# The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds

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**Abstract**

We present an error analysis for a general semilinear stochastic evolution equation in $d$ dimensions based on pathwise approximation. We discretize in space by a Fourier Galerkin method and in time by a stochastic exponential integrator. We show that for spatially regular (smooth) noise the number of nodes needed for the noise can be reduced and that the rate of convergence degrades as the regularity of the noise reduces (and the noise is rougher).

*Key words:* Numerical solution of stochastic PDEs, stochastic exponential integrator, pathwise convergence
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## 1 Introduction

We consider the pathwise numerical approximation of the stochastic evolution equation

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]\, dt + dW(t), \qquad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{1}$$

on the Hilbert space $H = L^2([a,b]^d)$. Here, $-A$ is the generator of an analytic semigroup $(e^{-tA}, t \geq 0)$ on $H$, $u(0) \in D(A)$, $W = (W(t), t \geq 0)$ is a $Q$-Wiener process on $(\Omega, \mathcal{A}, \mathbf{P})$ with values in $H$ and the mapping $F : H \to H$ is nonlinear, precise assumptions are given in Section 2.1. Finally, we assume that $A$ and the covariance operator $Q$ of the Wiener process have the same eigenfunctions $\phi_n$, i.e.

$$A\phi_n = \alpha_n \phi_n, \qquad Q\phi_n = \lambda_n \phi_n, \qquad n \in \mathbb{N}^d,$$

where $\alpha_n, \lambda_n \geq 0$ and $\phi_n$, $n \in \mathbb{N}^d$, is an orthonormal basis of $H$. In particular, we have the representation

$$W(t) = \sum_{n \in \mathbb{N}^d} \lambda_n^{1/2} \beta_n(t) \cdot \phi_n, \qquad t \geq 0,$$

with independent scalar Brownian motions $\beta_n$, $n \in \mathbb{N}^d$.

Typical examples for equations of the above type are the stochastic cable equation

$$du(t) = [\Delta u(t) - u(t)]\, dt + dW(t)$$

or the stochastic Allen-Cahn equation

$$du(t) = [\nu \Delta u(t) + u(t) - u(t)^3]\, dt + dW(t)$$

with periodic boundary conditions, where $\Delta$ denotes the Laplace operator and $\nu > 0$ is a parameter. However, our assumptions cover also the case that $A$ is e.g. a fractional power of the Laplacian. This paper builds on the error analysis for the exponential integrator method, introduced in [8], [9] for equation (1) with $A$ being the one-dimensional Laplacian. $H^1$ error bounds for smooth Gevrey noise, i.e. with exponential spatial correlation, were derived in [8], an $L^2$ and $H^m$ error analysis for a post processing variant of the exponential integrator scheme is given in [9], in the case of an arbitrary driving infinite dimensional Wiener process $W$. Here, we extend these results in the following way. We consider a general differential operator $A$ in $d$-dimensions instead of the one-dimensional Laplacian and we derive pathwise error bounds for this exponential integrator scheme. To do this we first derive error bounds in the $p$-th mean for all $p \geq 1$. Then by a Borel-Cantelli type argument, which has been used in a similar way e.g. in [2], [7], [10], we obtain the pathwise convergence rates.

Pathwise error bounds for the approximation of SDEs have been studied in several articles. However, pathwise approximation of SPDEs with an infinite-dimensional Wiener process has been considered so far mainly for stochastic parabolic PDEs with multiplicative space-time white noise, i.e. for equations with one space dimension, see e.g. [5], [3] and [4]. In these articles the pathwise convergence rates of several finite difference schemes are determined. Moreover – simultaneously to the preparation of this article – pathwise convergence rates for an exponential type approximation scheme for equation (1), which uses linear functionals of the driving noise, have been derived in [6].

## 2 Numerical Scheme

We now describe our numerical scheme for the approximation of (1). For this, recall that $\phi_n$ are the eigenvectors of $A$, so that $A\phi_n = \alpha_n \phi_n$, $n \in \mathbb{N}^d$, and moreover that the driving Wiener process is given by

$$W(t) = \sum_{n\in\mathbb{N}^d} \lambda_n^{1/2} \beta_n(t) \cdot \phi_n. \tag{2}$$

So, consider the mild solution of equation (1), i.e.

$$u(t) \;=\; e^{-tA}u(0) + \int_0^t e^{-(t-s)A}F(u(s))\,ds + \int_0^t e^{-(t-s)A}\,dW(s). \tag{3}$$

Writing the solution as a Fourier series $u(t) = \sum_{n\in\mathbb{N}^d} u_n(t)\cdot\phi_n$ we obtain the infinite system of coupled equations

$$u_n(t) \;=\; e^{-t\alpha_n}u_n(0) + \int_0^t e^{-(t-s)\alpha_n}F_n(u(s))\,ds + \int_0^t e^{-(t-s)\alpha_n}\lambda_n^{1/2}\,d\beta_n(s). \tag{4}$$

Here $F_n(u)$ denotes the $n$-th Fourier coefficient of $F(u)$, that is we have $F(u) = \sum_{n\in\mathbb{N}^d} F_n(u)\cdot\phi_n$.

Now let $\Delta t > 0$ denote the time step and $N$ the size of the Galerkin truncation. Consider the discretization of (1) at times $t_k = k\Delta t$ given by

$$\begin{aligned} \widehat{u}_n\Big(t_{k+1}\Big) &= e^{-\Delta t\alpha_n}\Big(\widehat{u}_n(t_k) + \Delta t F_n(\widehat{u}(t_k)) + \lambda_n^{1/2}\Delta B_{k,n}\Big), \\ \widehat{u}_n(0) &= u_n(0), \end{aligned} \tag{5}$$

where $|n| \leq N$ and $\Delta B_{k,n} = \beta_n(t_{k+1}) - \beta_n(t_k)$. The time continuous version of this scheme is given by

$$\widehat{u}_n(t) = e^{-t\alpha_n} u_n(0) + \int_0^t e^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} F_n(\widehat{u}(\lfloor s \rfloor_{\Delta t}))\, ds + \int_0^t e^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} \lambda_n^{1/2}\, d\beta_n(s). \tag{6}$$

Here we use the notation $\lfloor s \rfloor_{\Delta t} = \max_{k \in \mathbb{N}}\{t_k : t_k \leq s\}$. We study a version of the post processing method introduced in [9]:

$$\begin{aligned} \widehat{u}_n(t_{k+1}) &= e^{-\Delta t \alpha_n}\Big(\widehat{u}_n(t_k) + \Delta t F_n(\widehat{u}(t_k)) + 1_{\{|n| \leq N_w\}} \lambda_n^{1/2} \Delta B_{k,n}\Big), \\ \widehat{u}_n(0) &= u_n(0), \end{aligned} \tag{7}$$

where $|n| \leq N$. The constant $N_w$ describes the number of modes used to approximate the Wiener process $W$. If the noise is smooth, then fewer modes for the approximation of the noise than for the approximation of the nonlinearity can be used, see Corollary 3.4 in [9].

For the numerical analysis we use the following interpolant of $\widehat{u}_n(t_k)$ in time:

$$\widehat{u}_n(t) = e^{-t\alpha_n} u_n(0) + \int_0^t e^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} F_n(\widehat{u}(\lfloor s \rfloor_{\Delta t}))\, ds + 1_{\{|n| \leq N_w\}} \int_0^t e^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} \lambda_n^{1/2}\, d\beta_n(s). \tag{8}$$

So, finally our approximation of $u(t)$ is given by $\widehat{u}(t) = \sum_{|n| \leq N} \widehat{u}_n(t)$ for $t \geq 0$. Note that $\widehat{u}(t)$ depends on $N$, the size of the Galerkin truncation, on $N_w$, the number of the Fourier modes for the approximation of the noise $W$, and on the stepsize $\Delta t$.

