# POST PROCESSING FOR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

GABRIEL J LORD[†§] AND TONY SHARDLOW[‡¶]

**Abstract.** We investigate the strong approximation of stochastic parabolic partial differential equations with additive noise. We introduce post-processing in the context of a standard Galerkin approximation, although other spatial discretizations are possible. In time, we follow [20] and use an exponential integrator. We prove strong error estimates and discuss the best number of post-processing terms to take. Numerically, we evaluate the efficiency of the methods and observe rates of convergence. Some experiments with the implicit Euler–Maruyama method are described.

**Key words.** Stochastic exponential integrator, post-processing, numerical solution of stochastic PDEs.

**AMS subject classifications.** 60H15,65M12,65M15,65M60

**1. Introduction.** We consider the numerical approximation of the stochastic evolution equation

$$du = \Big[\Delta u + F(u)\Big]\,dt + dW(t), \qquad \text{given} \qquad u(0) = u_0, \tag{1.1}$$

with periodic boundary conditions on $[0, 2\pi)$, where $W(t)$ is a $Q$ Wiener process [3] on $L^2(0, 2\pi)$ and $F$ is nonlinear (precise assumptions are given in §3.1).

Suppose that $\phi_n$ are eigenvectors of the Laplacian $\Delta$ with periodic boundary conditions, so that $\Delta\phi_n = -n^2\phi_n$, $n \in \mathbb{Z}$. We assume that $Q$ has eigenfunctions $\phi_n$ with corresponding eigenvalues $\lambda_n \geq 0$, in which case

$$W(t) = \sum_{n\in\mathbb{Z}} \lambda_n^{1/2}\phi_n\beta_n(t), \tag{1.2}$$

for independent Brownian motions $\beta_n$. We do not consider the existence of solutions to (1.1) here, instead we call on [3]. We will investigate the effect on numerics of the spatial regularity of the noise, determined from the decay of $\lambda_n$.

There is a growing literature on numerical methods for stochastic PDEs and the majority of these analyse convergence in the strong or root mean squared sense. Finite difference approximations have been examined by a number of authors, see for example [25], [11], [12], [4] and finite element methods have also been considered, e.g. [29]. Galerkin approximations and strong Taylor schemes were considered in [10] with a scalar Wiener process. Strong convergence of the implicit Euler–Maruyama method was investigated in [18]. A more general analysis is found in [14], which considers different types of spatial discretizations (Galerkin as well as collocation, finite differences, finite elements, and wavelet based schemes) for similar forms of noise considered here. [24] analyses convergence and complexity through the number of random samples of the Wiener process. Spatially smooth noise is considered in [20] and [26] and these papers also consider Fourier based spatial discretizations. In

[†]Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK, gabriel@ma.hw.ac.uk

[‡]School of Mathematics, Oxford Road, University of Manchester M13 9PL, UK. shardlow@maths.man.ac.uk

[§]Supported in part by EPSRC grant GR/S60921/01.

[¶]Supported in part by EPSRC grant GR/R78725/01.

[26], a Taylor based discretization is taken and efficient methods for approximating the Wiener process are considered. In [20] strong convergence of an exponential integrator (see also [23]) is examined and we consider this scheme further in this paper, see §3.

The purpose of this paper is to study Galerkin post processing methods for (1.1), prove their convergence in the strong sense and evaluate their efficiency. We have restricted attention to reaction diffusion equations with homogeneous diffusion and additive noise, and plan to return to the general case in further work.

§2 is an introduction to post-processing methods for deterministic PDEs. §3 describes our Galerkin post processing scheme and §3.1 a theorem on the convergence of the method. §4 investigates the numerical behaviour of the method for the stochastic Allen-Cahn equation. We evaluate the efficiency of the methods, compare the rates of convergence to those predicted by the theorem, and illustrate numerically that post-processing is efficient for other time-stepping algorithms by experimenting with implicit Euler–Maruyama. We summarise our results and conclude in §5. The proof of the theorem is given in §6, with the proof of two lemmas left to the Appendix.

**2. A review of deterministic post processing.** Post-processing methods originate from analytical results on inertial manifolds for PDEs, see for example [6], where it can be shown that the dynamics of infinite dimensional PDEs converge to a finite dimensional system in large time. Typically, a graph $\Phi$ is obtained that "enslaves" the high Fourier modes (fine scale dynamics) to a finite number of low Fourier modes (large scale dynamics). For example, if $P$ denotes the projection onto the first $N$ Fourier modes and $u = p + q = Pu + (I - P)u$, we can write the deterministic PDE

$$u_t = \Delta u + F(u) \qquad \text{as} \qquad p_t = \Delta p + PF(p+q), \qquad q_t = \Delta q + (I - P)F(p+q).$$

The dynamics on the inertial manifold can be re-written as

$$p_t = \Delta p + PF(p+q), \qquad q(t) = \Phi(p).$$

Numerically the nonlinear Galerkin methods, also called approximate inertial manifolds (AIM) methods, make an approximation to the graph. In these methods, the evolution on a coarse mesh (i.e., low Fourier modes) uses information from the fine scale (i.e., high modes) at each time step, where a simpler form of equation is solved.

To deal with deterministic PDEs with non-smooth initial data, long transients or highly oscillatory time dependent forcing, He and Mattheij [28] introduced a dynamic form of post-processing, where the following system is approximated

$$p_t = \Delta p + PF(p), \qquad q_t = \Delta q + (1 - P)F(p).$$

It extends the approach of [7], where a fine mesh solution is found at the end of the computations. For the dynamic post-processing approach, both the coarse and fine mesh approximations are evolved in time and, unlike a traditional approximate inertial manifold approach, there is no communication from fine to coarse mesh until the end of the computation. Indeed, this communication was one of the main reasons that the AIM approach was computationally less efficient than a standard Galerkin method; see [7, 8].

He and Mattheij [28] discretized the PDEs in space by a Galerkin method and in time by implicit Euler and examined stability and convergence of the scheme and propose this as a computationally more efficient method. In [21] the post-processing method is examined from a truncation analysis point of view. From a perturbation expansion for the high modes and by keeping terms to different orders, they obtain systems that correspond to the post-processed Galerkin method and this yields convergence theory. Furthermore, from numerics based on Burgers equation with highly oscillatory forcing, they show that post-processing methods are more efficient and have an improved rate of convergence. These results suggest that post-processing may be advantageous for a stochastically forced PDE.

Although inertial manifolds have been shown to exist for stochastic PDEs [2], we do not attempt to approximate this directly here. Instead we base our method on the post-processing approaches of [28] and [21].

**3. Numerical Scheme.** We will consider a Fourier based Galerkin discretization, although other spatial discretizations are possible. The time discretization may be thought of as a stochastic version of an exponential integrator proposed by [19]; for a review of these methods in the deterministic case see [22] and for an application using a finite difference spatial discretization see [16]. In the stochastic context such schemes are considered in [20, 23] and related schemes by [27, 17] which are of the exponential time differencing type.

