of Theorem 2.** In the proof of Theorem 1 we showed that the identity (14) holds for rectangular lattices. Therefore it is sufficient to prove only the parts of the theorem concerning  $\mu_1$ .

We consider first (v) and (vi). When  $\Delta = 0$ , exact formulae for  $\eta_1$  and  $\omega_1$  (and indeed for  $\wp(.)$  itself) are available [6, Section 18.12], and (v) follows easily from these. For (vi), (8)  $\Rightarrow \mu_1(g_2, g_3) \sim \mu_1(1, 0)g_2^{1/2}$  as  $g_2 \rightarrow \infty$  with  $g_3$  fixed. Moreover (11) implies that  $\mu_1(1, 0) < 0$ . Therefore (vi) holds.

We turn now to (ii). From (7) we have

$$\Delta \frac{\partial \mu_1}{\partial g_3} = -\left[3\left(g_2\mu_1 + \frac{3}{2}g_3\right)^2 + \frac{1}{4}\Delta\right]/g_2.$$
(27)

Since  $g_2 > 0$  and  $\Delta > 0$  for a rectangular lattice, this implies (ii).

Finally, we consider (i). We begin by proving two preliminary inequalities. Differentiating (1) and multiplying by  $\wp$  gives

$$6\wp^{3} - \frac{1}{2}g_{2}\wp = \wp\wp'' = (\wp\wp')' - (\wp')^{2}$$
  
$$\Rightarrow \wp^{3} = \frac{1}{10}(\wp\wp')' + \frac{3}{20}g_{2}\wp + \frac{1}{10}g_{3}$$
(28)

using (1). Substituting (28) into (1) gives

$$2g_2\wp + 3g_3 = 2(\wp\wp')' - 5(\wp')^2.$$
<sup>(29)</sup>

Now  $\wp(z)$  is real-valued when Im  $z = \omega_3/i$ , so that  $(\wp')^2 \ge 0$ . Therefore integrating (29) between  $\omega_3$  and  $\omega_3 + 2\omega_1$  along a path parallel to the real axis shows that

$$2g_2\mu_1 + 3g_3 \le (1/\omega_1) \left[\wp \wp'\right]_{\omega_3}^{\omega_3 + 2\omega_1} = 0.$$
(30)

Note that the validity of (30) is lattice-dependent. In particular, it does not hold in general for a rhombic lattice: the derivation fails because there is no path for the integration of (29) along which  $\wp$  is real-valued and non-singular.

Our second preliminary inequality follows immediately from (ii) and (v). For a given value of  $g_2 > 0$ , and for  $g_3$  such that  $\Delta > 0$ ,

$$\mu_1 \left( g_2, - \left( g_2^3 / 27 \right)^{1/2} \right) > \mu_1(g_2, g_3) > \mu_1 \left( g_2, \left( g_2^3 / 27 \right)^{1/2} \right)$$
  
i.e.  $- \left( g_2 / 12 \right)^{1/2} < \mu_1(g_2, g_3) < \left( g_2 / 12 \right)^{1/2}$   
 $\Rightarrow \mu^2 < g_2 / 12.$  (31)

Substituting (30) and (31) into the formula for  $\partial \mu_1/dg_2$  in (7) gives

$$\partial \mu_1 / \partial g_2 < \Big[ \frac{3}{8} g_2 g_3 + \frac{1}{2} g_2^2 (-3g_3/2g_2) + \frac{9}{2} g_3 (g_2/12) \Big] / \Delta = 0.$$

Proof of Theorem 3. We consider first (i). From (7) we have

$$\frac{\partial \mu_1}{\partial g_2} = g_3 \left( \frac{9}{2\Delta} \left( \mu_1 + \frac{g_2^2}{18g_3} \right)^2 - \frac{g_2}{72g_3^2} \right)$$

from which (i) follows immediately.

For (ii), we consider first the case of  $g_2 < 0$  fixed. Then (7) and (12) imply that when  $g_3 = 0$ ,  $\partial \mu_1 / \partial g_3 \approx -0.0934/g_2 > 0$ . But (8) implies that  $\mu_1(g_2, g_3) \sim g_3^{1/3} \mu_1(0, 1)$  as  $g_3 \rightarrow \infty$  with  $g_2$  fixed, and (13)  $\Rightarrow \mu_1(0, 1) < 0$ . Therefore  $\partial \mu_1 / \partial g_3 < 0$  for  $g_3$  sufficiently large and positive. Hence by continuity there is a value of  $g_3 > 0$  for which  $\partial \mu_1 / \partial g_3 = 0$  and  $\partial^2 \mu_1 / \partial g_3^2 \leq 0$ . It remains to exclude the possibility that  $\partial^2 \mu_1 / \partial g_3^2 = 0$ , which we do by contradiction. Differentiation of the equation for  $\partial \mu_1 / \partial g_3$  in (7) implies that when  $\partial \mu_1 / \partial g_3 = 0$ ,  $\partial^2 \mu_1 / \partial g_3^2 = -9\mu_1 / \Delta$ . Thus  $\partial \mu_1 / \partial g_3 = \partial^2 \mu_1 / \partial g_3^2 = 0 \Rightarrow \mu_1 = 0 \Rightarrow \partial \mu_1 / \partial g_3 \neq 0$ , using (7). A similar argument applies for  $g_3 > 0$  fixed. Then (7) implies that when  $g_n = 0$ ,  $\partial \mu_1 / \partial g_n = -\mu^2 / 6g_n < 0$  since  $\mu_1 < 0$ . But (8)