### *2.1 Error bounds in the p-th mean*

We make the following assumptions on the nonlinearity $F$ and on the operators $A$ and $Q$:

**Assumption 1** *Let $F \in \mathcal{C}^2(H; H)$, i.e. the mapping $F : H \to H$ is twice continuously Fréchet-differentiable, and there exist constants $K_0, K_1, K_2 > 0$ such that*

$$\|F(u)\|_H \leq K_0(1 + \|u\|_H) \tag{9}$$

*and*

$$\|dF(u)\|_{L(H;H)} \leq K_1, \tag{10}$$
$$\|d^2F(u)\|_{L(H\times H;H)} \leq K_2. \tag{11}$$

*for all $u \in H$.*

Moreover, we have the following assumption on the eigenvalues of the covariance operator $Q$, which is by definition self-adjoint and positive.

**Assumption 2** *There exist $\gamma \geq 0$ and constants $C_1, C_2 > 0$ such that*

$$C_1 \cdot |n|^{-\gamma} \leq \lambda_n \leq C_2 \cdot |n|^{-\gamma}$$

*for $n \in \mathbb{N}^d$.*

Note that for $\gamma > d$ we have the so called trace class noise and $Q = id$ is included in the case $\gamma = 0$. For the eigenvalues of the operator $A$ we assume that they are strictly positive and have a polynomial growth.

**Assumption 3** *The operator $A : H \rightarrow H$ is self-adjoint and positive. Moreover, $\alpha_n > 0$ for $n \in \mathbb{N}^d$, $\alpha_m \leq \alpha_n$ for $|m| \leq |n|$ and there exists a $\kappa > 0$ and constants $C_3, C_4 > 0$ such that*

$$C_3 \cdot |n|^{\kappa} \leq \alpha_n \leq C_4 \cdot |n|^{\kappa}$$

*for $n \in \mathbb{N}^d$.*

Thus $-A$ generates in particular an analytical semigroup $(e^{-tA}, t \geq 0)$ on $H$, see [11].

Under the above assumptions, we have the following theorem, which in particular describes the smoothness of the solution in terms of the parameters $\gamma$ and $\kappa$. Its proof is given in the appendix:

**Theorem 4** *Let Assumptions 1, 2 and 3 hold, $u(0) \in D(A)$ and let $\gamma + \kappa > d$ and $T > 0$. Then equation (1) has a unique mild solution $(u(t), t \in [0, T])$, which satisfies*

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|_H^p < \infty \tag{12}$$

*for all $p \geq 1$.*

*Moreover, let* $\theta^* := \frac{\gamma+\kappa-d}{2\kappa}$. *Then we have* $u(t,\omega) \in D(A^\theta)$, $t \in [0,T]$, *for all* $\theta < \min\{1,\theta^*\}$ *and almost all* $\omega \in \Omega$. *Finally, for all* $p \geq 1$ *and all* $\theta < \min\{1,\theta^*\}$ *we have*

$$\sup_{t\in[0,T]} \mathbf{E}\|A^\theta u(t)\|_H^p < \infty \tag{13}$$

*and there exist constants* $K_{p,T,\theta} > 0$ *such that*

$$\Big(\mathbf{E}\|u(t) - u(s)\|_H^p\Big)^{1/p} \leq K_{p,T,\theta}|t-s|^\theta \tag{14}$$

*for all* $s,t \in [0,T]$ *and all* $\theta < \min\{1/2,\theta^*\}$.

Our main result for the convergence rates in the $p$-th mean is as follows:

**Theorem 5** *Let Assumptions 1, 2 and 3 hold and let* $\gamma+\kappa > d$ *and* $u_0 \in D(A)$. *Then for all* $\varepsilon > 0$, $T > 0$ *and* $p \geq 1$ *there exists a constant* $C_{\varepsilon,T,p} > 0$ *such that*

$$\sup_{t\in[0,T]} \Big(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\Big)^{1/p} \leq C_{\varepsilon,T,p}\Big(\Delta t^{\min\{1,\theta^*\}-\varepsilon} + N^{-\kappa} + N_w^{-\kappa\theta^*}\Big).$$

**Proof.** This is given in §4.

□

To balance the error contributions of the different parts, we have to consider two cases: (i) $\theta^* \geq 1$: Here it is optimal to choose

$$N_w = \lceil c_w \cdot N^{1/\theta^*} \rceil$$

with $c_w > 0$, so we can use fewer modes to approximate the noise. Furthermore, balancing the $\Delta t$-terms gives

$$\Delta t = c_{\Delta t} \cdot N^{-\kappa}$$

with $c_{\Delta t} > 0$. So, for $\widehat{u}$ with such a choice of $\Delta t, N, N_w$ we have

$$\sup_{t\in[0,T]} \Big(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\Big)^{1/p} \leq \widetilde{C}_{\varepsilon,T,p} \cdot N^{-\kappa+\varepsilon}.$$

(ii) $\theta^* < 1$: Here we can not save modes for the noise and have to choose

$$N_w = \lceil c_w \cdot N \rceil$$

with $c_w > 0$. Balancing again the $\Delta t$-terms gives

$$\Delta t = c_{\Delta t} \cdot N^{-\kappa\theta^*}$$

with $c_{\Delta t} > 0$. So, here we obtain

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq \widetilde{C}_{\varepsilon,T,p} \cdot N^{-\kappa\theta^*+\varepsilon}.$$

Summarizing, we have

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq \widetilde{C}_{\varepsilon,T,p} \cdot N^{-\kappa \min\{1,\theta^*\}+\varepsilon} \tag{15}$$

with

$$N_w = \lceil c_w \cdot N^{\min\{1,1/\theta^*\}} \rceil, \qquad \Delta t = c_{\Delta t} \cdot N^{-\kappa \min\{1,\theta^*\}}. \tag{16}$$

In the case that $-A$ is the one-dimensional Laplacian these error bounds coincide (up to the arbitrarily small $\varepsilon > 0$ ) with the results of Corollary 3.4 in [9].

### *2.2 Pathwise convergence rates*

For the pathwise convergence rates, we need the following lemma, which is a straightforward consequence of the Borel-Cantelli-Lemma, see e.g. [7].

**Lemma 1** *Let $\alpha > 0$ and $C_p \in [0,\infty)$ for $p \geq 1$. In addition, let $Z_n$, $n \in \mathbb{N}$, be a sequence of real-valued random variables such that*

$$(\mathbf{E}|Z_n|^p)^{1/p} \leq C_p \cdot n^{-\alpha}$$

*for all $p \geq 1$ and all $n \in \mathbb{N}$. Then for all $\varepsilon > 0$ there exists a random variable $\eta_\varepsilon$ such that*

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\alpha+\varepsilon} \qquad \mathbf{P}\text{-}a.s.$$

*for all $n \in \mathbb{N}$. Moreover, $\mathbf{E}|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.*

Applying this lemma we get the following result:

**Corollary 1** *Let Assumptions 1, 2 and 3 hold and let $\gamma + \kappa > d$ and $u_0 \in D(A)$. Moreover let $N, N_w$ and $\Delta t$ satisfy (16). Then for all $T > 0$ and $\varepsilon > 0$,* *there exists a random variable $\eta_{\varepsilon,T} > 0$ such that*

$$\|u(T,\omega) - \widehat{u}(T,\omega)\|_H \leq \eta_{\varepsilon,T}(\omega) \cdot N^{-\kappa \min\{1,\theta^*\}+\varepsilon}$$

*for almost all $\omega \in \Omega$.*

Since $\|h\|_H^2 = \int_{[a,b]^d} h(x)^2\, dx$ another application of the Borel-Cantelli-Lemma yields:

**Corollary 2** *Let the same assumptions as in the previous corollary hold and assume additionally that $\gamma + \kappa > d + 2$ and $\kappa > 1$. Then we have*

$$\widehat{u}(T,x,\omega) \overset{N\to\infty}{\longrightarrow} u(T,x,\omega)$$

*for almost all $\omega \in \Omega$ and almost all $x \in [a,b]^d$.*

So, in the case of the $d$-dimensional Laplacian, i.e. $\kappa = 2$ and trace class noise, i.e. $\gamma > d$, the exponential integrator scheme converges for almost all $\omega \in \Omega$ and almost all $x \in [a,b]^d$.