We describe our numerical scheme for (1.1). Represent $u(t)$ as a Fourier series $u(t) = \sum_n u_n(t)\phi_n$ and obtain the infinite system of coupled equations

$$u_n(t) \;=\; e^{-tn^2}u_n(0) + \int_0^t e^{-(t-s)n^2}F_n\big(u(s)\big)\,ds + \int_0^t e^{-(t-s)n^2}\lambda_n^{1/2}\,d\beta_n(s), \qquad (3.1)$$

where $F_n$ is the $n$th component of $F$, so that $F(u) = \sum_n F_n(u)\phi_n$. Let $\Delta t > 0$ denote the time step and $N$ the size of the Galerkin truncation. Consider the discretization of (1.1) at times $t_k = k\Delta t$ given by

$$u_n^N(t_{k+1}) = e^{-\Delta t n^2}\left(u_n^N(t_k) + \Delta t F_n\big(u^N(t_k)\big) + \lambda_n^{1/2}\Delta B_{k,n}\right), \qquad (3.2)$$

where $|n| \le N$, the noise terms $\Delta B_{k,n} = \beta_n(t_{k+1}) - \beta_n(t_k)$, and initial data $u_n^N(0) = u_n(0)$. The relationship between (3.2) and (3.1) is quite obvious when we iterate (3.2): for $t = k\Delta t$,

$$u_n^N(t) = e^{-tn^2}u_n^N(0) + \sum_{k=0}^{\lfloor t/\Delta t\rfloor - 1} e^{-(t-t_k)n^2}\left(\Delta t F_n(u^N(t_k)) + \lambda_n^{1/2}\Delta B_{k,n}\right) \qquad (3.3)$$

(no terms in the sum for $0 \le t < \Delta t$). This approximation has been studied in detail in [20] for Gevrey (exponentially smooth) noise.

We study a generalisation of this method, which incorporates post-processing terms and flexibility in the approximation of $W(t)$. The generalised method has the following form: for $|n| \le N$,

$$u_n^N(t_{k+1}) = e^{-n^2\Delta t}\Big(u_n^N(t_k) + \Delta t F_n(u^N(t_k)) + \mathbf{1}_{\{|n|\le N_w\}}\lambda_n^{1/2}\Delta B_{k,n}\Big), \qquad (3.4)$$

with initial data $u_n^N(0) = u_n(0) = u_{0,n}$, where $\mathbf{1}_X$ equals 1 if $X$ holds, 0 otherwise. The constant $N_w$ describes the number of modes used to approximate $W(t)$; this is the first generalisation and we will show the advantages in taking $N_w < N$ in certain applications. As in [20], the analysis depends on an interpolant of $u_n^N(t_k)$ in time: let

$$u_n^N(t) = e^{-n^2 t}u_n^N(0) + \sum_{k=0}^{\lfloor t/\Delta t\rfloor - 1} e^{-(t-t_k)n^2}\Big(\Delta t F_n(u^N(t_k)) + \lambda_n^{1/2}\mathbf{1}_{\{|n|\le N_w\}}\Delta B_{k,n}\Big), \qquad (3.5)$$

and note that the two definitions of $u_n^N(t_k)$ agree.

Now we introduce post processing. Given knowledge of $u^N$ the following are efficiently computed

$$q_n^N(t_{k+1}) = e^{-n^2\Delta t}\Big(q_n^N(t_k) + \Delta t \mathbf{1}_{\{|n|\le N_p\}} F_n(u^N(t_k)) + \lambda_n^{1/2}\mathbf{1}_{\{|n|\le N_w\}}\Delta B_{k,n}\Big), \quad (3.6)$$

with initial data $q_n^N(0) = u_n(0)$ for $N < |n| \le N_p$, where $N_p$ describes the number of nonlinear terms. Again in the analysis §6, we use an interpolant

$$\begin{aligned} q_n^N(t) &= e^{-n^2 t} q_n^N(0) \\ &+ \sum_{k=0}^{\lfloor t/\Delta t\rfloor -1} e^{-(t-t_k)n^2}\Big(\Delta t \mathbf{1}_{\{|n|\le N_p\}} F_n(u^N(t_k)) + \lambda_n^{1/2}\mathbf{1}_{\{|n|\le N_w\}}\Delta B_{k,n}\Big). \end{aligned} \quad (3.7)$$

We seek to estimate the error in approximating $u(t)$ by $u^N(t) + q^N(t)$, where $u^N = \sum_{|n|\le N}\phi_n u_n^N$ and $q^N = \sum_{N<|n|\le \max\{N_p,N_w\}}\phi_n q_n^N$, and in particular to understand the best choice of $N_w$ and $N_p$.

**3.1. Statement of Main Theorem.** Let $\|\cdot\|$ denote the standard $L^2(0,2\pi)$ norm. Denote the $H^m(0,2\pi)$ Sobolev norm for $u = \sum_n u_n\phi_n$ by

$$\|u\|_m = \|(I-\Delta)^{m/2}u\| = \Big(\sum_{n\in\mathbb{Z}}(1+n^2)^m u_n^2\Big)^{1/2}.$$

We make the following assumption of $f$ and $Q$:

Assumption 3.1. *For $u_1, u_2, u \in L^2(0,2\pi)$, for some constant $K_0$ and some $m, r \ge 0$,*

$$\|F(u_1) - F(u_2)\|_r \le K_0\|u_1-u_2\|_r, \quad (3.8)$$

$$\|F(u)\|_r \le K_0(1+\|u\|_r) \quad (3.9)$$

*and*

$$\|F(u_1) - F(u_2)\|_m \le K_0\|u_1-u_2\|_m, \quad (3.10)$$

$$\|F(u)\|_m \le K_0(1+\|u\|_m). \quad (3.11)$$

*There exists a constant $K_1$ such that for $u \in L^2(0,2\pi)$ and $\delta, \delta_1, \delta_2 \in H^m(0,2\pi)$,*

$$\|dF(u)\delta\|_m \le K_1\|\delta\|_m, \quad (3.12)$$

$$\|d^2F(u)(\delta_1,\delta_2)\|_m \le K_1\|\delta_1\|_m\|\delta_2\|_m. \quad (3.13)$$

*The covariance $Q$ of $W(t)$ satisfies* $\mathrm{Tr}(I-\Delta)^\gamma Q < \infty$*; i.e.,*

$$\sum_{n\in\mathbb{Z}}(1+n^2)^\gamma\lambda_n < \infty. \quad (3.14)$$

We have introduced three regularity parameters: $\gamma$ describes regularity of the noise; $r$ gives the regularity of the solution $u(t)$; $m$ indicates the norm for our error analysis.

Theorem 3.2. *Let $u_0 \in H^2(0,2\pi)$, $m < \min\{r,2\}$, $0 \le r \le \gamma+1$ and $\gamma > -1$. For some $\nu > 0$, consider $\Delta t \to 0$ and $N \to \infty$ with $\Delta t N^2 \le \nu$. For each $T > 0$, there exists $K > 0$ such that*

$$\begin{aligned} &\Big(\mathbf{E}\Big[\sup_{0<t_k\le T}\|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\Big)^{1/2} \\ &\qquad \le K\Big(\Delta t + N^{-2} + \mathbf{1}_{N\le N_w}N^{-1-\gamma} + N_p^{-2-r+m} + \Delta t N_w^{1-\gamma+m} + N_w^{-1-\gamma+m}\Big), \end{aligned}$$

*where* $u^N = \sum_{|n|\le N} \phi_n u_n^N$ *and* $q^N = \sum_{N<|n|\le \max\{N_p,N_w\}} \phi_n q_n^N$ *with components defined by* (3.5)–(3.7).

*Proof.* This is given in §6. □

Note that we take limits in $\Delta t, N$ with $\Delta t N^2 \le \nu$ but employ no restriction on $\nu$. If an explicit Euler time integrator was used, we would require $\nu \le \frac{1}{2}$ [11], and the absence of this restriction is a clear advantage to the exponential time integrator.

The theorem is stated under the global Lipschitz assumption on the nonlinearity. This is the simplest setting in which to work and allows us to focus attention on post processing. The global Lipschitz assumption excludes many important cases, including the Allen-Cahn equation we discuss in §4. The first approach to this problem is to change the nonlinearity without affecting the underlying model: for example, in the Allen-Cahn equation, the variable $u$ describes the phase of some material and is only physically meaningful inside a bounded set. If we smooth out the nonlinear term at infinity, the essential features of the model remain. In §4, we discover our results are demonstrated without such a modification. The second approach is to develop the mathematics to include ever wider classes of nonlinearities. Approaches of this type include [15] for finite dimensional SDEs, who use moment conditions to control the behaviour of $u$ at infinity and gain rates of convergence, and [11] who shows convergence in probability, without rates, for very general classes of $f$. The inclusion of these approaches in the present paper would obscure the main idea, which is post processing.