that when  $g_2 = 0$ ,  $\partial \mu_1 / \partial g_2 = -\mu_1^2 / 6g_3 < 0$  since  $\mu_1 < 0$ . But (8) implies that  $\mu_1(g_2, g_3) \sim (-g_2)^{1/2} \mu_1(-1, 0)$  as  $g_2 \rightarrow -\infty$  with  $g_3 > 0$  fixed. But (12)  $\Rightarrow \mu_1(-1, 0) < 0$ . Therefore  $\partial \mu_1 / \partial g_2 > 0$  for  $g_2$  sufficiently large and negative. Hence by continuity there is a value of  $g_2 < 0$  for which  $\partial \mu_1 / \partial g_2 = 0$  and  $\partial^2 \mu_1 / \partial g_2^2 \leq 0$ . To exclude the possibility that  $\partial^2 \mu_1 / \partial g_3^2 = 0$ , we first consider the sign of  $\mu_1$ . The first equation in (7) implies that if  $\mu_1 = 0$  then  $\partial \mu_1 / \partial g_2 = 3g_2g_3/8\Delta > 0$ . However  $\mu_1 < 0$  when  $g_2 = 0$ . Therefore  $\mu_1 < 0$  must hold for all  $g_2 < 0$  and  $g_3 > 0$ . Now when  $\partial \mu_1 / \partial g_2 = 0$ , (7) implies that  $\partial^2 \mu_1 / \partial g_2^2 = (\frac{3}{8}g_3 + g_2\mu_1)/\Delta$ . Therefore  $\partial \mu_1 / \partial g_3 = \partial^2 \mu_1 / \partial g_3^2 = 0 \Rightarrow \mu_1 = -3g_3/8g_2 > 0$ , a contradiction.  $\Box$ 

**Remark.** We note that Eq. (27), and the monotonicity results (ii) and (iv) that follow from it, appear on page 315 of the 1886 edition of Halphen's book [13]. This is the only published reference to the  $\mu_j$ 's of which we are aware.

#### 7. Numerical computation of $\mu_1$ and $\mu_3$

The mean values  $\mu_1$  and  $\mu_3$  can be computed by numerical evaluation of the defining integrals (5). However this is a slightly laborious approach, and error estimation is difficult. Our results suggest two other approaches that are more robust. Again we restrict attention in this section to  $g_2$ ,  $g_3 \in \mathbb{R}$ , with computation of the  $\mu_i$ 's for complex invariants discussed in Section 8.

Numerical solution of (7). For  $g_2, g_3 \in \mathbb{R}$ , numerical solution of the differential equations (7) is an efficient method for calculating the  $\mu_j$ 's. A suitable starting point is required for the solution. For a rectangular lattice, one can use the lemniscatic case (11), while for a rhombic lattice, potential starting points are provided by the pseudo-lemniscatic case (12) or the equianharmonic case (13). As one would expect from the form of (7), numerical integration becomes progressively more difficult as one approaches  $\Delta = 0$ , and very small increments in the  $g_i$ 's are required.

Numerical evaluation of the exact solutions. The exact solutions (20), (22) and (25) can also be used for numerical calculation of the  $\mu_j$ 's. Hypergeometric and Legendre functions are preprogrammed in many mathematical software packages including MAPLE [30], making numerical evaluation of these solutions relatively straightforward.

We have used both of these numerical methods successfully. We found the use of the solutions (20) and (22) in terms of hypergeometric series to be the most efficient for real  $g_2$  and  $g_3$ . For example, using the MAPLE function evalf/hypergeom/kernel [30] with Digits=10, one can perform about 2000 evaluations of  $\mu_j$ per second on a typical desktop computer. Notice that whilst  $\mu_3$ is strictly complex for rhombic lattices (Theorem 1), its computation is no less efficient than that for  $\mu_1$  because it involves only the real  $_2F_1$ -series, or it reduces to a computation of  $\mu_1$  itself (via formula (14)).

Fig. 1 shows contour plots of  $\mu_1$  and  $\mu_3$  in the  $g_2$ - $g_3$  plane for rectangular lattices; note that only the part of the plane for which  $\Delta > 0$  is relevant. One notable feature of Fig. 1b is the



**Fig. 1.** Numerically calculated values of  $\mu_1$  and  $\mu_3$  for rectangular lattices; then  $\mu_1$ ,  $\mu_3 \in \mathbb{R}$ . (a, c) Contours of  $\mu_1$  (a) and  $\mu_3$  (c) in the  $g_2-g_3$  plane. The thick curves denote the boundaries  $\Delta = 0$  of the parameter region covered by rectangular lattices. Note that  $\mu_1$  takes positive as well as negative values, but  $\mu_1 > 0$  only in a thin layer adjacent to the left hand branch of the  $\Delta = 0$  curve; the opposite is true for  $\mu_3$ . This is illustrated in (b, d), which show  $\mu_1$  (b) and  $\mu_3$  (d) as a function of  $g_3$  for  $g_2 = 5$ . Note the steepness of the curve at the left/right (b/d) hand boundary of the  $g_3$  range; the arrows indicate the values of  $\mu_1$  on these boundaries.

steepness of  $\mu_1$  when  $g_3$  is just above its minimum value of  $-(g_2^3/27)^{1/2}$ . Detailed calculation of this behaviour using (7) shows that  $\mu_1 - \sqrt{g_2/12} \sim 2(g_2/3)^{1/2}/\log\left[(g_2^3/27)^{1/2} + g_3\right]$  as  $g_3 \rightarrow -(g_2^3/27)^{1/2+}$ . The corresponding steepness of  $\mu_3$  for  $g_3$  just below  $(g_2^3/27)^{1/2}$  is implied by (14); this is illustrated in Fig. 1d.

For rhombic lattices, Fig. 2 shows  $\mu_1$  and the real and imaginary parts of  $\mu_3$  as functions of  $g_2$  for  $g_3 = \pm 1$ , and Fig. 3 shows them as functions of  $g_3$  for  $g_2 = \pm 1$ . The parts of the graphs in which nothing is plotted are those for which  $\Delta \ge 0$ .