## 3 Numerical illustration

Consider the Allen–Cahn equation in two-dimensions

$$du(t) = [\nu \Delta u(t) + u(t) - u(t)^3]\, dt + dW(t)$$

with periodic boundary conditions on $[0,2\pi) \times [0,2\pi)$. Here we have the $d = 2$ dimensional Laplacian operator, so that $\kappa = 2$ in Assumption 3. We take noise that is white in time and vary the spatial regularity through the parameter $\gamma$ in Assumption 2. With these values we see that $\theta^* = \gamma/4$ and we have a critical value of $\gamma = 4$. We integrate using (7) to a final time $T = 2$ with a time step of $\Delta t = 0.005$. For our numerical calculations, we take the diffusion coefficient $\nu = 0.004$. To test the numerics, "true" solutions were computed using $256 \times 256$ modes and two sample "true" solutions at $T = 2$ are plotted in Figure 1. These solutions are computed with the same path and it is only the regularity of the noise that varies, in (a) $\gamma = 4$ and (b) $\gamma = 3$, and visually this is reflected in the regularity of the solution. In Figure 2 we show that our results agree with the theoretical results and for $\gamma = 4$ we see convergence like $N^{-2}$ (numerically we observe in the figure $-2.05$) both for a single realization and for the mean over 10 realizations. For $\gamma = 3$ we have convergence like the predicted $N^{-3/2}$ (numerically we observe in the figure $-1.53$) again for a single realization and for the mean over 10 realizations.

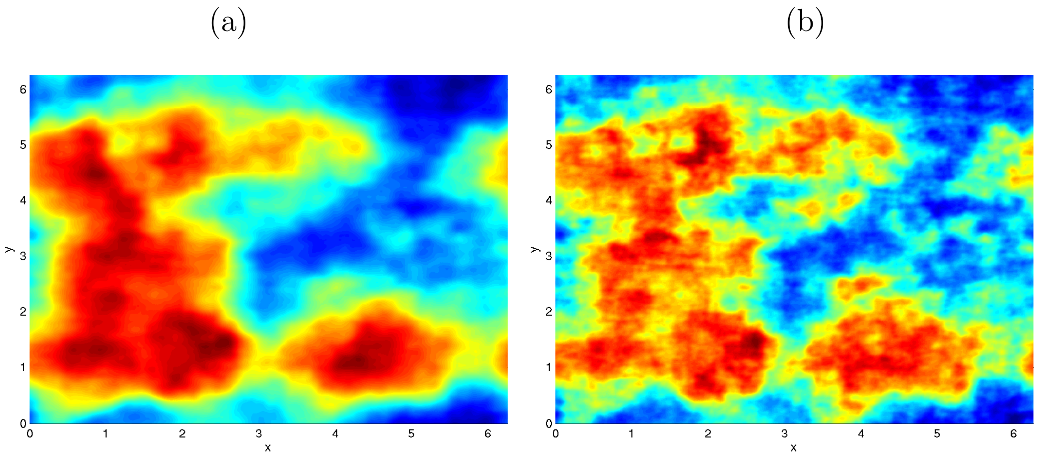


Fig. 1. Plot of two sample true solutions with $256 \times 256$ modes at time $T = 2$, (a) $\gamma = 4$ and (b) $\gamma = 3$. Note the solution in (a) is smoother than the solution in (b) as the regularity of the noise decreases.

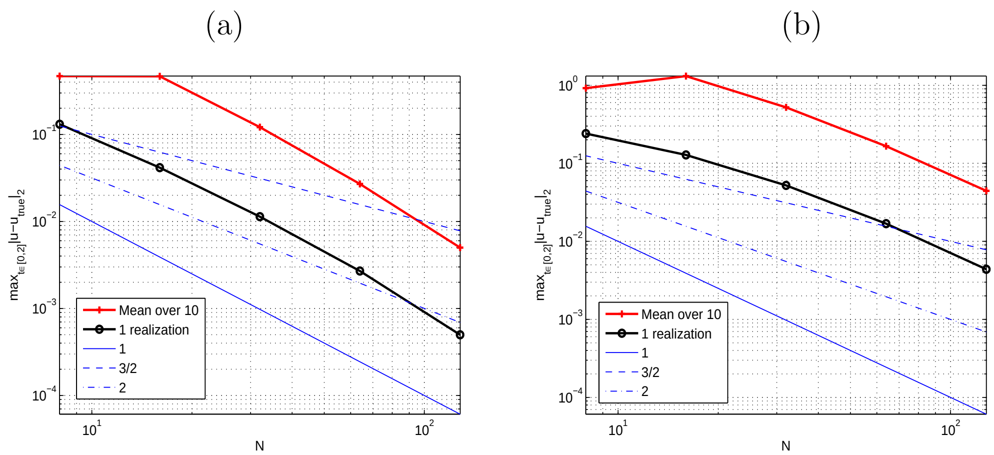


Fig. 2. Convergence in space for (a) $\gamma = 4$ and (b) $\gamma = 3$. Plot is a log log plot of the maximum $L^2$ error on $[0, 2]$, i.e. $\max_{t\in[0,2]} \|u(t) - u^N(t)\|_H^2$, as the system size $N$ is changed. Results are plotted for a single realization and for the mean over 10 realizations. In (a) we see the predicted rate of $N^{-2}$ and in (b) $N^{-3/2}$.

## 4 Proof of the convergence result

We prove Theorem 5 by estimating

$$e(\tau) = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\Big]^{1/p}$$

and applying Gronwall's Lemma.

We first recall some basic facts of stochastic integration with respect to a $Q$-Wiener process. Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ be a filtered probability space and let $W = (W(t), t \in [0,T])$ be a $Q$-Wiener process on this space with respect to the filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0,T])$. Denote by $L_2^0 := HS(Q^{1/2}(H), H)$ the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to $H$ and by $\|\cdot\|_{L_2^0}$ the corresponding norm given by

$$\|C\|_{L_2^0}^2 = \mathrm{Tr}(C^*QC) := \sum_{n\in\mathbb{N}^d} \langle C^*QC\varphi_n, \varphi_n\rangle,$$

where $\varphi_n, n \in \mathbb{N}^d$, is an arbitrary orthonormal basis of $H$. Moreover denote by $L_{\mathcal{F}}^2 := L_{\mathcal{F}}^2([0,T]; L_2^0)$ the space of all predictable stochastic processes $X = (X(t), t \in [0,T])$ with values in $L_2^0$ such that

$$\|X\|_{L_{\mathcal{F}}^2} := \Big(\int_0^T \mathbf{E}\|X(t)\|_{L_2^0}^2\, dt\Big)^{1/2} < \infty.$$

Then for $X \in L_{\mathcal{F}}^2$ the stochastic integral

$$\int_0^T X(t)\, dW(t)$$

is well defined as an element of $H$ and we have the following Itô isometry:

$$\mathbf{E}\Big\|\int_0^T X(t)\, dW(t)\Big\|_H^2 = \int_0^T \mathbf{E}\|X(t)\|_{L_2^0}^2\, dt. \tag{17}$$

(A process $X$ with values in $L_2^0$ is called predictable, if $X : [0,T]\times\Omega \to L_2^0$ is a $\mathcal{P}_T - \mathcal{B}(L_2^0)$ measurable mapping, where $\mathcal{P}_T$ is the $\sigma$-field generated by the sets $]s,t] \times F$, with $s,t \in [0,T]$, $F \in \mathcal{F}_s$ and $\{0\}\times F$ with $F \in \mathcal{F}_0$.)