To understand post processing, we state two corollaries (using that $\Delta t N^2 = \nu$). The first describes convergence for the method (3.2) for non-smooth problems (extending work done in [20]). The second gives the values $N_w, N_p$ that yield the best convergence rates.

Corollary 3.3 (no postprocessing). *Under the assumptions of Theorem 3.2 with* $N = N_w = N_p$,

$$\Big(\mathbf{E}\Big[\sup_{0<t_k\le T} \|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\Big)^{1/2} \le K\Big(N^{-2} + N^{-2-r+m} + N^{-1-\gamma+m}\Big).$$

For example, with $\gamma = -1/2$ (space-time white noise), the $L^2(0, 2\pi)$ error (case $m = 0$) converges like $N^{-1/2}$. This is consistent with related results in the literature (e.g., [13],[18]). For Gevrey noise and a smooth nonlinearity, the parameters $r$ and $\gamma$ may be chosen arbitrarily large and we recover the result of [20]: for any $z > 0$, there exists a constant $K$ such that

$$\mathbf{E}\Big[\sup_{0<t_k\le T} \|u(t_k) - u^N(t_k)\|_1\Big] \le K(N^{-z} + \Delta t). \tag{3.15}$$

This is faster convergence than any polynomial, although not the exponential rate found [5] for the deterministic case.

Now we turn to post-processing.

Corollary 3.4 (post processing). *Let the assumptions of Theorem 3.2 hold.*

1. *If* $\gamma \ge 1$ *and* $m < \gamma - 1$, *then*

$$\Big(\mathbf{E}\Big[\sup_{0<t_k\le T} \|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\Big)^{1/2} \le K N^{-2}$$

*with* $N_p = N$ *and* $N_w = \lceil N^{2/(1+\gamma-m)} \rceil$.

*2. If $\gamma \ge 1$ and $m \ge \gamma - 1$, then*

$$\Big(\mathbf{E}\Big[\sup_{0<t_k\le T}\|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\Big)^{1/2} \le K N^{-1-\gamma+m}$$

*with $N_p = N$ and $N_w = N$.*

*3. If $-1 < \gamma < 1$, then*

$$\Big(\mathbf{E}\Big[\sup_{0<t_k\le T}\|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\Big)^{1/2} \le K N^{-1-\gamma}$$

*with $N_p = N$ and $N_w = \lceil N^{(1+\gamma)/(1+\gamma-m)}\rceil$.*

*These choices of $N_p$ and $N_w$ provide the best convergence rate (up to scalar multiplication).*

*Proof.* We wish to choose $N_p$ and $N_w$ in terms of $N$ to achieve the best convergence rate with $N$ by balancing terms in the estimate provided in Theorem 3.2. We ignore multiplying constants which do not affect the rate.

1. We can achieve an $N^{-2}$ convergence rate by balancing $N_p^{-2-r+m}$, $\Delta t N_w^{1-\gamma-m}$, and $N_w^{-1-\gamma+m}$ with $N^{-2}$. The condition $N_p^{-2-r+m} = N^{-2}$ yields $N_p = N^{2/(2+r-m)}$ and as $N^{2/(2+r-m)} \le N$ for $m \le r$, we choose $N_p = N$. Under assumption $m < \gamma - 1$, $\Delta t N_w^{1-\gamma-m} < N^{-2}$ and so the value $N_w$ is found by solving $N^{-2} = N_w^{-1-\gamma+m}$.
2. In the case $m > \gamma - 1$, the accuracy is limited by the term $\Delta t N_w^{1-\gamma+m}$. The condition $\Delta t N_w^{1-\gamma+m} = N_w^{-1-\gamma+m}$ implies $N_w = N$. The condition $N_p^{-2-r+m} = N_w^{-1-\gamma+m}$ implies $N_p = N^{(2+r-m)/(1+\gamma-m)}$. Because we have $N^{(2+r-m)/(1+\gamma-m)} > N$ for $r > \gamma - 1$, the choice $N_p = N$ terms is optimal.
3. We achieve an $N^{-1-\gamma}$ rate by choosing $\Delta t N_w^{1-\gamma-m}$ and $N_w^{-1-\gamma+m}$ less than $N^{-1-\gamma}$. This is achieved by taking

$$N_w \ge \max\{N^{(1-\gamma)/(1-\gamma+m)}, N^{(1+\gamma)/(1+\gamma-m)}\}.$$

   As $m \ge 0$, we take $N_w = N^{(1+\gamma)/(1+\gamma-m)}$. Balancing the terms $N_p^{-2-r+m}$ and $N^{-1-\gamma}$ provides $N_p = N^{(1+\gamma)/(2+r-m)}$. As $m < r$ and $\gamma < 1$, we have $N^{(1+\gamma)/(2+r-m)} < N$ and choose $N_p = N$.

□

There are a number of issues to consider: the rate of convergence, the constant for this rate, and the efficiency of the scheme. We can improve the rate of convergence by choice of $N_w$ and there are two cases to consider. For smooth noise $\gamma \ge 1 + m$, the optimal value is $N_w < N$, which saves computing random numbers for many of the components $u_n^N$. This has been used with good effect in [26] for a Gevrey smooth noise. Note that $N_w \to 1$ as $\gamma \to \infty$. In practise, it is important for $N_w \to \infty$ as we ask for more accuracy and to take to enough modes to resolve the noise.

For non-smooth noise ($\gamma < 1$), the optimal $N_w > N$, which implies that the post-processing corrections $q_n^N$ are Gaussian processes

$$q_n^N(t_{k+1}) = e^{-n^2\Delta t}\Big(q_n^N(t_k) + \lambda_n^{1/2}\mathbf{1}_{\{|n|\le N_w\}}\Delta B_{k,n}\Big). \tag{3.16}$$

Thus, computing the post-processing update is straightforward and cheap. To compare solutions for a single realisation of $W(t)$, $q_n^N$ must be found by time stepping. For weak approximation, it will be more efficient to compute and sample from the Gaussian distribution at the final time.

Our analysis predicts no improvement in the rate of convergence from post-processing the nonlinear term. This contrasts with results on post-processing in the deterministic case, where there is a gain in the rate of convergence [7, 8] (though this gain is often out weighed by extra computational cost).

**4. Numerics.** Consider the one–dimensional Allen-Cahn equation with noise:

$$du = \left[\alpha u_{xx} + u - u^3\right] dt + dW(t), \qquad u(0) = u_0, \tag{4.1}$$

with periodic boundary conditions on $[0, 2\pi)$. For numerical calculations, we take the diffusion coefficient $\alpha = 1/36$. We always take noise white in time and vary the spatial regularity $\gamma$, see (3.14).

To test the numerics, "true" solutions were computed by a standard Galerkin approximation with $N = 2^{11}$ modes and a time step $\Delta t = 5 \times 10^{-6}$. To avoid aliasing errors, the nonlinear term was computed with $2N$ terms (more than the optimal number of terms suggested by the 2/3 rule [1]). For a discussion of the role of aliasing in post-processing (in the deterministic case) see [9].