# 8. Extension to $g_2, g_3 \in \mathbb{C}$

In Sections 5–7 we have focussed on the case of real invariants  $g_2$  and  $g_3$ . However our definition of the mean values  $\mu_1$  and  $\mu_3$ , and the differential equations (7), are valid for all  $g_2, g_3 \in \mathbb{C}$ . In this section we discuss the extension of our results in Section 5 and Section 7 to this more general setting.

# 8.1. Analytical solutions of (7) for $g_2, g_3 \in \mathbb{C}$

No assumption of reality for  $g_2$  and  $g_3$  is needed in the derivation of solutions (20) and (22) for the  $\mu_i$ 's in terms of hypergeometric functions, and these solutions also apply for complex  $g_2$  and  $g_3$ . The hypergeometric functions  ${}_2F_1$  and noninteger powers are then multi-valued; the former is obtained by analytic continuation of the hypergeometric series. Therefore the solutions for  $\mu_i$  are similarly multi-valued; this issue is discussed in detail in Section 8.2. Single-valued analytic functions can be obtained on suitably cut planes. The principal branch of  ${}_2F_1$ is defined by introducing a cut along  $[1, \infty)$  [1, Section 15.2], while the principal branch of a non-integer power is given by a cut along  $(\infty, 0]$  [1, Section 4.2(iv)]. If we assume these principal branches, then (20) is valid for  $s \in \mathbb{C} \setminus [1, \infty)$  and  $g_3 \in \mathbb{C} \setminus (-\infty, 0]$ , while (22) is valid for  $s \in \mathbb{C} \setminus [0, 1]$  and  $g_2 \in \mathbb{C} \setminus (-\infty, 0]$ . In fact the branch cuts for  $g_3$  in (20) and  $g_2$  in (22) can be relocated to any half lines  $\arg(g_j) = \theta$  via the substitutions  $C^* = e^{2i(\theta - \pi)/3}C$  and  $\widetilde{C}^* = e^{i(\theta - \pi)/2} \widetilde{C}$ . It is important to emphasise that (20) and (22) are not analytic continuations of one another. In practice, it is natural to use (20) when |s| > 1 and (22) when |s| < 1, because then the principal branches of the hypergeometric functions are given simply by hypergeometric series.

The solution (25) is also valid for complex  $g_2$  and  $g_3$ , provided that one redefines  $\xi = 3\sqrt{3}g_3g_2^{-3/2}$  (i.e. provided one omits the modulus sign around  $g_3$ ). Again these solutions are multi-valued. The principal branch of Legendre's functions are given by a cut along  $(-\infty, 1]$ , and therefore if one assumes this branch then (25) is valid for  $\xi \in \mathbb{C} \setminus (-\infty, 1]$  and  $g_2 \in \mathbb{C} \setminus (-\infty, 0]$ .

#### 8.2. The multi-valuedness of $\mu_i$ for $g_2, g_3 \in \mathbb{C}$

The hypergeometric and Legendre functions that appear in our exact solutions for  $\mu_j$  are not single-valued: this is a ramification of the singularities of Eqs. (18) and (24). Therefore the mean values  $\mu_j(g_2, g_3)$  are multi-valued, when considered as analytic functions on  $\mathbb{C} \times \mathbb{C}$ . To explain this, it is necessary to return to the modular inversion problem mentioned in Section 3. This is the problem of determining half-periods  $\omega_1$  and  $\omega_3$  for given values of the invariants  $g_2$  and  $g_3$ ; it is discussed in detail in many textbooks (see for example [2, Section 21.73]), [16,5] and we give here only a very brief summary of the key results. The central player in the theory is the period ratio  $\tau = \omega_3/\omega_1$ , and  $\omega = \omega_1$  and  $\tau$  are the most natural parameters for the Weierstrassian functions in this context. For  $g_2$  and  $g_3$ ,  $\tau$  is determined by the transcendental equation

$$I(\tau) = g_2^3 / \left(g_2^3 - 27g_3^2\right), \tag{32}$$

where J(.) is Klein's modular function, introduced previously in Section 5.1 [31], [3, Section 10], [1, Section 23.15.7]. Once  $\tau$  is known, the  $\omega_j$ 's are determined uniquely apart from an arbitrary choice of the signs of the  $\omega_j$ 's. This arbitrariness of sign is clear from the basic formulae (2), which imply

$$\omega_1^2 = \frac{7}{3} \frac{g_2}{g_3} \frac{\sum (2m+2n\tau)^{-6}}{\sum (2m+2n\tau)^{-4}}$$
(33)

(here the summations are over  $m, n \in \mathbb{Z}$  with m = n = 0excluded) [32, Section II.4]<sup>2</sup> and does not affect the values of the  $\mu_i$ 's. However the solution of (32) for  $\tau$  is not unique in  $\mathbb{H}^+$ . (Here we use  $\mathbb{H}^+$  to denote the upper half of the complex plane.) Consequently there is multi-valuedness in the  $\omega_i$ 's, and this is inherited by the  $\mu_i$ 's. The branch cuts mentioned in Section 8.1 implicitly imply one choice for the  $\omega_i$ 's, but it is not the standard choice. Rather, one conventionally specifies the solution of (32) by requiring  $\tau$  to lie in  $\mathfrak{T} \subset \mathbb{H}^+$ , known as the "fundamental region". There are different conventions for the choice of  $\mathfrak{T}$ , and we follow [16] by defining  $\mathfrak{T}$  by the four conditions (i) Im ( $\tau$ ) > 0, (ii)  $-1/2 \leq \text{Re}(\tau) < 1/2$ , (iii)  $|\tau| \geq 1$ , (iv)  $\text{Re}(\tau) \leq 0$  if  $|\tau| = 1$  (see Fig. 4). The key result is that given any  $g_2, g_3 \in \mathbb{C}$  such that  $\Delta \neq 0$ , the transcendental equation  $J(\tau) = c$  has a unique solution with  $\tau \in \mathfrak{T}$  for any given  $c \in \overline{\mathbb{C}}$  [32, Section II.4.3, Thm. 3, p. 211]; see also [33, Section II.3.2]. Therefore the restriction  $\tau \in \mathfrak{T}$ specifies the  $\omega_i$ 's uniquely (up to sign change). The definition (5) then uniquely determines the  $\mu_i$ 's. Finally, it should be noted that for  $g_2, g_3 \in \mathbb{R}$ , the lattices implied by fixing  $\tau \in \mathfrak{T}$  differ (in some cases) from those specified in Section 4. One great advantage of the lattices given in Section 4 is that the corresponding  $\mu_i$ 's vary continuously with  $g_2$  and  $g_3$  away from  $\Delta = 0$ ; also  $\mu_1$  is continuous across  $\Delta = 0$ . In contrast, specifying  $\tau \in \mathfrak{T}$  gives discontinuities in the  $\mu_i$ 's as functions of  $g_2$  and  $g_3$ . These arise from discontinuities in  $\tau$  at the boundaries of  $\mathfrak{T}$ , and also, in some cases, when  $g_2$  and  $g_3$  change sign.