The Itô integral satisfies the following stability property, see e.g. Proposition 4.15 in [1]: Let $G : D(G) \to H$ be a closed operator, where $D(G)$ is a Borel subset of $H$ and let moreover $X \in L_{\mathcal{F}}^2$ such that $\mathbf{P}(X(t) \in D(G)$ for all $t \in [0,T]) = 1$ and $GX \in L_{\mathcal{F}}^2$. Then, we have

$$\mathbf{P}\Big(\int_0^T X(s)\, dW(s) \in D(G)\Big) = 1$$

and

$$G\int_0^T X(s)\, dW(s) = \int_0^T GX(s)\, dW(s) \qquad \mathbf{P}\text{-a.s.}$$

Moreover, one has the following version of the Burkholder-Davis-Gundy inequality, see e.g. Lemma 7.2 in [1]: For any $r \geq 1$ and any $X \in L_{\mathcal{F}}^2$ there exist constants $C_r > 0$ such that

$$\mathbf{E}\Big\| \int_0^T X(s)\, dW(s)\Big\|^{2r} \le C_r\, \mathbf{E}\Big( \int_0^T \|X(s)\|_{L_2^0}^2\, ds\Big)^r. \tag{18}$$

We need the following version of the stochastic Fubini theorem, see e.g. Theorem 4.18 in [1]: Let $Y : \Omega \times [0,T] \to L_2^0$ be a $\mathcal{P}_T \times \mathcal{B}([0,T]) - \mathcal{B}(L_2^0)$−measurable mapping such that

$$\int_0^T \left( \mathbf{E} \int_0^T \|Y(t,s)\|_{L_2^0}^2\, dt \right)^{1/2} ds < \infty.$$

Then we have $\mathbf{P}$-a.s.

$$\int_0^T \int_0^T Y(t,s)\, dW(t)\, ds = \int_0^T \int_0^T Y(t,s)\, ds\, dW(t). \tag{19}$$

We also require the following properties of the operator $A$ and the semigroup $e^{-tA}$, see e.g. Theorem 6.13 in Chapter 2 in [11].

**Lemma 2** *For arbitrary $\delta_1 \ge 0$, $0 \le \delta_2 \le 1$ there exist constants $C_5, C_6 > 0$ such that we have*

$$\|A^{\delta_1} e^{-At}\|_{L(H;H)} \le C_5 t^{-\delta_1} \tag{20}$$

*and*

$$\|A^{-\delta_2}(\mathrm{id} - e^{-At})\|_{L(H;H)} \le C_6 t^{\delta_2} \tag{21}$$

*for any $t \in (0,T]$.*

We denote by $\mathbb{P}_N : H \to H$ the orthogonal projection of $H$ to the subspace generated by $\{\phi_n : |n| \le N\}$, i.e.

$$\mathbb{P}_N u = \sum_{|n| \le N} c_n \cdot \phi_n$$

for $u = \sum_{n \in \mathbb{N}^d} c_n \cdot \phi_n \in H$. Clearly, we have

$$\|\mathbb{P}_N u\|_H^2 = \sum_{|n| \le N} |c_n|^2$$

and

$$\|(\mathrm{id} - \mathbb{P}_N)u\|_H^2 = \sum_{|n| > N} |c_n|^2,$$

for $u = \sum_{n \in \mathbb{N}^d} c_n \cdot \phi_n$, which we use several times in the following. We also have

$$\|(\mathrm{id} - \mathbb{P}_N)e^{-At}\|_{L(H;H)} \le e^{-\min\{\alpha_n : |n| = N\} t} \tag{22}$$

for $t \in [0, T]$.

Finally, we require the following estimate, which can be obtained by straightforward calculations. Let $\delta > d$. Then, there exist constants $C_7, C_8 > 0$ which depend only on $d$ and $\delta$ such that

$$C_7 \cdot N^{-\delta+d} \leq \sum_{|n|>N} |n|^{-\delta} \leq C_8 \cdot N^{-\delta+d}. \tag{23}$$

After these preparations, we can now start with the error analysis. To estimate terms, we use a generic constant $C$ which varies between instances but is independent of $\Delta t$, $N$, $N_w$ and $t \in [0, T]$. Moreover, we write $\|\cdot\|$ instead of $\|\cdot\|_H$, $\|\cdot\|_{L(H;H)}$ respectively $\|\cdot\|_{L_2^0}$, if no misunderstanding is possible.

### *4.2 The initial value*

For the error of the approximation of the initial value we have

$$\textbf{INITIAL} = \sup_{t\in[0,\tau]} \|e^{-At}(u(0) - \widehat{u}(0))\|.$$

Since

$$\sup_{t\in[0,\tau]} \|e^{-At}(u(0) - \widehat{u}(0))\|^2 = \sup_{t\in[0,\tau]} \sum_{|n|>N} e^{-2\alpha_n t}|u_n(0)|^2 = \sum_{|n|>N} |u_n(0)|^2$$

and $u(0) \in D(A)$ it follows that

$$\textbf{INITIAL} = \Big(\sum_{|n|>N} |u_n(0)|^2\Big)^{1/2} \leq \frac{1}{\alpha_N^*}\Big(\sum_{|n|>N} |\alpha_n u_n(0)|^2\Big)^{1/2} \leq \frac{1}{\alpha_N^*}\|Au(0)\|,$$

where $\alpha_N^* = \min\{\alpha_n : |n| = N\}$. So, we obtain

$$\textbf{INITIAL} \leq C \cdot N^{-\kappa} \tag{24}$$

by Assumption 3.

### *4.3 The noise terms*

For estimating the noise terms recall that

$$W(t) = \sum_{n\in\mathbb{N}^d} \lambda_n^{1/2}\beta_n(t) \cdot \phi_n.$$

(i) Consider first the noise with modes $|n| \leq N_w$.

We have

$$\mathbf{NOISE}_1 = \sup_{t\in[0,\tau]} \left[\mathbf{E}\left\| \sum_{|n|\leq N_w} \lambda_n^{1/2}\Big( \int_0^t \left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right) d\beta_n(s)\Big) \cdot \phi_n \right\|^p\right]^{1/p}.$$

Since

$$\sum_{|n|\leq N_w} \lambda_n^{1/2}\Big( \int_0^t e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\, d\beta_n(s)\Big) \cdot \phi_n = \int_0^t \varphi(t,s)\, dW(s)$$

with

$$\varphi(t,s) = \sum_{|n|\leq N_w} \left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right) \cdot \phi_n,$$

an application of the Burkholder-Davis-Gundy inequality (18) yields

$$\mathbf{NOISE}_1 \leq C \sup_{t\in[0,\tau]} \left[ \int_0^t \|\varphi(t,s)\|^2_{L_2^0}\, ds\right]^{1/2}.$$

However,

$$\|\varphi(t,s)\|^2_{L_2^0} = \sum_{|n|\leq N_w} \lambda_n \left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2$$

and thus

$$\mathbf{NOISE}_1 \leq C \sup_{t\in[0,\tau]} \left[ \int_0^t \sum_{|n|\leq N_w} \lambda_n \left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds\right]^{1/2}.$$

Since for every $\theta \in [0,1]$ we have

$$|e^{-x} - e^{-y}| \leq |x-y|^\theta, \qquad x, y \geq 0,$$

we obtain

$$\begin{aligned}\int_0^t \left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds &\leq \int_0^t e^{-2(t-s)\alpha_n}\left(1 - e^{-(s-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds \\ &\leq \Delta t^{2\theta}\alpha_n^{2\theta} \int_0^t e^{-2(t-s)\alpha_n}\, ds \leq C\Delta t^{2\theta}\alpha_n^{2\theta-1}.\end{aligned}$$

for $\theta \in (0,1)$. Hence we have

$$\mathbf{NOISE}_1 \leq C\Delta t^\theta \Big( \sum_{|n|\leq N_w} \lambda_n \alpha_n^{2\theta-1}\Big)^{1/2} \leq C\Delta t^\theta \Big( \sum_{|n|\leq N_w} |n|^{-\gamma-\kappa+2\theta\kappa}\Big)^{1/2},$$

since

$$0 \leq \lambda_n \alpha_n^{2\theta-1} \leq C \cdot |n|^{-\gamma-\kappa+2\theta\kappa}$$

by Assumptions 2 and 3. Now (23) gives

$$\mathbf{NOISE}_1 \leq C \cdot \Delta t^\theta \tag{25}$$

for $\theta < \min\{1, \theta^*\}$ where $\theta^* = \frac{\gamma+\kappa-d}{2\kappa}$.