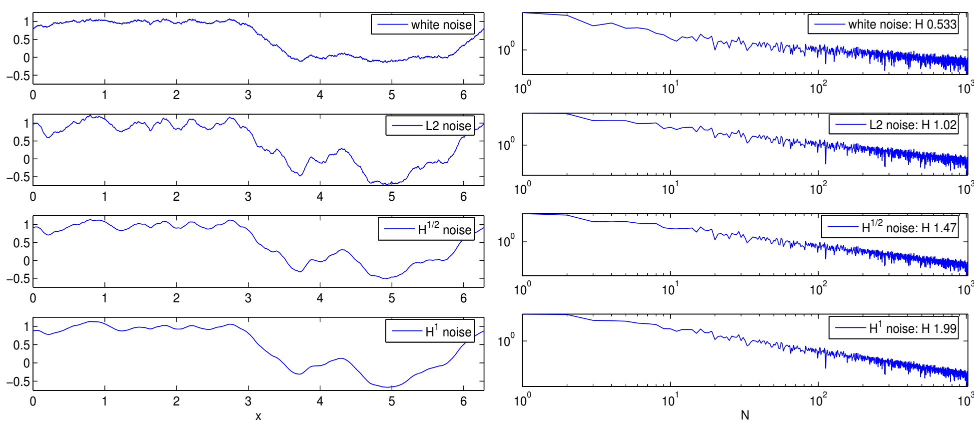


FIG. 4.1. *Plot (left) of "true" solutions at time $t = 1$ for $\gamma = -0.5, 0, 0.5, 1.0$ for one realization of the noise. Plot (right) is the corresponding log log plot of the Fourier coefficients at time $t = 1$ which shows that for $\gamma > 0$ the solutions are in a Sobolev space $H^r$ with $r = 0.5, 1, 1.5, 2$.*

Sample "true" solutions are plotted in Fig 4.1, this shows (left) the effect of different spatial regularity in real space and (right) the corresponding loglog plot in Fourier space. In real space, the solutions are smoother as the regularity of the noise increases. This is confirmed by the decay of the Fourier modes and we see numerically that $r = \gamma + 1$, consistent with the results of Lemma A.1. Essentially we gain a derivative on the regularity of the solution over the noise.

Let $\hat{N}$ denote a parameter for post-processing (either $2N$, $4N$, $8N$, or $N^2$ in experiments). The "true" solutions were used to compute errors for the following approximations:

***Galerkin*:** A standard Galerkin approximation, from solving (3.4) with $N_w = N$.

***PP Full*:** A full post-processed solution, from solving (3.4) and (3.6) with $N_w = N_p = \hat{N}$.

***PP Noise*:** A post-processed solution on noise only, from solving (3.4) and (3.16) with $N_p = N$, $N_w = \hat{N}$.

We examine the rate of convergence and efficiency by a mean cputime time. From a practical point of view, plots of cputime versus error can be interpreted in two ways: either fix a desired accuracy and see how long it would take to achieve, or fix a time and see how accurate a solution can be computed in that time. The expectation is computed from 10 samples and we examine the root mean square of the error at time $t = 1$ in an appropriate norm. Normally we take the $L^2$ norm ($m = 0$) or $H^1$ norm ($m = 1$).

On the plots below we draw a line with slope equal to the predicted rate of convergence for *Galerkin*. We also report in the legend the observed slope from the data for the rate of convergence.

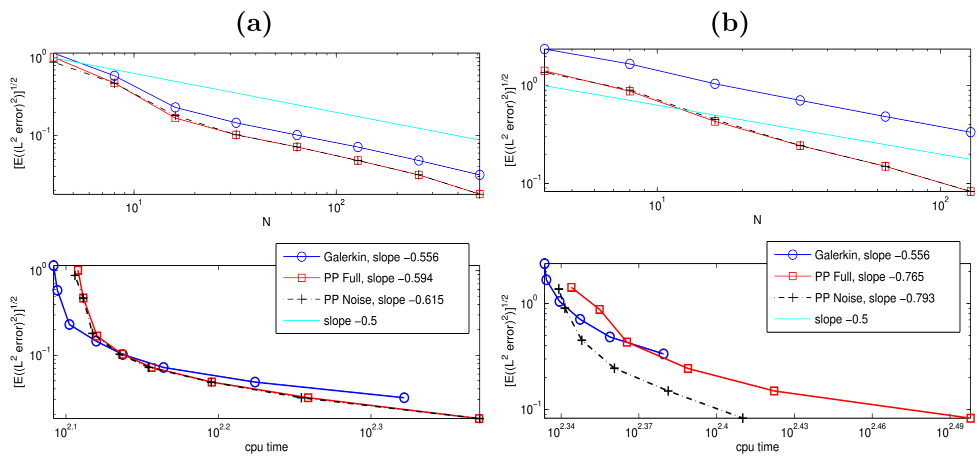


FIG. 4.2. *Space-time white noise (a) with $\hat{N} = 2N$ and (b) with $\hat{N} = 8N$. Plots show the $L^2$ error (top) rate of convergence and (below) plot of efficiency (cputime).*

We examine the rates of convergence and computational efficiency for $W(t)$ defined by (1.2) with $\lambda_n = (1 + n^2)^{-\gamma}|n|^{-1}$, $n \neq 0$ and $\lambda_0 = 0$. We consider $\gamma = -1/2$ (*space time white noise*) , $\gamma = 0$ ($L^2$ *noise*), and $\gamma = 1/2, 1, 2$ ($H^\gamma$ *noise*). Our predictions for the numerics are based on Theorem 3.2 where, motivated by Lemma A.1, we assume that $r = \gamma + 1$.

**4.1. Space-time white noise: $\gamma = -\frac{1}{2}$.** We observe in Fig 4.2(top) the theoretically predicted rates of convergence for *Galerkin*: the $L^2$ error decays like $N^{-1/2}$. There is no convergence for $H^1$ error.

Post processing is not expected to improve the rate of convergence in the $L^2$ norm, as $N_w = N$ in Corollary 3.4. With $\hat{N} = 2N$, this is supported by computations: see Fig 4.2 (a) (top) where the post-processing has no beneficial effect and the observed rate is the same as *Galerkin*. However, there is an improvement in the error constant and for $N > 32$ modes post-processing is more efficient; see Fig 4.2 (a) (bottom). Taking this further and using more modes for the post-processing, Fig 4.2 (b) shows *PP Full* and *PP Noise* with $\hat{N} = 8N$. The numerics suggest a rate of convergence faster than the theoretical one. This is encouraging, although the resolution is coarse and the theoretical rate may reappear for larger $N$.

We clearly see the computational advantage of *PP Noise* compared to *PP Full* and *Galerkin* in Fig 4.2 (a) and (b) bottom. Post-processing on the noise terms only is far more efficient.

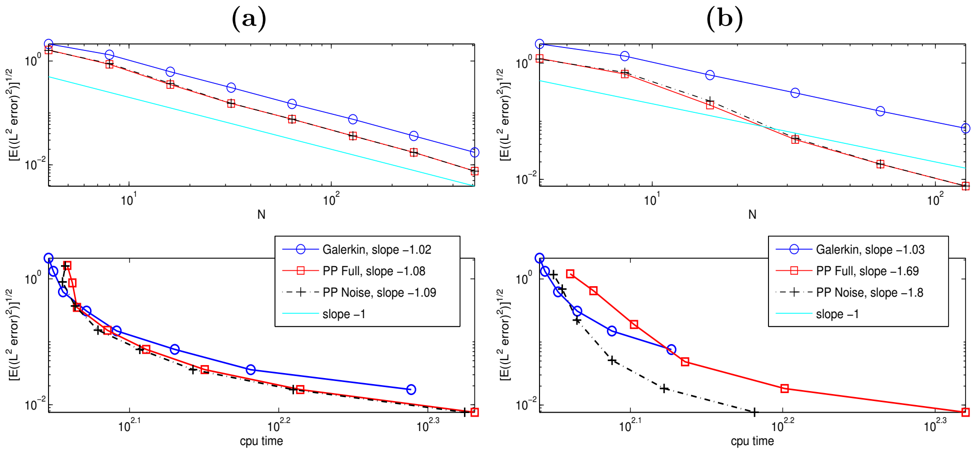


FIG. 4.3. *For $L^2$ noise, we plot (a) the $L^2$ error with $\hat{N} = 2N$ and (b) the $L^2$ error with $\hat{N} = 8N$. Top shows error against $N$ and below error against average cputime.*

**4.2. $L^2$ noise.** This is similar to white noise: for *Galerkin*, the $L^2$ error decays like $N^{-1}$, which is observed in Fig 4.3 (a) and (b), and the $H^1$ error does not converge. In theory, post-processing offers no improvement. In practise, there is an improvement in the error constant and an improvement in efficiency for $\hat{N} = 2N$ and further improvement for $\hat{N} = 8N$. See Fig 4.3(a) and (b).