<sup>&</sup>lt;sup>2</sup> Note that this step in the modular inversion solution, i. e. determination of  $\omega_1$ , is given incorrectly in Section 11 of [3]. Note also that the Weierstrass–Eisenstein series (33) is entirely unsuited for numerical computations, and in practice one uses instead Lambert series; see [32, Section II.4.2, p. 210, 222].



**Fig. 2.** Numerically calculated values of  $\mu_1$  and  $\mu_3$  as functions of  $g_2$  with  $g_3 = \pm 1$ , for rhombic lattices. The dashed lines indicate the boundaries ( $\Delta = 0$ ) of the  $g_2$  range given by rhombic lattices, and the arrows indicate the values on these boundaries.



**Fig. 3.** Numerically calculated values of  $\mu_1$  and  $\mu_3$  as functions of  $g_3$  with  $g_2 = \pm 1$ , for rhombic lattices. For  $g_2 = 1$ , the dashed lines indicate the boundaries ( $\Delta = 0$ ) of the  $g_3$  range given by rhombic lattices, and the arrows indicate the values on these boundaries; for  $g_2 = -1$  there is no restriction on the value of  $g_3$ .

# 8.3. Numerical computation of $\mu_1$ and $\mu_3$ for $g_2, g_3 \in \mathbb{C}$

When  $g_2$  and  $g_3$  are complex, numerical solution of the governing differential equation (7) is not a very convenient means of computing the  $\mu_j$ 's. However the exact solutions (20), (22) and (25) all provide straightforward approaches to computation; we found the Legendrian form (25) to be the most computationally efficient for  $g_2$ ,  $g_3 \in \mathbb{C}$ .

Another numerical approach is also available, different from those discussed in Section 8.3, that builds on the comments in Section 8.2. This involves numerical solution of the elliptic inversion problem. For  $g_2$  and  $g_3$  both non-zero, Klein's equation (32) has the solution

$$\tau = i \frac{P_{-1/6} \left(-3\sqrt{3}g_3/g_2^{3/2}\right)}{P_{-1/6} \left(3\sqrt{3}g_3/g_2^{3/2}\right)}$$

[15, Thm. 8.1] which can be evaluated numerically using standard mathematical software such as MAPLE [30]. Separate formulae are required for  $g_2 = 0$  or  $g_3 = 0$ ; these are the lemniscatic and equianharmonic cases. Using these solutions for  $\tau$ , the series

$$\eta_j = \frac{1}{\omega_j} \cdot 2\pi^2 \left\{ \frac{1}{24} - \sum_{k=1}^{\infty} \frac{e^{2k\pi i\tau}}{(1 - e^{2k\pi i\tau})^2} \right\}, \quad \text{where } \tau = \frac{\omega_3}{\omega_1} \quad (34)$$

can be used to numerically calculate  $\eta_j$ ;  $\mu_j = -\eta_j/\omega_j$  then follows. The computational efficiency of this approach depends on the value of Im  $\tau$ . However, fixing  $\tau$  to be in the fundamental region  $\mathfrak{T}$  defined in Section 8.2 causes the convergence to be very rapid. For example, even the "worst" purely equianharmonic point Im  $\tau = \sqrt{3}/2 \approx 0.866025$  corresponds to a very rapidly convergent series. Formula (34) shows that this series converges almost in the manner of the geometric progression  $\sum q^k$  with the very small exponent  $q = \exp(-\pi\sqrt{3}) \approx 0.004333$  (see the two last sentences in [32, Section II.7.2, p. 249]).



**Fig. 4.** An illustration of the fundamental region  $\mathfrak{T}$ . The significance of this region is that given any  $g_2$ ,  $g_3 \in \mathbb{C}$  such that  $\Delta \neq 0$ , there is a unique solution of  $J(\tau) = c$  with  $\tau \in \mathfrak{T}$  for any given  $c \in \overline{\mathbb{C}}$  (see Section 8.2 for more details).

Figs. 5 and 6 show plots of the  $\mu_j$ 's as functions of complexvalued  $g_2$  and  $g_3$  respectively. These figures have  $g_3 \equiv 1$  and  $g_2 \equiv 1$ respectively; however the behaviour for other fixed values of  $g_3$  and  $g_2$  can be inferred from (8). The discontinuities discussed in Section 8.2 are clearly visible; however, note that the plots suggest that Im  $\mu_1$  is continuous as a function of  $g_3$  (for fixed  $g_2$ ).