(ii) Now consider the noise with modes $|n| > N_w$, i.e.

$$\begin{aligned}\mathbf{NOISE}_2 &= \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|>N_w} \int_0^t \lambda_n^{1/2} e^{-(t-s)\alpha_n} d\beta_n(s)\cdot\phi_n\Big\|^p\Big]^{1/p} \\ &= \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\|(\mathrm{id}-\mathbb{P}_{N_w})\int_0^t e^{-A(t-s)}\, dW(s)\Big\|^p\Big]^{1/p}.\end{aligned}$$

Using the stability of the Itô integral and the Burkholder-Davis-Gundy inequality, see Subsection 4.1, we have

$$\mathbf{NOISE}_2 \le \sup_{t\in[0,\tau]} C\Big(\sum_{|n|>N_w} \lambda_n \int_0^t e^{-2(t-s)\alpha_n} ds\Big)^{1/2} \le C\Big(\sum_{|n|>N_w} \frac{\lambda_n}{\alpha_n}\Big)^{1/2}.$$

Assumptions 2 and 3 and the estimate (23) now give

$$\mathbf{NOISE}_2 \le C\cdot N_w^{(-\gamma-\kappa+d)/2}. \tag{26}$$

### *4.4 Nonlinear terms: modes $|n| > N$*

Consider now the nonlinear terms of $F$ not contributing to $\widehat{u}$: Using Jensen's inequality, estimate (22) and Assumption 1 we have

$$\begin{aligned}\mathbf{TAIL} &= \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|>N} \int_0^t e^{-(t-s)\alpha_n} F_n(u(s))\, ds\cdot\phi_n\Big\|^p\Big]^{1/p} \\ &= \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\|\int_0^t (\mathrm{id}-\mathbb{P}_N) e^{-(t-s)A} F(u(s))\, ds\Big\|^p\Big]^{1/p} \\ &\le \sup_{t\in[0,\tau]} \int_0^t \Big[\mathbf{E}\Big\|(\mathrm{id}-\mathbb{P}_N) e^{-(t-s)A} F(u(s))\Big\|^p\Big]^{1/p} ds \\ &\le C \sup_{t\in[0,\tau]} \int_0^t e^{-(t-s)\alpha_N^*} [\mathbf{E}(1+\|u(s)\|)^p]^{1/p}\, ds,\end{aligned}$$

where $\alpha_N^* = \min\{\alpha_n : |n| = N\}$. Since

$$\sup_{s\in[0,T]} \mathbf{E}(1+\|u(s)\|)^p < \infty$$

by Theorem 4, we have

$$\sup_{t\in[0,\tau]} \int_0^t e^{-(t-s)\alpha_N^*} [\mathbf{E}(1+\|u(s)\|)^p]^{1/p}\, ds \le C\frac{1}{\alpha_N^*}$$

and thus we obtain by Assumption 3 that

$$\mathbf{TAIL} \le C \cdot N^{-\kappa}. \tag{27}$$

### *4.5 Nonlinear Terms: modes $|n| \le N$*

We have

$$\sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(e^{-\alpha_n(\lfloor s\rfloor_{\Delta t}-s)}F_n(u(s)) - F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t}))\Big)\, ds \cdot \phi_n\Big\|^p\Big]^{1/p}$$
$$\le C\cdot(\mathbf{NL}_1 + \mathbf{NL}_2 + \mathbf{NL}_3),$$

where

$$\mathbf{NL_1} = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(F_n(u(s)) - F_n(u(\lfloor s\rfloor_{\Delta t}))\Big)\, ds \cdot \phi_n\Big\|^p\Big]^{1/p},$$
$$\mathbf{NL_2} = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(F_n(u(\lfloor s\rfloor_{\Delta t})) - F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t}))\Big)\, ds \cdot \phi_n\Big\|^p\Big]^{1/p},$$
$$\mathbf{NL_3} = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(e^{-\alpha_n(\lfloor s\rfloor_{\Delta t}-s)} - 1\Big)F_n(u(s))\, ds \cdot \phi_n\Big\|^p\Big]^{1/p}.$$

(i) The first term. Note that

$$\sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(F_n(u(s)) - F_n(u(\lfloor s\rfloor_{\Delta t}))\Big)\, ds \cdot \phi_n$$
$$= \mathbb{P}_N\Big[\int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})}\Big(F(u(s)) - F(u(\lfloor s\rfloor_{\Delta t}))\Big)\, ds\Big].$$

Moreover, we have

$$u(s) - u(\lfloor s\rfloor_{\Delta t}) = \delta_s^i + \delta_s^d + \delta_s^w$$

with

$$\delta_s^i = (e^{-A(s-\lfloor s\rfloor_{\Delta t})} - \mathrm{id})u(\lfloor s\rfloor_{\Delta t}),$$
$$\delta_s^d = \int_{\lfloor s\rfloor_{\Delta t}}^s e^{-A(s-\tau)}F(u(\tau))\, d\tau,$$
$$\delta_s^w = \int_{\lfloor s\rfloor_{\Delta t}}^s e^{-A(s-\tau)}\, dW(\tau)$$

and

$$F(u(s)) - F(u(\lfloor s\rfloor_{\Delta t})) = dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i + dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d + dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^w + r_s,$$

where

$$\|r_s\| \le C\|\delta_s\|^2$$

by Assumption 1. Thus, we have

$$\begin{aligned}
&\left[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(F_n(u(s)) - F_n(u(\lfloor s\rfloor_{\Delta t}))\Big)\, ds \cdot \phi_n \Big\|^p\right]^{1/p} \\
&\quad = \left[\mathbf{E}\Big\| \mathbb{P}_N \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})}\Big(F(u(s)) - F(u(\lfloor s\rfloor_{\Delta t}))\Big)\, ds\Big\|^p\right]^{1/p} \\
&\quad \le C\left[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i \, ds\Big\|^p\right]^{1/p} \\
&\qquad + C\left[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d \, ds\Big\|^p\right]^{1/p} \\
&\qquad + C\left[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^w \, ds\Big\|^p\right]^{1/p} \\
&\qquad + C\left[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} r_s \, ds\Big\|^p\right]^{1/p}.
\end{aligned}$$

For the first term note that $\mathbf{P}\Big(u(\lfloor s\rfloor_{\Delta t}) \in D(A^\theta) \text{ for all } s \in [0,T]\Big) = 1$ by Theorem 4 and thus $\mathbf{P}$-a.s.

$$\|(e^{-A(s-\lfloor s\rfloor_{\Delta t})} - \mathrm{id})u(\lfloor s\rfloor_{\Delta t})\| \le \|A^{-\theta}(e^{-A(s-\lfloor s\rfloor_{\Delta t})} - \mathrm{id})A^\theta u(\lfloor s\rfloor_{\Delta t})\|,$$

since $A^\theta$ and the semigroup $e^{-tA}$ commute. Now, Lemma 2 gives $\mathbf{P}$-a.s.

$$\|A^{-\theta}(e^{-A(s-\lfloor s\rfloor_{\Delta t})} - \mathrm{id})A^\theta u(\lfloor s\rfloor_{\Delta t})\| \le C|s - \lfloor s\rfloor_{\Delta t}|^\theta \|A^\theta u(\lfloor s\rfloor_{\Delta t})\|.$$

So we obtain by the assumptions on the nonlinearity $F$ and the boundedness of the semigroup generated by $-A$ that $\mathbf{P}$-a.s.

$$\begin{aligned}
\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i \, ds\Big\| &\le C\Delta t^\theta \int_0^t \|e^{-A(t-\lfloor s\rfloor_{\Delta t})}\| \|A^\theta u(\lfloor s\rfloor_{\Delta t})\|\, ds \\
&\le C\Delta t^\theta \int_0^t \|A^\theta u(\lfloor s\rfloor_{\Delta t})\|\, ds
\end{aligned}$$

for $t \in [0,T]$. An application of Hölder's inequality and Theorem 4 yield that

$$\sup_{t\in[0,\tau]} \left[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i \, ds\Big\|^p\right]^{1/p} \le C\cdot \Delta t^\theta \tag{28}$$

for all $\theta < \min\{1, \theta^*\}$.