**4.3. $H^{1/2}$ noise.** Corollary 3.3 predicts convergence of the $L^2$ error like $N^{-3/2}$ and the $H^1$ error like $N^{-1/2}$ for *Galerkin* and these rates are observed in Fig 4.4 (a) and (b). With post-processing, the optimal rate for the $L^2$ error is not changed and the $H^1$ error is like $N^{-3/2}$ if $N_w = N^3$. It is impractical to calculate with $N^3$ post processing terms for large $N$, and instead we look at $\hat{N} = 2N, 4N, 8N$. Fig 4.4 shows the effect of increasing $\hat{N}$ for $L^2$ error (left) and $H^1$ error (right) with $\hat{N}$ increasing top to bottom. For $L^2$ and $H^1$ errors, increasing $\hat{N}$ improves the error and seems to improve the rate of convergence – although this is not expected from the analysis for $L^2$ and we are a long way from taking the predicted $N^3$ modes for $H^1$. We clearly see that *PP Noise* is the most efficient method.

**4.4. $H^1$ noise.** Corollary 3.3 predicts that the *Galerkin* $L^2$ error decays like $N^{-2}$ and $H^1$ error decays like $N^{-1}$ as observed in Fig 4.5. This is the limiting case in Corollary 3.4, where we find $N^{-2}$ convergence by taking $N_w = N$ for $L^2$ error and $N_w = N^2$ for $H^1$ error; the solution is smooth in space and accuracy is now limited by time stepping. It is impractical to calculate with $N^2$ post processing terms for large $N$, and instead we look at $\hat{N} = 8N$: Fig 4.5 shows (a) the $L^2$ error and (b) the $H^1$ error. The post-processing methods give smaller errors and are more efficient, in particular *PP Noise*.

**4.5. $H^2$ noise.** The optimal number of modes is $N_w = N^{2/3}$ for the $L^2$ error, giving $N^{-2}$ convergence. We see in Fig 4.6 (a) that the $L^2$ error is converging faster than the theoretical rate, close to $N^{-3}$. Here we see a limitation of the analysis: the theoretical convergence rate is limited to an $N^{-2}$ rate because of time stepping and regularity of the initial data. In this case, the error is dominated by the spatial approximation of a smooth problems, which may decrease like $N^{-3}$, similar to rates described in (3.15).

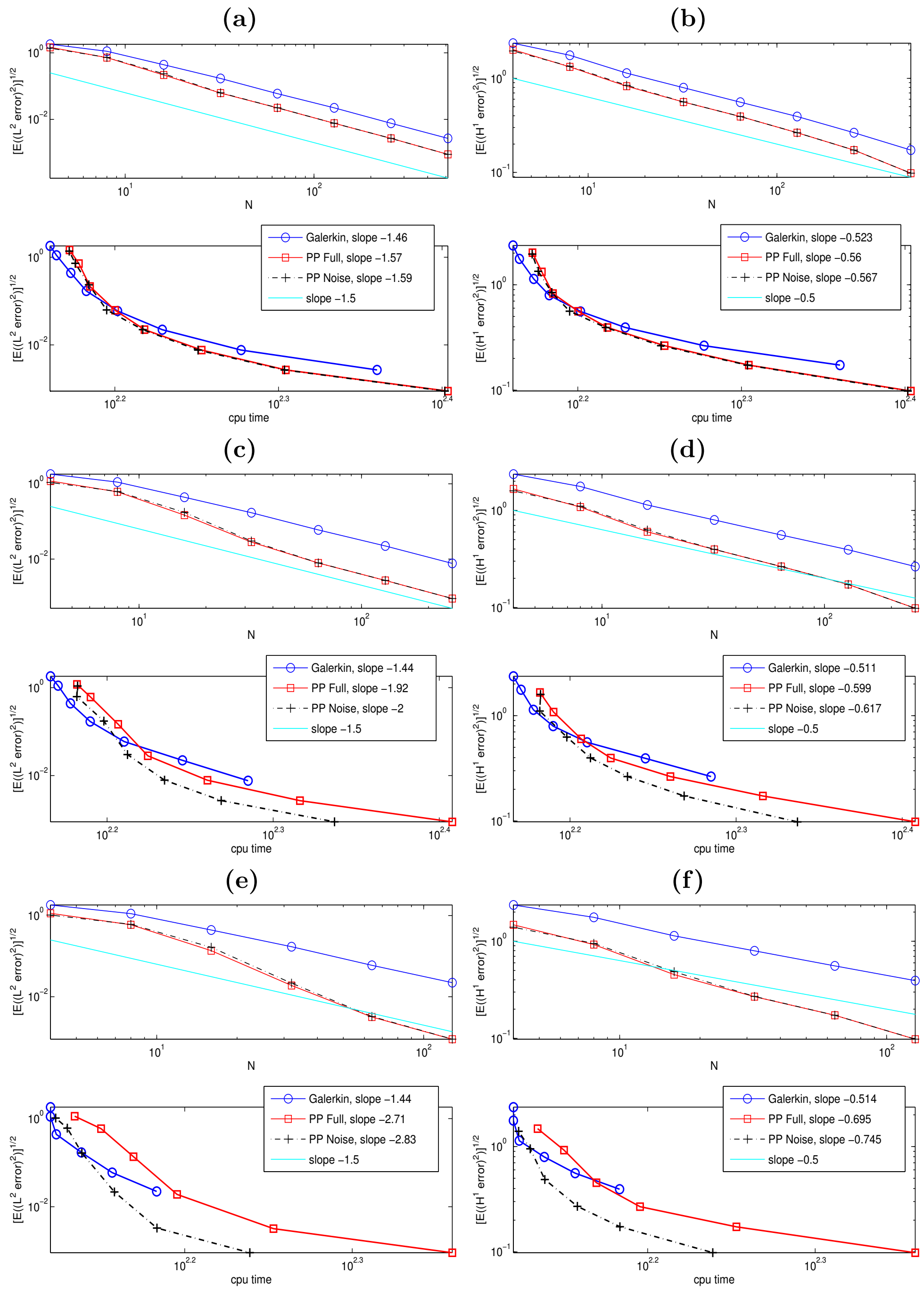


FIG. 4.4. *For $H^{1/2}$ noise, we examine the number of post-processing terms $\hat{N}$. In (a) (b) we take $\hat{N} = 2N$, (c) (d) $\hat{N} = 4N$, (e) (f) $\hat{N} = 8N$ with $L_2$ error (left) and $H^1$ error (right). For each case, we show plots of error against $N$ (above) and error against cputime (below).*