# 9. Example application: vegetation patterns

An example of the application of our results comes from selforganised patterns of vegetation. These are a common feature of semi-arid landscapes, and on hillsides they consist of stripes of vegetation running parallel to the contours, separated by stripes of bare ground. The plants involved can range from grasses to shrubs and trees, with typical wavelengths for the latter being about 1km. Patterns of this type occur in many parts of the world and are particularly well documented in Africa, Australia and Mexico (see [34,35] for ecological reviews). The harsh environmental conditions make field work difficult, and there are no laboratory replicates of the pattern-forming process. Consequently mathematical modelling is an important research tool; for reviews of the various models that have been proposed, see [36,37]. One of the oldest and most influential models is due to Klausmeier [9], and has the dimensionless form

$$\partial u/\partial t = \widetilde{wu^2}^{\text{plant}} - \widetilde{Bu}^{\text{plant}} + \widetilde{\partial^2 u}/\partial x^2$$
(35)



**Fig. 5.** Numerically calculated values of  $\mu_1$  and  $\mu_3$  as functions of  $g_2 \in \mathbb{C}$  for  $g_3 \equiv 1$ . Here the lattice generators are specified by the requirement  $\tau \in \mathfrak{T}$  (see Section 8.2). The discontinuities correspond either to  $\tau$  lying on the boundary of  $\mathfrak{T}$  or to  $g_2 = 0$ .



**Fig. 6.** Numerically calculated values of  $\mu_1$  and  $\mu_3$  as functions of  $g_3 \in \mathbb{C}$  for  $g_2 \equiv 1$ . Here the lattice generators are specified by the requirement  $\tau \in \mathfrak{T}$  (see Section 8.2). The discontinuities correspond either to  $\tau$  lying on the boundary of  $\mathfrak{T}$  or to  $g_3 = 0$ .

$$\frac{\partial w}{\partial t} = \underbrace{A}_{\substack{\text{rain-}\\\text{fall}}} - \underbrace{w}_{\substack{\text{ovap-}\\\text{oration}}} - \underbrace{wu^2}_{\substack{\text{uptake}\\\text{by plants}}} + \underbrace{v \frac{\partial w}{\partial x}}_{\substack{\text{flow}\\\text{downhill}}}.$$
(36)

Here u(x, t) is plant density, w(x, t) is water density, t is time and x is a one-dimensional space variable running in the uphill direction. A key term in this model is the nonlinear uptake rate of water by plants,  $wu^2$ . On bare ground, much of the water that falls as rain simply runs off, but on vegetated ground the higher levels of organic matter in the soil, and the presence of roots, both increase the proportion of rain water infiltrating into the soil [38,39]. Consequently when vegetation biomass is larger there is greater water availability, and thus increased per capita uptake by plants and increased plant growth.

Many empirical studies show that on a time-scale of decades, striped vegetation patterns move uphill; speeds of 0.3–0.8 m/year are typical [34]. Intuitively, this pattern migration is due to higher moisture levels on the uphill edge of the vegetation bands than on their downhill edge, leading to reduced plant death and greater seedling density [40,41]. Numerical solutions of (35) and (36) reflect this migration. Starting from randomly generated initial conditions, numerical solutions typically evolve to spatially periodic solutions that move in the positive x direction at a constant speed (Fig. 7). Therefore the appropriate solution ansatz for patterns is  $u(x, t) = U(\xi)$ ,  $w(x, t) = W(\xi)$ ,  $\xi = x - ct$  where the new parameter c > 0 is the migration speed. Substituting these

solution forms into (35) and (36) gives

$$U'' + c U' + WU^{2} - BU = 0$$
  
(v + c)W' + A - W - WU^{2} = 0. (37)

The value of  $\nu$  is typically large; this follows from the nondimensionalisation [9,42] and the fact that the advection rate of water is much larger than the plant dispersal rate. For instance, Klausmeier [9] estimated  $\nu = 182.5$ , with *A* and *B* lying in the ranges 0.1–3.0 and 0.05–2.0 respectively, depending on vegetation type. Therefore it is natural to study the asymptotic form of solutions of (37) for large  $\nu$ . This depends on how *c* scales with  $\nu$ , but if one considers the case  $c \ll 1$  then to leading order for large  $\nu$ ,  $W = W_0$ , a constant, and

$$U'' + W_0 U^2 - BU = 0. (38)$$

The constant  $W_0$  is determined by the rainfall parameter *A* and the migration speed [11]. Substituting  $\xi = \tilde{\xi}/\sqrt{B}$  and

$$U = B\left(\frac{1}{2} - 6\widetilde{U}\right) / W_0 \tag{39}$$

into (38) gives  $d^2 \widetilde{U}(\widetilde{\xi})/d\widetilde{\xi}^2 = 6\widetilde{U}^2 - \frac{1}{24}$ . Therefore  $\widetilde{U}(\widetilde{\xi})$  satisfies (1) with  $g_2 = 1/12$ . Calculations of higher order terms in the asymptotic expansion of U [11] show that  $g_3$  corresponds to the leading order migration speed, and can take any value in



**Fig. 7.** A typical example of a pattern solution of the model (35) and (36) for vegetation in semi-arid environments. The alternating peaks and troughs of *u* correspond to vegetation bands and the gaps between them, respectively. The solution is plotted at three different times, 3 dimensionless time units apart, to illustrate the uphill migration of the pattern. The parameter values are A = 2.4, B = 0.5 and v = 182.5. The spatial domain is of length 100 with periodic boundary conditions. The initial conditions are small random perturbations (amplitude  $\pm 5\%$ ) to the vegetated steady state  $u = 2B/[A + (A^2 - 4B^2)^{1/2}]$ ,  $w = [A + (A^2 - 4B^2)^{1/2}]/2$ . The first solution is plotted after 2400 dimensionless time units; this long time ensures that transients have decayed. The equations were solved numerically using a finite difference scheme in which the diffusion terms were evaluated semi-implicitly, with explicit evaluation of the reaction and advection terms, using upwinding for the latter. The spatial grid had a uniform spacing of 0.025 and 1 used a time step of  $1.096 \times 10^{-4}$ ; these imply a CFL number of 0.8, and give an error of about 0.06% in the solution.