Now to the second term. Here we have by the assumptions on the nonlinearity $F$ and the boundedness of the semigroup generated by $-A$ that

$$\|\delta_s^d\| \le C\int_{\lfloor s\rfloor_{\Delta t}}^s (1 + \|u(\tau)\|)\, d\tau.$$

So we obtain

$$\Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d \, ds \Big\| \le C \int_0^t \int_{\lfloor s\rfloor_{\Delta t}}^{\lfloor s\rfloor_{\Delta t}+\Delta t} (1+\|u(\tau)\|)\, d\tau\, ds$$

and it again follows by Theorem 4 and an application of Hölder's inequality that

$$\sup_{t\in[0,\tau]} \Big[ \mathbf{E} \Big\| \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d \, ds \Big\|^p \Big]^{1/p} \le C\cdot \Delta t. \tag{29}$$

The third term: Since

$$\delta_s^w = \int_{\lfloor s\rfloor_{\Delta t}}^s e^{-A(s-\tau)}\, dW(\tau)$$

we have

$$\begin{aligned}
&\int_0^t \Big[ e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t}))\Big] \int_{\lfloor s\rfloor_{\Delta t}}^s e^{-A(s-\tau)}\, dW(\tau)\, ds \\
&\quad = \int_0^t \int_{\lfloor s\rfloor_{\Delta t}}^s e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\, dW(\tau)\, ds \\
&\quad = \int_0^T \int_0^T 1_{[\lfloor s\rfloor_{\Delta t},s]}(\tau) 1_{[0,t]}(s) e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\, dW(\tau)\, ds
\end{aligned}$$

using the stability of the Itô integral, see Subsection 4.1. By the stochastic Fubini Theorem, see again Subsection 4.1, it follows $\mathbf{P}$-a.s.

$$\begin{aligned}
&\int_0^T \int_0^T 1_{[\lfloor s\rfloor_{\Delta t},s]}(\tau) 1_{[0,t]}(s) e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\, dW(\tau)\, ds \\
&\quad = \int_0^T \int_0^T 1_{[\tau,\lceil \tau\rceil_{\Delta t}]}(s) 1_{[0,t]}(\tau) e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\, ds\, W(\tau) \\
&\quad = \int_0^t \Big[ \int_\tau^{\lceil \tau\rceil_{\Delta t}} e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\, ds \Big] dW(\tau), .
\end{aligned}$$

where $\lceil \tau\rceil_{\Delta t} = \min_{k\in\mathbb{N}}\{t_k : t_k \ge \tau\}$. Since

$$e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\phi_n = e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})} dF_n(u(\lfloor s\rfloor_{\Delta t})) e^{-\alpha_n(s-\tau)}\phi_n$$

we have

$$\begin{aligned}
\|e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\|^2_{L_2^0} &= \sum_{n\in\mathbb{N}^d} \lambda_n e^{-2\alpha_n(t-\lfloor s\rfloor_{\Delta t}+s-\tau)} |dF_n(u(\lfloor s\rfloor_{\Delta t}))|^2 \\
&\le \sum_{n\in\mathbb{N}^d} \lambda_n e^{-2\alpha_n(t-\tau)} |dF_n(u(\lfloor s\rfloor_{\Delta t}))|^2
\end{aligned}$$

and thus

$$\|e^{-A(t-\lfloor s\rfloor_{\Delta t})} dF(u(\lfloor s\rfloor_{\Delta t})) e^{-A(s-\tau)}\|^2_{L_2^0} \le C \sum_{n\in\mathbb{N}^d} \lambda_n e^{-2\alpha_n(t-\tau)}$$

by the assumptions on $F$. Hence it follows

$$\Big\| \int_\tau^{\lceil \tau \rceil_{\Delta t}} e^{-A(t-\lfloor s \rfloor_{\Delta t})} dF(u(\lfloor s \rfloor_{\Delta t})) e^{-A(s-\tau)}\, ds \Big\|_{L_2^0}^2 \le C\Delta t^2 \sum_{n\in\mathbb{N}^d} \lambda_n e^{-2\alpha_n (t-\tau)}.$$

Thus, we obtain by the Burkholder-Davis-Gundy inequality (18) that

$$\begin{aligned} &\sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \int_0^t \int_\tau^{\lceil \tau \rceil_{\Delta t}} e^{-A(t-\lfloor s \rfloor_{\Delta t})} dF(u(\lfloor s \rfloor_{\Delta t})) e^{-A(s-\tau)}\, ds\, dW(\tau)\Big\|^p\Big]^{1/p} \\ &\quad \le C\Delta t \Big( \sum_{n\in\mathbb{N}^d} \frac{\lambda_n}{\alpha_n}\Big)^{1/2}. \end{aligned}$$

Since $\gamma + \kappa > d$ we have $\sum_{n\in\mathbb{N}^d} \frac{\lambda_n}{\alpha_n} < \infty$ and hence

$$\sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s \rfloor_{\Delta t})} dF(u(\lfloor s \rfloor_{\Delta t})) \delta_s^w\, ds\Big\|^p\Big]^{1/p} \le C\cdot \Delta t. \qquad (30)$$

Finally, for the remainder term we obtain by straightforward estimations and Theorem 4 that

$$\sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \int_0^t e^{-A(t-\lfloor s \rfloor_{\Delta t})} r_s\, ds\Big\|^p\Big]^{1/p} \le C\cdot \Delta t^\theta + C\cdot \Delta t \qquad (31)$$

for all $\theta < 1$. Thus combining the estimates (28)–(31) yields

$$\mathbf{NL_1} \le C\cdot \Delta t^\theta \qquad (32)$$

for all $\theta < \min\{1, \theta^*\}$.

(ii) The second term. Here we have

$$\mathbf{NL}_2 = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \int_0^t \sum_{|n|\le N} e^{-\alpha_n(t-\lfloor s \rfloor_{\Delta t})} \Big(F_n(u(\lfloor s \rfloor_{\Delta t})) - F_n(\widehat{u}(\lfloor s \rfloor_{\Delta t}))\Big)\, ds\cdot \phi_n\Big\|^p\Big]^{1/p}.$$

Again, we can write

$$\mathbf{NL}_2 = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\|\mathbb{P}_N \int_0^t e^{-A(t-\lfloor s \rfloor_{\Delta t})} \Big(F(u(\lfloor s \rfloor_{\Delta t})) - F(\widehat{u}(\lfloor s \rfloor_{\Delta t}))\Big)\, ds\Big\|^p\Big]^{1/p}.$$

So we obtain by Jensen's inequality, the Lipschitz continuity of $F$ and the boundedness of $e^{-tA}$ that

$$\mathbf{NL}_2 \le C \int_0^\tau \sup_{t\in[0,s]} \Big[\mathbf{E}\|u(t) - \widehat{u}(t)\|^p\Big]^{1/p} ds. \qquad (33)$$

(iii) The third nonlinear term.

$$\mathbf{NL}_3 = \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\Big(e^{\alpha_n(s-\lfloor s\rfloor_{\Delta t})} - 1\Big) F_n(u(s))\, ds \cdot \phi_n \Big\|^p\Big]^{1/p}.$$

Rewriting this expression using the projection operator and applying Jensen's inequality we have

$$\begin{aligned}\mathbf{NL}_3 &= \sup_{t\in[0,\tau]} \Big[\mathbf{E}\Big\|\mathbb{P}_N \int_0^t e^{-A(t-\lfloor s\rfloor_{\Delta t})}\Big(e^{A(s-\lfloor s\rfloor_{\Delta t})} - \mathrm{id}\Big) F(u(s))\, ds\Big\|^p\Big]^{1/p} \\ &\le \sup_{t\in[0,\tau]} \int_0^t \Big[\mathbf{E}\Big\| e^{-A(t-\lfloor s\rfloor_{\Delta t})}\Big(\mathrm{id} - e^{-A(s-\lfloor s\rfloor_{\Delta t})}\Big) F(u(s))\Big\|^p\Big]^{1/p} ds \\ &\le \sup_{t\in[0,\tau]} \int_0^t \Big[\mathbf{E}\Big\| A^\theta e^{-A(t-\lfloor s\rfloor_{\Delta t})} A^{-\theta}\Big(\mathrm{id} - e^{-A(s-\lfloor s\rfloor_{\Delta t})}\Big) F(u(s))\Big\|^p\Big]^{1/p} ds \\ &\le \sup_{t\in[0,\tau]} \int_0^t \Big\| A^\theta e^{-A(t-\lfloor s\rfloor_{\Delta t})}\Big\| \Big\| A^{-\theta}\Big(\mathrm{id} - e^{-A(s-\lfloor s\rfloor_{\Delta t})}\Big)\Big\| \Big[\mathbf{E}\Big\| F(u(s))\Big\|^p\Big]^{1/p} ds.\end{aligned}$$

Now Theorem 4 and Lemma 2 give

$$\mathbf{NL}_3 \le C\Delta t^\theta \sup_{t\in[0,\tau]} \int_0^t (t - \lfloor s\rfloor_{\Delta t})^{-\theta}\Big[\mathbf{E}\Big\|F(u(s))\Big\|^p\Big]^{1/p} ds$$

for all $\theta < 1$. So using Assumption 1 and Theorem 4 we have

$$\mathbf{NL}_3 \le C \cdot \Delta t^\theta \tag{34}$$

for all $\theta < 1$.