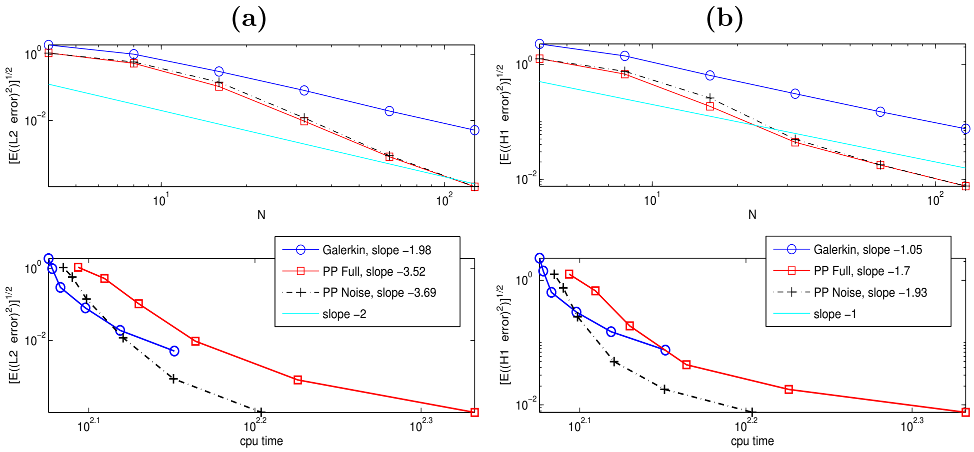


FIG. 4.5. *For $H^1$ noise with $\hat{N} = 8N$, we plot (a) $L^2$ error and (b) $H^1$ error. Top shows error against $N$ and bottom shows error against cputime.*

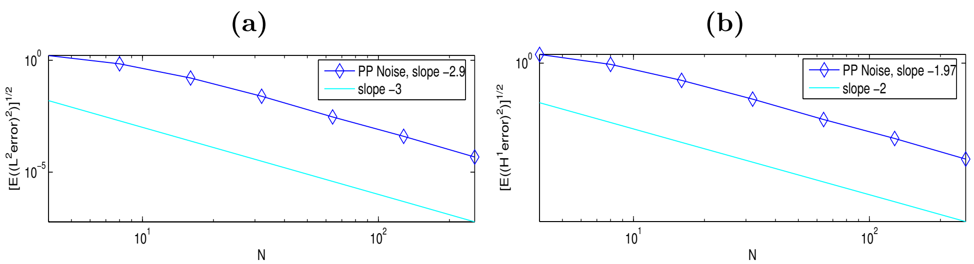


FIG. 4.6. *For $H^2$ noise, we see (a) faster than the predicted rate of convergence for the $L^2$ error with the optimal value $\hat{N} = N^{2/3}$ and (b) the $N^{-2}$ convergence rate is achieved for the $H^1$ error, with $\hat{N} = N^{2/3}$ rather than the theoretical rate of $N_w = N$.*

The $H^1$ error in (b) shows $N^{-2}$ convergence – although only $\hat{N} = N^{2/3}$ modes are used rather than the theoretical optimum value $N$. In this case, the accuracy is determined by approximation of the deterministic terms and we are unable to increase the number of modes to see the theoretical optimal number for $N_w$ bite.

**4.6. Post-processing implicit Euler-Maruyama.** Post-processing is effective for other time stepping algorithms. In Fig 4.7, we plot results of experiments with the implicit Euler-Maruyama scheme. We take $\hat{N} = 8N$ and plot (a) the $L^2$ error for white noise, (b) the $L^2$ error with $L^2$ noise, (c) $L^2$ error with $H^1$ noise, and (d) $H^1$ error with $H^1$ noise. Again *PP Noise* is the most efficient of the methods and there appears to be an improvement in the rate of convergence in addition to the constant. These trends are identical to those found in Theorem 3.2 and shown in Fig 4.2–Fig 4.6.

**5. Conclusions.** Theorem 3.2 shows that post-processing can improve the rate of convergence over a standard Galerkin method for stochastic PDEs. For non-smooth forcing, the best number of modes is greater than the standard Galerkin method. For smooth noise, as observed in [26], the optimal number of modes is smaller. With the smooth nonlinearity in (4.1), it is flexibility in the number of modes that approximate $W(t)$ that is key. This was confirmed in numerics. We found post-processing on the

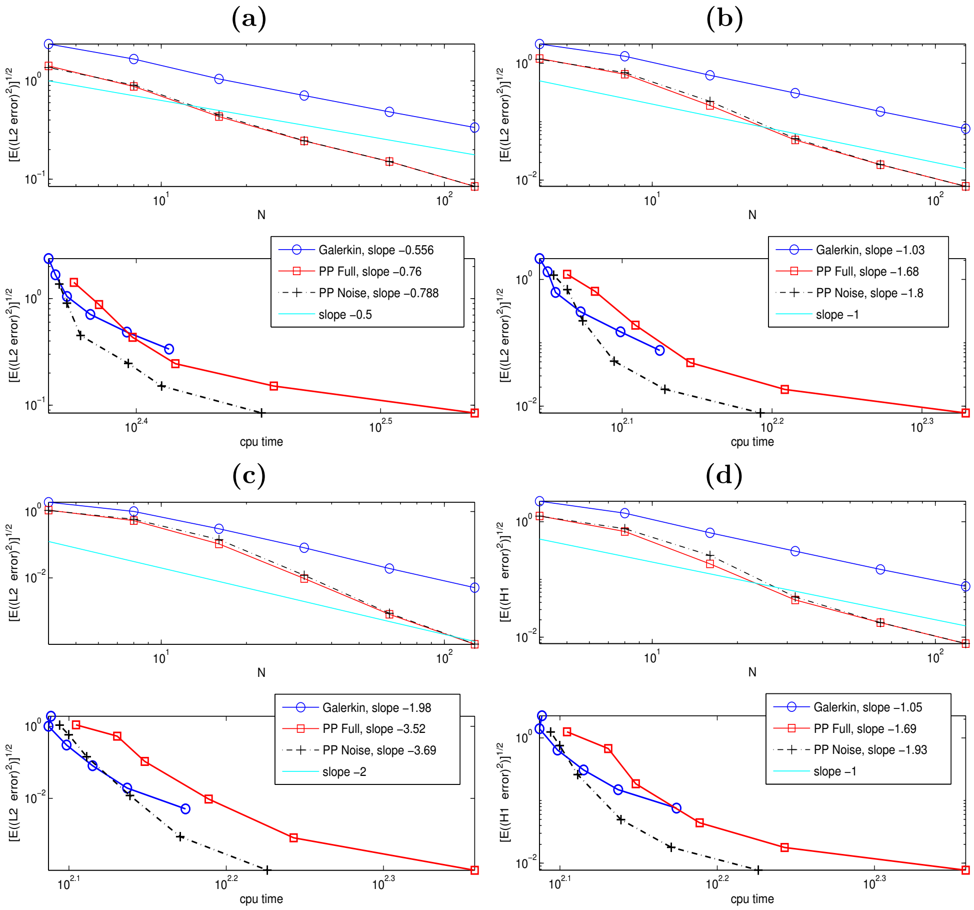


FIG. 4.7. *Post-processing for the implicit Euler–Maruyama method. In (a) white noise and $L^2$ error, (b) $L^2$ noise and $L^2$ error, (c) $H^1$ noise and $L^2$ error, (d) again $H^1$ noise but with $H^1$ error.*

noise improves on the convergence and efficiency of the standard Galerkin approximation and that the contribution from the (smooth) nonlinearity in the post-processing is negligible. This improvement in efficiency over the standard Galerkin method holds true for all spatial regularities of the noise that we tested.

It is often computationally prohibitive to use the the number of modes suggested by the theorem. From a practical point of view, improvements were noted with $N_w = 2N$ even when the theoretical optimum number of nodes is $N_w = N^2$. For non–smooth noise, we found numerically that taking $N_w = 8N$ gave a good compromise between the extra effort involved and accuracy. Indeed it seems we get a rate of convergence not predicted by the theory.