 $(-1/216, 1/216) \implies \Delta > 0)$ . Ecological realism demands that the solution is real-valued and without singularities. Taking  $\omega_1, i\omega_3 \in \mathbb{R}$  as in Section 4, it follows that

$$\widetilde{U}(\widetilde{\xi}) = \wp(\widetilde{\xi} + \omega_3 + \xi_0)$$

where  $\xi_0 \in \mathbb{R}$  is arbitrary.

One important reason for studying vegetation patterns in semiarid regions is their potential vulnerability to a transition to total desert [43]. In this context, a key solution measure is the mean vegetation density, and (39) and (40) imply that, to leading order for large  $\nu$ , this is given by

$$U_{\text{mean}} \equiv B\left(\frac{1}{2} - 6\mu_1\left(\frac{1}{12}, g_3\right)\right) / W_0.$$

The results presented in this paper can then be combined with details of the relationship between  $g_3$  and the migration speed c to make predictions on the relationship between  $U_{\text{mean}}$  and c. For example, for  $c \ll 1/\nu$  as  $\nu \rightarrow \infty$ , it is shown in [11] that c is a decreasing function of  $g_3$  when  $\nu$  is sufficiently large. Theorem 2(ii) then implies that  $\partial \mu_1 / \partial c > 0$ , i.e. the mean vegetation level is negatively correlated with the speed of pattern migration. This prediction is experimentally testable, by combining field data on vegetation cover (e.g. [44]) with inferences about pattern migration from satellite images [45].

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# Appendix A. Derivation of (3)

In this appendix we give a derivation of (3), based on the presentations in [14,12]. Let  $G(z|\omega_1, \omega_3)$  be any doubly periodic function, i.e.

 $G(z + 2n\omega_1 + 2m\omega_3|\omega_1, \omega_3) = G(z|\omega_1, \omega_3).$ 

Differentiating this identity with respect to z,  $\omega_1$  and  $\omega_3$  gives

$$\frac{\partial G}{\partial z}(z + 2n\omega_1 + 2m\omega_3) = \frac{\partial G}{\partial z}(z)$$
(A.1)
$$2n\frac{\partial G}{\partial z}(z + 2n\omega_1 + 2m\omega_3) + \frac{\partial G}{\partial \omega_1}(z + 2n\omega_1 + 2m\omega_3) = \frac{\partial G}{\partial \omega_1}(z)$$
(A.2)
$$2m\frac{\partial G}{\partial z}(z + 2n\omega_1 + 2m\omega_3)$$

$$+\frac{\partial G}{\partial \omega_3}(z+2n\omega_1+2m\omega_3)=\frac{\partial G}{\partial \omega_3}(z). \tag{A.3}$$

We now define

(40)

$$\Phi_1[G] = \omega_1 \partial_{\omega_1} G + \omega_3 \partial_{\omega_3} G + z \partial_z G.$$

Multiplying (A.1), (A.2) and (A.3) by z,  $\omega_1$  and  $\omega_3$  respectively and adding gives

$$\omega_1 \frac{\partial G}{\partial \omega_1} (z + 2n\omega_1 + 2m\omega_3) + \omega_3 \frac{\partial G}{\partial \omega_3} (z + 2n\omega_1 + 2m\omega_3) + (z + 2n\omega_1 + 2m\omega_3) \frac{\partial G}{\partial z} (z + 2n\omega_1 + 2m\omega_3) = \omega_1 \frac{\partial G}{\partial \omega_1} (z) + \omega_3 \frac{\partial G}{\partial \omega_3} (z) + z \frac{\partial G}{\partial z} (z);$$

that is  $\Phi_1[G](z + 2n\omega_1 + 2m\omega_3|\omega_1, \omega_3) = \Phi_1[G](z|\omega_1, \omega_3)$ . Similarly the combination

$$\Phi_2[G] = \eta_1 \partial_{\omega_1} G + \eta_3 \partial_{\omega_3} G + \zeta(z) \partial_z G$$

is doubly periodic in the periods of *G*.

We now fix  $G = \wp$ . Near z = 0

$$\wp(z) = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + \cdots.$$
(A.4)

[1, Sections 23.9.2, 23.9.4]. Therefore  $-\frac{1}{2}\Phi_1[\wp]$  is an elliptic function with (minimal) periods  $\omega_1$  and  $\omega_3$ , with a double pole at z = 0 whose principal part is  $1/z^2$ , and with  $-\frac{1}{2}\Phi_1[\wp] - 1/z^2$  being zero at z = 0 and analytic near z = 0. These properties uniquely define  $\wp(z)$ , so that

$$\omega_1 \frac{\partial \wp}{\partial \omega_1} + \omega_3 \frac{\partial \wp}{\partial \omega_3} + z \frac{\partial \wp}{\partial z} = -2\wp.$$
(A.5)

Similarly, using the Laurent expansion

$$\zeta(z) = \frac{1}{z} - \frac{1}{60}g_2 z^3 - \frac{1}{140}g_3 z^5 + \cdots$$

[1, Sections 23.9.3, 23.9.4] one sees that  $-\frac{1}{2}\Phi_2[\wp] + \frac{1}{6}g_2$  is an elliptic function with (minimal) periods  $\omega_1$  and  $\omega_3$ , with a quadruple pole at z = 0 whose principal part is  $1/z^4$ , and with  $-\frac{1}{2}\Phi_2[\wp] + \frac{1}{6}g_2 - 1/z^4$  being zero at z = 0 and analytic near z = 0. These properties uniquely define  $\wp(z)^2$ , so that

$$\eta_1 \frac{\partial \wp}{\partial \omega_1} + \eta_3 \frac{\partial \wp}{\partial \omega_3} + \zeta(z) \frac{\partial \wp}{\partial z} = -2\wp^2 + \frac{1}{3}g_2.$$
(A.6)