(v) Now, combining (32)–(34), we have

$$\mathbf{NL} \le C \int_0^\tau \sup_{t\in[0,s]} \Big[\mathbf{E}\Big\|u(t) - \widehat{u}(t)\Big\|^p\Big]^{1/p} ds + C\cdot \Delta t^\theta. \tag{35}$$

for all $\theta < \min\{1, \theta^*\}$.

### *4.6 Conclusion*

Combining the estimates (24)–(27) and (35) we have achieved the following inequality

$$\begin{aligned}&\sup_{s\in[0,\tau]} \Big[\mathbf{E}\Big\|u(s) - \widehat{u}(s)\Big\|^p\Big]^{1/p} \\ &\qquad \le C \int_0^\tau \sup_{t\in[0,s]} \Big[\mathbf{E}\Big\|u(t) - \widehat{u}(t)\Big\|^p\Big]^{1/p} ds + C\cdot N^{-\kappa} + C\cdot N_w^{(-\gamma-\kappa+d)/2} + C\cdot \Delta t^\theta.\end{aligned}$$

for all $\theta < \min\{1, \theta^*\}$. Gronwall's Lemma provides now the assertion of Theorem 5.

## A Proof of Theorem 4

We first show the following lemma:

**Lemma 3** *Let $\kappa+\gamma > d$, $\theta < \theta^* = \frac{\gamma+\kappa-d}{2\kappa}$ and $\vartheta \in [0, 1/2]$ such that $\vartheta+\theta < \theta^*$. Then there exist constants $C_9, C_{10}, C_{11} > 0$, which are independent of $s, t \in [0, T]$, such that*

$$\int_0^t \|e^{-Au}\|_{L_2^0}^2 \, du \leq C_9, \tag{A.1}$$

$$\int_s^t \|A^\theta e^{-A(t-u)}\|_{L_2^0}^2 \, du \leq C_{10} \cdot |t-s|^{2\vartheta} \tag{A.2}$$

*and*

$$\int_0^s \|A^\theta \Big( e^{-A(t-u)} - e^{-A(s-u)} \Big)\|_{L_2^0}^2 \, du \leq C_{11} \cdot |t-s|^{2\vartheta}. \tag{A.3}$$

**Proof.** Throughout this proof, we will denote constants, which are independent of $s, t \in [0, T]$, by $C$ regardless of their value.

(i) Recall that here $L_2^0$ denotes the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to $H$ and $\| \cdot \|_{L_2^0}$ is the corresponding norm given by $\|C\|_{L_2^0}^2 = \mathrm{Tr}(C^* Q C)$. Since $e^{-Au}$ is self-adjoint with eigenvalues $e^{-\alpha_j u}$ and eigenvectors $\phi_j$ and since moreover $Q$ is selfadjoint with eigenvalues $\lambda_j$ and eigenvectors $\phi_j$ and $\phi_j$, $j \in \mathbb{N}^d$, is an orthonormal basis of $H$, we have

$$\mathrm{Tr}(e^{-Au} Q e^{-Au}) = \sum_{j \in \mathbb{N}^d} \langle e^{-Au} Q e^{-Au} \phi_j, \phi_j \rangle = \sum_{j \in \mathbb{N}^d} e^{-2\alpha_j u} \lambda_j.$$

Thus we obtain

$$\int_0^T \|e^{-As}\|_{L_2^0}^2 \, ds = \sum_{j \in \mathbb{N}^d} \int_0^T e^{-2\alpha_j s} \lambda_j \, ds \leq \sum_{j \in \mathbb{N}^d} \frac{\lambda_j}{\alpha_j}.$$

Since

$$0 \leq \frac{\lambda_j}{\alpha_j} \leq C \cdot |j|^{-\gamma-\kappa}$$

by Assumptions 2 and 3, estimate (23) yields that

$$\int_0^T \|e^{-As}\|_{L_2^0}^2\, ds \le C \sum_{j\in\mathbb{N}^d} n^{-\gamma-\kappa} < \infty$$

for $\kappa + \gamma > d$.
(ii) We have similar that

$$\|A^\theta e^{-A(t-u)}\|_{L_2^0}^2 = \sum_{j\in\mathbb{N}^d} e^{-2\alpha_j(t-u)} \lambda_j \alpha_j^{2\theta}$$

and hence

$$\int_s^t \|A^\theta e^{-A(t-u)}\|_{L_2^0}^2\, du \le \sum_{j\in\mathbb{N}^d} \lambda_j \alpha_j^{2\theta-1}\Big(1 - e^{-2\alpha_j(t-s)}\Big).$$

Since for every $\theta \in [0,1]$ we have

$$|e^{-x} - e^{-y}| \le |x-y|^\theta, \qquad x, y \ge 0,$$

it follows

$$\int_s^t \|A^\theta e^{-A(t-u)}\|_{L_2^0}^2\, du \le |t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \lambda_j \alpha_j^{2(\theta+\vartheta)-1} \le C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} |j|^{-\gamma-\kappa+2\kappa(\theta+\vartheta)}$$

for $\vartheta \in [0, 1/2]$ by Assumptions 2 and 3. Moreover, $\vartheta + \theta < \theta^*$ yields

$$2\kappa(\theta+\vartheta) < \gamma + \kappa - d$$

and thus (23) gives

$$\sum_{j\in\mathbb{N}^d} |j|^{-\gamma-\kappa+2\kappa(\theta+\vartheta)} < \infty.$$

(iii) Similar to (i) we obtain

$$\begin{aligned}\|A^\theta\Big(e^{-A(t-u)} - e^{-A(s-u)}\Big)\|_{L_2^0}^2 &= \|A^\theta e^{-A(s-u)}\Big(e^{-A(t-s)} - \mathrm{id}\Big)\|_{L_2^0}^2 \\ &= \sum_{j\in\mathbb{N}^d} \alpha_j^{2\theta} \lambda_j e^{-2\alpha_j(s-u)}\Big(e^{-\alpha_j(t-s)} - 1\Big)^2\end{aligned}$$

and thus

$$\|A^\theta\Big(e^{-A(t-u)} - e^{-A(s-u)}\Big)\|_{L_2^0}^2 \le C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \alpha_j^{2(\theta+\vartheta)} \lambda_j e^{-2\alpha_j(s-u)},$$

respectively

$$\int_0^s \|A^\theta\Big(e^{-A(t-u)} - e^{-A(s-u)}\Big)\|_{L_2^0}^2\, du \le C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \lambda_j \alpha_j^{2(\theta+\vartheta)-1}.$$

Now we can proceed as in (ii).

$\square$

**Proof of Theorem 4.**

We will again denote constants, which are independent of $s, t \in [0,T]$, by $C$ regardless of their value.

(i) Note first that the stochastic integrals

$$W_A(t) = \int_0^t e^{-(t-s)A}\, dW(s), \qquad t \in [0,T],$$

are well defined, if

$$\int_0^T \|e^{-sA}\|_{L_2^0}\, ds < \infty,$$

see Theorem 5.2 in [1]. The latter is true due to Lemma 3 for $\kappa + \gamma > d$. Existence of a unique mild solution of equation (1) with $\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^p < \infty$ for all $p \geq 1$ follows now from a straightforward generalization of Theorem 7.6 in [1].