For smooth noise, our numerics suggest a convergence rate faster than that predicted by the theorem. From [20], it is known that for exponentially smooth noise a faster than polynomial convergence is available for smooth problems. Such techniques have not been used in the present paper and the results we give are optimal for the $H^2$ initial data and time stepping method studied.

Finally, although our analysis is for the scheme (3.2), this approach works equally well for other time-stepping methods, such as the implicit Euler–Maruyama time stepping scheme. Our presentation is for a Galerkin based approximation, however post-processing can easily be extended to other spatial discretizations using, for example, two grids.

**Acknowledgements.** We are grateful to Yubin Yan for helpful discussions on this work. We would also like to thank Edriss Titi and Jacques Rougemont for their comments early on in this work.

**6. Proof of main Theorem.** We prove Theorem 3.2 by estimating

$$\mathbf{E}\Big[\sup_{0\le t_j\le t'}\|u(t_j)-u^N(t_j)-q^N(t_j)\|_m^2\Big]$$

for $0\le t'\le T$ and applying Gronwall's Lemma. To estimate terms, we use a generic constant $K$ which varies between instances but is independent of $\Delta t$ and $N$ (it may depend on (1.1) and the length of time integration $T$ and constant $\nu$). Consider the difference of the variation of constants formulae (3.1),(3.5), and (3.7). Split into Fourier modes with $|n|\le N_p$ and $|n|>N_p$ and by nonlinear and noise terms.

**Nonlinear Terms: modes $|n|\le N_p$.**

$$\begin{aligned}
&\mathbf{E}\sup_{0\le t_j\le t'}\sum_{|n|\le N_p}(1+n^2)^m\Big|\sum_{k=0}^{j-1}\\
&\qquad\times\int_{t_k}^{t_{k+1}}e^{-(t_j-t_k)n^2}(e^{(s-t_k)n^2}F_n(u(s))-F_n(u^N(t_k)))\,ds\Big|^2\\
=&\sum_{|n|\le N_p}\mathbf{E}\Big[\sup_{0\le t_j\le t'}\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}e^{-(t_j-t_k)n^2}(1+n^2)^{m/2}\Big(\\
&\times\Big(F_n(u(s))-F_n(u(t_k))\Big)+\Big(F_n(u(t_k))-F_n(u^N(t_k)+q^N(t_k))\Big)\\
&+\Big(F_n(u^N(t_k)+q^N(t_k))-F_n(u^N(t_k))\Big)+\Big((e^{(s-t_k)n^2}-1)F_n(u(s))\Big)\Big)\,ds\Big]^2\\
\le&K(\mathbf{NL}_1+\cdots+\mathbf{NL}_4),
\end{aligned}$$

where the four terms $\mathbf{NL}_i$ are analysed below.

**The first term** Fix $t_j$ and consider $k\le j-1$. Define

$$L_{k,n}=\int_{t_k}^{t_{k+1}}e^{-(t_j-t_k)n^2}(1+n^2)^{m/2}\Big(F_n(u(s))-F_n(u(t_k))\Big)\,ds,$$

and let

$$\mathbf{NL'}_1=\sum_{|n|\le N_p}\mathbf{E}\Big[\sum_{k=0}^{j-1}L_{k,n}\Big]^2. \qquad (6.1)$$

Write $U_k=u(t_k)$ and $u(s)=u(t_k)+\delta_s$ for $t_k\le s<t_{k+1}$, then

$$F_n(u(s))-F_n(u(t_k))=dF_n(U_k)\delta_s+\int_0^1\int_0^\eta d^2F_n(U_k+\xi\delta_s)(\delta_s,\delta_s)\,d\xi\,d\eta.$$

In the following argument we neglect the remainder term, which can be dealt with easily under (3.13). Denote by $\mathcal{F}_t$ the filtration for the Wiener process $W(t)$. For $k > i$, under (3.12), the cross terms in (6.1)

$$\begin{aligned}\sum_{|n|\le N_p} \mathbf{E} L_{k,n} L_{i,n} =& \sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2} \mathbf{E}\Big[dF_n(U_k)\delta_s|\mathcal{F}_{t_k}\Big]\,ds \\ &\times \int_{t_i}^{t_{i+1}} e^{-(t_j-t_i)n^2} dF_n(U_i)\delta_s\,ds\Big] + \text{higher order terms} \\ \le& K\Delta t^4,\end{aligned}$$

because $dF_n(U_i)\delta_s$ is $\mathcal{F}_{t_k}$ measurable and $\Big\|\mathbf{E}\Big[dF(U_k)\delta_s|\mathcal{F}_{t_k}\Big]\Big\|_m \le K\Delta t$. As

$$\Big[\int_{t_k}^{t_{k+1}} \phi(s)\,ds\Big]^2 \le (t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} \phi(s)^2\,ds, \text{ for } \phi \in L^2(0,T),$$

$$\sum_{|n|\le N_p} \mathbf{E} L_{k,n}^2 \le \Delta t \sum_{|n|\le N_p} \int_{t_k}^{t_{k+1}} \mathbf{E}\Big[e^{-(t_j-t_k)n^2}(1+n^2)^{m/2} dF_n(U_k)\delta_s\Big]^2\,ds + \text{h.o.t.}$$

Here

$$\begin{aligned}\sum_{|n|\le N_p} \int_{t_k}^{t_{k+1}} \mathbf{E}\Big[e^{-(t_j-t_k)n^2}(1+n^2)^{m/2} dF_n(U_k)\delta_s\Big]^2\,ds \\ \le \int_{t_k}^{t_{k+1}} \mathbf{E}\Big[\|dF_n(U_k)\|_m^2 \cdot \|\delta_s\|_m^2\Big]\,ds.\end{aligned}$$

Because $\mathbf{E}\|u^N(t)-u^N(s)\|_m^2 \le K|t-s|\|u_0\|_m^2$ and (3.12) holds, we conclude that

$$\sum_{|n|\le N_p} \int_{t_k}^{t_{k+1}} \mathbf{E}\Big[e^{-(t_j-t_k)n^2}(1+n^2)^{m/2} dF_n(U_k)\delta_s\Big]^2\,ds \le K\Delta t^2.$$

Thus, we may estimate

$$\mathbf{NL'}_1 \le \sup_{0\le t_j\le t'} \sum_{|n|\le N_p} \Big\{\sum_{k=0}^{j-1} \mathbf{E}\Big[L_{k,n}\Big]^2 + \sum_{k,i=0,\,k\ne i}^{j-1} \mathbf{E} L_{k,n} L_{i,n}\Big\} \le K\Delta t^2.$$

Apply the Doob martingale inequality, to get

$$\mathbf{E}\Big[\sup_{0\le t_j\le t'} NL_1'\Big] \le 4K\Delta t^2.$$

**The second term**

$$\begin{aligned}\mathbf{NL}_2 =& \sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_j\le t'} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2} \\ &\times \Big(|F_n(u(t_k)) - F_n(u^N(t_k)+q^N(t_k))|\Big)\,ds\Big]^2 \\ \le& \int_0^{t'} \sum_{|n|\le N_p} \mathbf{E}\Big[\sup_{0\le t_k\le t} (1+n^2)^m |F_n(u(t_k)) - F_n(u^N(t_k)+q^N(t_k))|^2\Big]\,dt.\end{aligned}$$

Using (3.10),

$$\mathbf{NL}_2 \le K \int_0^{t'} \mathbf{E}\Big[\sup_{0\le t_k\le t} \|u(t_k) - u^N(t_k) - q^N(t_k)\|_m^2\Big]\, dt.$$