We now differentiate (A.4) termwise with respect to  $\omega_j$  (j = 1, 3) giving

$$\frac{\partial \wp}{\partial \omega_j} = \frac{1}{20} \frac{\partial g_2}{\partial \omega_j} \cdot z^2 + \frac{1}{28} \frac{\partial g_3}{\partial \omega_j} \cdot z^4 + \cdots$$

Substituting these expansions and (A.4) into (A.5) and (A.6), and equating coefficients of  $z^2$  and  $z^4$  gives

$$\begin{split} &\omega_1 \frac{\partial g_2}{\partial \omega_1} + \omega_3 \frac{\partial g_2}{\partial \omega_3} = -4g_2, \quad \eta_1 \frac{\partial g_2}{\partial \omega_1} + \eta_3 \frac{\partial g_2}{\partial \omega_3} = -6g_3, \\ &\omega_1 \frac{\partial g_3}{\partial \omega_1} + \omega_3 \frac{\partial g_3}{\partial \omega_3} = -6g_3, \quad \eta_1 \frac{\partial g_3}{\partial \omega_1} + \eta_3 \frac{\partial g_3}{\partial \omega_3} = -\frac{1}{3}g_2^2. \end{split}$$

Using Legendre's identity  $\omega_3\eta_1 - \omega_1\eta_3 = \frac{1}{2}\pi i$  [1, Section 23.2.14], these linear equations for  $\partial g_i/\partial \omega_j$  can be solved to give

$$\begin{aligned} \frac{\partial g_2}{\partial \omega_1} &= \frac{i}{\pi} (12g_3\omega_3 - 8g_2\eta_3),\\ \frac{\partial g_2}{\partial \omega_3} &= -\frac{i}{\pi} (12g_3\omega_1 - 8g_2\eta_1),\\ \frac{\partial g_3}{\partial \omega_1} &= \frac{i}{\pi} (\frac{2}{3}g_2^2\omega_3 - 12g_3\eta_3),\\ \frac{\partial g_3}{\partial \omega_3} &= -\frac{i}{\pi} (\frac{2}{3}g_2^2\omega_1 - 12g_3\eta_1). \end{aligned}$$

Hence for  $F(\omega_1, \omega_3) = \widetilde{F}(g_2, g_3)$ , the usual linear chain relations

$$\frac{\partial F}{\partial \omega_j} = \frac{\partial g_2}{\partial \omega_j} \frac{\partial F}{\partial g_2} + \frac{\partial g_3}{\partial \omega_j} \frac{\partial F}{\partial g_3}$$
  
(j = 1, 3) imply  
$$\frac{\partial F}{\partial \omega_1} = \frac{2}{3} \frac{i}{\pi} \left\{ 6 (3g_3\omega_3 - 2g_2\eta_3) \frac{\partial \widetilde{F}}{\partial g_2} + (g_2^2\omega_3 - 18g_3\eta_3) \frac{\partial \widetilde{F}}{\partial \theta_3} \right\}$$
$$\frac{\partial F}{\partial \omega_3} = -\frac{2}{3} \frac{i}{\pi} \left\{ 6 (3g_3\omega_1 - 2g_2\eta_1) \frac{\partial \widetilde{F}}{\partial g_2} + (g_2^2\omega_1 - 18g_3\eta_1) \frac{\partial \widetilde{F}}{\partial g_2} \right\}$$

which implies the first of both of the double equalities in (3). The second equality follows easily by applying the chain rule for differentiation and using Legendre's identity. For completeness we also give the inverse formulae:

$$\Delta \frac{\partial \tilde{F}}{\partial g_2} = -\frac{1}{4} \left\{ \left( g_2^2 \omega_1 - 18g_3 \eta_1 \right) \frac{\partial F}{\partial \omega_1} + \left( g_2^2 \omega_3 - 18g_3 \eta_3 \right) \frac{\partial F}{\partial \omega_3} \right\}, \\ \Delta \frac{\partial \tilde{F}}{\partial g_3} = \frac{3}{2} \left\{ \left( 3g_3 \omega_1 - 2g_2 \eta_1 \right) \frac{\partial F}{\partial \omega_1} + \left( 3g_3 \omega_3 - 2g_2 \eta_3 \right) \frac{\partial F}{\partial \omega_3} \right\}.$$

An alternative derivation of these formulae is given in the Halphen's book [13, p. 302–307 and p. 319–320]. Halphen uses series expansions of the Weierstrass sigma function; in fact he works with variants of  $\sigma(z)$  which he denotes by  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$ ; these are defined on page 189 of the 1886 edition of his book.

# Appendix B. Condition for (25) $\in \mathbb{R}$ when $g_2 \in \mathbb{R}^-$

In this appendix we show that when  $g_2$ ,  $g_3$  are real with  $g_2 < 0$ , solution (25) for  $\mu_j$  is also real provided that the constant of integration  $\widehat{C}$  satisfies the condition

$$\left|\widehat{C}\right| = 2\pi \cos\left[\arg\left(\widehat{C}\right) - \pi/6\right].$$

This defines one of the two real quantities  $|\widehat{C}|$  and  $\arg \widehat{C}$  as a function of the other. Therefore there is one remaining real free parameter, which depends on the lattice and *j*, and which can be calculated by reference to the lemniscatic  $(g_2, 0)$  or equianharmonic  $(0, g_3)$  cases. Note that since  $|\widehat{C}| \ge 0$ ,  $\arg(\widehat{C})$  is restricted to the interval  $[-\pi/3, 2\pi/3]$ .