(ii) Now recall that $\theta^* = \frac{\gamma+\kappa-d}{2\kappa}$, let $\theta < \theta^*$ and consider

$$W_A(t) - W_A(s) = \int_s^t e^{-A(t-u)}\, dW(u) + \int_0^s \left(e^{-A(t-u)} - e^{-A(s-u)}\right) dW(u).$$

By Lemma 3 and the stability of the Itô integral, see Subsection 4.1, we have that $A^\theta(W_A(t) - W_A(s))$ is $\mathbf{P}$-a.s. well defined. Moreover, by the Burkholder-Davis-Gundy inequality (18) and the above lemma we obtain

$$\begin{aligned}
&\Big[\mathbf{E}\|A^\theta(W_A(t) - W_A(s))\|^p\Big]^{1/p} \\
&\quad\leq C\left[\mathbf{E}\left\|\int_s^t A^\theta e^{-A(t-u)}\, dW(u)\right\|^p\right]^{1/p} + C\left[\mathbf{E}\left\|\int_0^s A^\theta\Big(e^{-A(t-u)} - e^{-A(s-u)}\Big)\, dW(u)\right\|^p\right]^{1/p} \\
&\quad\leq C\left[\int_s^t \left\|A^\theta e^{-A(t-u)}\right\|_{L_2^0}^2 du\right]^{1/2} + C\left[\int_0^s \left\|A^\theta\Big(e^{-A(t-u)} - e^{-A(s-u)}\Big)\right\|_{L_2^0}^2 du\right]^{1/2} \\
&\quad\leq C|t-s|^\vartheta
\end{aligned}$$

for all $\vartheta \in [0, 1/2]$ such that $\vartheta + \theta < \theta^*$. The Kolmogorov-Chentsov Theorem now implies that there exists a modification $\widetilde{W}_A$ of $W_A$ such that

$$\widetilde{W}_A(\cdot, \omega) \in \bigcap_{\theta<\theta^*} C\Big([0,T]; D(A^\theta)\Big)$$

for almost all $\omega \in \Omega$. Moreover, we have

$$\sup_{t\in[0,T]} \Big[\mathbf{E}\|A^\theta \widetilde{W}_A(t)\|^p\Big]^{1/p} < \infty, \qquad \Big[\mathbf{E}\|\widetilde{W}_A(t) - \widetilde{W}_A(s)\|^p\Big]^{1/p} \leq C|t-s|^{\min\{1/2,\theta\}}$$

for all $s, t \in [0, T]$ and $\theta < \theta^*$.

(iii) Finally consider $A^\theta(u(t) - u(s))$, $s, t \in [0, T]$. We have $\mathbf{P}$-a.s.

$$A^\theta(u(t) - u(s)) = A^\theta(e^{-At}u(0) - e^{-As}u(0)) + A^\theta \int_s^t e^{-A(t-\tau)} F(u(\tau))\, d\tau \\ + A^\theta \int_0^s \Big(e^{-A(t-\tau)} - e^{-A(s-\tau)}\Big) F(u(\tau))\, d\tau + A^\theta(\widetilde{W}_A(t) - \widetilde{W}_A(s))$$

for all $s, t \in [0, T]$. So it follows

$$\Big[\mathbf{E}\Big\|A^\theta(u(t) - u(s))\Big\|^p\Big]^{1/p} \leq \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} + \mathbf{I_4}$$

with

$$\begin{aligned}
\mathbf{I_1} &= \Big\|A^\theta e^{-As}(e^{-A(t-s)} - \mathrm{id})u(0)\Big\|, \\
\mathbf{I_2} &= \Big[\mathbf{E}\Big\|A^\theta \int_s^t e^{-A(t-\tau)} F(u(\tau))\, d\tau\Big\|^p\Big]^{1/p}, \\
\mathbf{I_3} &= \Big[\mathbf{E}\Big\|A^\theta \int_0^s e^{-A(s-\tau)}\Big(e^{-A(t-s)} - \mathrm{id}\Big) F(u(\tau))\, d\tau\Big\|^p\Big]^{1/p}, \\
\mathbf{I_4} &= \Big[\mathbf{E}\Big\|A^\theta(\widetilde{W}_A(t) - \widetilde{W}_A(s))\Big\|^p\Big]^{1/p}.
\end{aligned}$$

Since $u(0) \in D(A)$ we have by Lemma 2 that

$$\begin{aligned}
\mathbf{I_1} &= \|A^\theta e^{-As}(e^{-A(t-s)} - \mathrm{id})u(0)\| \\
&\leq \|e^{-As} A^{\theta-1}(e^{-A(t-s)} - \mathrm{id})Au(0)\| \\
&\leq \|e^{-As}\| \|A^{\theta-1}(e^{-A(t-s)} - \mathrm{id})\| \|Au(0)\| \\
&\leq C|t - s|^{1-\theta}
\end{aligned}$$

for all $\theta < 1$. Moreover, by step (ii) we have

$$\mathbf{I_4} \leq C|t - s|^\vartheta$$

for all $\vartheta \in [0, 1/2]$ such that $\vartheta + \theta < \theta^*$. For the second term we obtain by Jensen's inequality and the stability of the integral that

$$\begin{aligned}
\mathbf{I_2} &= \Big[\mathbf{E}\Big\|\int_s^t A^\theta e^{-A(t-\tau)} F(u(\tau))\, d\tau\Big\|^p\Big]^{1/p} \\
&\leq \int_s^t \|A^\theta e^{-A(t-\tau)}\|\, [\mathbf{E}\|F(u(\tau))\|^p]^{1/p}\, d\tau.
\end{aligned}$$

Hence Assumption 1 and Lemma 2 give

$$\mathbf{I_2} \leq C \int_s^t |t - \tau|^{-\theta}\Big(1 + \sup_{t \in [0,T]} [\mathbf{E}\|u(t)\|^p]^{1/p}\Big)\, d\tau.$$

Since $\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^p < \infty$ by part (i) of the proof, it follows

$$\mathbf{I_2} \le C|t-s|^{1-\theta}.$$

Finally, consider the third term. Here we have, proceeding as above,

$$\begin{aligned}
\mathbf{I_3} &= \Big[\mathbf{E}\Big\| \int_0^s A^\theta e^{-A(s-\tau)}\Big(e^{-A(t-s)} - \mathrm{id}\Big) F(u(\tau))\, d\tau \Big\|^p\Big]^{1/p} \\
&\le C \int_0^s \Big\|A^{\theta+\delta} e^{-A(s-\tau)}\Big\| \, \Big\|A^{-\delta}\Big(e^{-A(t-s)} - \mathrm{id}\Big)\Big\| \, \Big(1 + \sup_{t\in[0,T]} [\mathbf{E}\|u(t)\|^p]^{1/p}\Big)\, d\tau \\
&\le C|t-s|^\delta
\end{aligned}$$

for $\delta < 1-\theta$.

Combining the estimates for $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3}$ and $\mathbf{I_4}$ we obtain

$$\Big[\mathbf{E}\Big\|A^\theta(u(t)-u(s))\Big\|^p\Big]^{1/p} \le C|t-s|^\vartheta + C|t-s|^\delta,$$

for all $\vartheta \in [0,1/2]$ such that $\theta + \vartheta < \theta^*$ and $\delta \in [0,1]$ such that $\delta < 1-\theta$. Hence by the Kolmogorov-Chentsov Theorem it follows that there exists a modification $\widetilde{u}$ of $u$ such that

$$\widetilde{u}(\cdot,\omega) \in \bigcap_{\theta < \min\{1,\theta^*\}} C\Big([0,T]; D(A^\theta)\Big)$$

for almost all $\omega \in \Omega$. Furthermore, the above estimates give

$$\sup_{t\in[0,T]} \Big[\mathbf{E}\|A^\theta \widetilde{u}(t)\|^p\Big]^{1/p} < \infty, \qquad \Big[\mathbf{E}\|\widetilde{u}(t) - \widetilde{u}(s)\|^p\Big]^{1/p} \le C|t-s|^{\min\{1/2,\theta\}}$$

for all $s,t \in [0,T]$ and $\theta < \min\{1,\theta^*\}$. □

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