This result is a simple corollary of:

**Proposition B.1.** For  $v \in \mathbb{R}$ , define

$$H(x) = \frac{i[CP_{\nu}(ix) + Q_{\nu}(ix)]}{CP_{\nu-1}(ix) + Q_{\nu-1}(ix)}.$$
(B.1)

Then  $H(x) \in \mathbb{R}$  for all  $x \in \mathbb{R}^+$  if and only if

$$|C| = \frac{\pi \cos\{\arg(C) - (1 - \nu)\pi\}}{\sin\{\nu\pi\}}.$$

**Remark.** For  $x \in \mathbb{R}^-$ , a different condition must be satisfied:

$$|C| = \frac{\pi \cos\{\arg(C) + (1 - \nu)\pi\}}{\sin\{\nu\pi\}}.$$

This follows immediately from Proposition B.1, by taking the complex conjugate of (B.1).

**Proof.** We begin by writing the Legendre functions  $P_{\nu}$  and  $Q_{\nu}$  in terms of hypergeometric functions, using formulae (22) and (40) of [17]; here we are following the approach of Dunster [46]. For  $x \in \mathbb{R}^+$ , these formulae give

$$P_{\nu}(ix) = \frac{\pi^{1/2}A_{\nu}(x)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\nu\right)} - i\frac{2\pi^{1/2}B_{\nu}(x)}{\Gamma\left(-\frac{1}{2}\nu\right)}$$
(B.2)  

$$Q_{\nu}(ix) = i\pi^{1/2}e^{-i\pi\nu/2}\left[\Gamma\left(1 + \frac{1}{2}\nu\right)B_{\nu}(x) - \frac{1}{2}\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right)A_{\nu}(x)\right]$$
where  $A_{\nu} = {}_{2}F_{1}\left(-\frac{1}{2}\nu, \frac{1}{2} + \frac{1}{2}\nu; \frac{1}{2}; -x^{2}\right)/\Gamma\left(1 + \frac{1}{2}\nu\right)$   
 $B_{\nu} = {}_{2}F_{1}\left(\frac{1}{2} - \frac{1}{2}\nu, 1 + \frac{1}{2}\nu; \frac{3}{2}; -x^{2}\right)x/\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right).$  (B.3)

Note that positivity of *x* is necessary for (B.3) though not for (B.2). Writing  $C = C_R + iC_l$  ( $C_R$ ,  $C_l \in \mathbb{R}$ ), it follows that

$$\operatorname{Re}\left[CP_{\nu}(ix)\right] = \frac{\pi^{1/2}C_{R}A_{\nu}(x)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\nu\right)} + \frac{2\pi^{1/2}C_{I}B_{\nu}(x)}{\Gamma\left(-\frac{1}{2}\nu\right)}$$
(B.4)

$$\operatorname{Im}\left[CP_{\nu}(ix)\right] = \frac{\pi^{1/2}C_{I}A_{\nu}(x)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\nu\right)} - \frac{2\pi^{1/2}C_{R}B_{\nu}(x)}{\Gamma\left(-\frac{1}{2}\nu\right)}$$
(B.5)

$$\operatorname{Re}\left[Q_{\nu}(ix)\right] = \pi^{1/2} \sin\left[\frac{1}{2}\pi\nu\right] \times \left[\Gamma\left(1+\frac{1}{2}\nu\right)B_{\nu}(x) - \frac{1}{2}\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)A_{\nu}(x)\right]$$
(B.6)

$$\operatorname{Im}\left[Q_{\nu}(ix)\right] = \pi^{1/2} \cos\left[\frac{1}{2}\pi\nu\right] \times \left[\Gamma\left(1+\frac{1}{2}\nu\right)B_{\nu}(x) - \frac{1}{2}\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)A_{\nu}(x)\right].$$

Now

 $H(x) \in \mathbb{R} \Leftrightarrow \operatorname{Re} \left[ CP_{\nu}(ix) + Q_{\nu}(ix) \right] \cdot \operatorname{Re} \left[ CP_{\nu-1}(ix) + Q_{\nu-1}(ix) \right] \\ + \operatorname{Im} \left[ CP_{\nu}(ix) + Q_{\nu}(ix) \right] \cdot \operatorname{Im} \left[ CP_{\nu-1}(ix) + Q_{\nu-1}(ix) \right] = 0.$ 

Substituting (B.4)–(B.7) into this condition and simplifying using the identity  $\Gamma(z + 1) = z\Gamma(z)$  gives

$$H(x) \in \mathbb{R} \Leftrightarrow [A_{\nu}(x)A_{\nu-1}(x) - 2\nu B_{\nu}(x)B_{\nu-1}(x)]H_0 = 0$$

(B.7)

where

$$H_{0} = \left\{ C_{R} + \frac{1}{2} \Gamma \left( 1 - \frac{1}{2} \nu \right) \Gamma \left( \frac{1}{2} \nu \right) \cos \left( \frac{1}{2} \pi \nu \right) \right\}$$

$$\times \left\{ C_{R} - \frac{1}{2} \Gamma \left( \frac{1}{2} - \frac{1}{2} \nu \right) \Gamma \left( \frac{1}{2} + \frac{1}{2} \nu \right) \sin \left( \frac{1}{2} \pi \nu \right) \right\}$$

$$+ \left\{ C_{I} - \frac{1}{2} \Gamma \left( 1 - \frac{1}{2} \nu \right) \Gamma \left( \frac{1}{2} \nu \right) \sin \left( \frac{1}{2} \pi \nu \right) \right\}$$

$$\times \left\{ C_{I} - \frac{1}{2} \Gamma \left( \frac{1}{2} - \frac{1}{2} \nu \right) \Gamma \left( \frac{1}{2} + \frac{1}{2} \nu \right) \cos \left( \frac{1}{2} \pi \nu \right) \right\}.$$

Therefore  $H(x) \in \mathbb{R}$  for all  $x \in \mathbb{R}^+$  if and only if  $H_0 = 0$ . Simplifying the condition  $H_0 = 0$  using the identity  $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$  gives the condition stated in the proposition.  $\Box$ 

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