**The third nonlinear term**

$$\begin{aligned}
\mathbf{NL}_3 = &\sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_j\le t'} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2} \\
&\times \Big(|F_n(u^N(t_k)+q^N(t_k)) - F_n(u^N(t_k))|\Big)\, ds\Big]^2 \\
\le &\sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_j\le t'} |F_n(u^N(t_j)+q^N(t_j)) - F_n(u^N(t_j))| \\
&\times \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2}\, ds\Big]^2 \\
\le &\sum_{0<|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_k\le t'} |F_n(u^N(t_k)+q^N(t_k)) - F_n(u^N(t_k))|\frac{1}{n^2}\Big]^2 \\
&+ \mathbf{E}\Big[\sup_{0\le t_k\le t'} |F_0(u^N(t_k)+q^N(t_k)) - F_0(u^N(t_k))|\Big].
\end{aligned}$$

Choose $m \le 2$, then using (3.8),

$$\begin{aligned}
\mathbf{NL}_3 \le &\sum_{|n|\le N_p} \mathbf{E}\Big[\sup_{0\le t_k\le t'} (1+n^2)^m |F_n(u^N(t_k)+q^N(t_k)) - F_n(u^N(t_k))|^2\Big] \\
\le &K \int_0^{t'} \mathbf{E}\Big[\sup_{0<t_k\le t} \|q^N(t_k)\|^2\Big]\, dt.
\end{aligned}$$

Finally, from Lemma A.2,

$$\mathbf{NL}_3 \le K(N^{2(-2)} + \mathbf{1}_{N\le N_w} N^{2(-1-\gamma)} + N^{2(-2-r)}).$$

**The fourth nonlinear term**

$$\begin{aligned}
&\mathbf{NL}_4 = \\
&= \sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_j\le t'} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2} \Big(|(e^{(s-t_k)n^2}-1)F_n(u(s))|\Big)\, ds\Big]^2 \\
&\le \sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le t_j\le t'} |F_n(u(t_j))|^2 \sum_{k=0}^{\lfloor t/\Delta t\rfloor -1} e^{-(t_j-t_k)n^2} K\Delta t^2 n^2\Big]^2.
\end{aligned}$$

Note that for $0 \le \Delta t n^2 \le \nu$

$$\int_{t_k}^{t_{k+1}} |e^{(s-t_k)n^2} - 1|\, ds \le \Big(\frac{e^{\Delta t n^2}-1}{n^2} - \Delta t\Big) \le n^{-2}(K\Delta t^2 n^4 e^{n^2\Delta t}) \le K\Delta t^2 n^2 e^{\nu}$$

and

$$\sum_{k=1}^{\infty} e^{-kn^2\Delta t} \le \frac{1}{1-e^{-n^2\Delta t}} \le \frac{K}{n^2\Delta t}.$$

Thus, using (3.11),

$$\begin{aligned}\mathbf{NL}_4 \le & K \sum_{|n|\le N_p} (1+n^2)^m \mathbf{E}\Big[\sup_{0\le s\le t'} |F_n(u(s))|^2 \Delta t\Big]^2 \\ \le & K\Delta t^2 \Big(1+\mathbf{E}\Big[\sup_{0\le s\le t'} \|u(s)\|_m\Big]^2\Big).\end{aligned}$$

By (3.10) and Lemma A.1,

$$\mathbf{NL}_4 \le K\,\Delta t^2.$$

**Nonlinear terms: modes $|n| > N_p$.** Consider now the tail of the expansion of $u(t)$; i.e., the modes not included in either $u^N$ or $q^N$. If $r > m$,

$$\begin{aligned}\mathbf{TAIL} = & \mathbf{E}\Big[\sup_{0\le t_j\le t'} \sum_{|n|>N_p} (1+n^2)^m \Big| \int_0^{t_j} e^{-(t_j-s)n^2} F_n(u(s))\, ds\Big|^2\Big] \\ \le & K\Big(\int_0^{t'} (1+N_p^2)^{-(r-m)/2} e^{-(t_j-s)N_p^2}\, ds\Big)^2 \mathbf{E}\Big[\sup_{0\le s\le t'} \|F(u(s))\|_r^2\Big].\end{aligned}$$

By (3.9) and Lemma A.1,

$$\mathbf{TAIL} \le K\, N_p^{2(m-2-r)}.$$

**Noise with modes $|n| \le N_w$.**

$$\begin{aligned}&\mathbf{NOISE}_1 = \\ =&\mathbf{E}\Big[\sup_{0<t_j\le t'} \sum_{|n|\le N_w} (1+n^2)^m \\ &\times \Big|\sum_{k=0}^{j-1}\Big(\int_{t_k}^{t_{k+1}} e^{-(t_j-s)n^2}\lambda_n^{1/2} d\beta_n(s) - e^{-(t-t_k)n^2}\lambda_n^{1/2}\Delta B_{k,n}\Big)\Big|^2\Big] \\ \le & \sum_{|n|\le N_w} (1+n^2)^m |\lambda_n| \mathbf{E}\Big[\sup_{0<t_j\le t'} \int_0^{t_j} (e^{-(t_j-s)n^2} - e^{-(t_j-\lfloor s/\Delta t\rfloor \Delta t)n^2}) d\beta_n(s)\Big]^2.\end{aligned}$$

By Doob's martingale inequality

$$\begin{aligned}\mathbf{NOISE}_1 \le & 4\sum_{|n|\le N_w} (1+n^2)^m |\lambda_n| \int_0^{t'} (e^{-(t_j-s)n^2} - e^{-(t_j-\lfloor s/\Delta t\rfloor \Delta t)n^2})^2\, ds \\ = & 4\sum_{|n|\le N_w} (1+n^2)^m |\lambda_n| \int_0^{t'} e^{-2(t_j-s)n^2} (1-e^{-(s-\lfloor s/\Delta t\rfloor \Delta t)n^2})^2\, ds.\end{aligned}$$

Note that $1-e^{-tn^2} \le tn^2$ for $0\le t\le \Delta t$ and

$$\int_0^{t'} e^{-2(t_j-s)n^2} (1-e^{-(s-\lfloor s/\Delta t\rfloor \Delta t)n^2})^2\, ds \le (\Delta t n^2)^2 \int_0^{t'} e^{-2(t_j-s)n^2}\, ds \le K\Delta t^2 n^2.$$

Hence

$$\begin{aligned}\mathbf{NOISE}_1 \le & 4 \sum_{|n|\le N_w} (1+n^2)^m |\lambda_n| \Delta t^2 n^2 \\ \le & K \Delta t^2 (1+N_w^2)^{(1+m-\gamma)} \sum_{|n|\le N_w} (1+n^2)^\gamma |\lambda_n| \\ \le & K \Delta t^2 (1+N_w^2)^{(1+m-\gamma)},\end{aligned}$$

under (3.14).

**Noise with modes $|n| > N_w$.**

$$\begin{aligned}\mathbf{NOISE}_2 = & \mathbf{E}\Big[\sup_{0\le t_j\le t'} \sum_{|n|>N_w} (1+n^2)^m \Big| \int_0^{t_j} e^{-(t_j-s)n^2} \lambda_n^{1/2} d\beta_n(s)\Big|^2\Big] \\ \le & 4(1+N_w^2)^{m-\gamma} \frac{1-e^{-t' N_w^2}}{N_w^2} \sum_{|n|\ge N_w} \lambda_n (1+n^2)^\gamma \le K\, N_w^{2(m-1-\gamma)},\end{aligned}$$

using (3.14).

**Conclusion.** We have achieved the following inequality

$$\begin{aligned}&\mathbf{E}\Big[\sup_{0\le t_j\le t'} \|u(t_j)-u^N(t_j)-q^N(t_j)\|_m^2\Big], \\ &\le K\Big(\Delta t^2 + (N^{2(-2)} + \mathbf{1}_{N\le N_w} N^{2(-1-\gamma)} + N^{2(-2-r)}) + N_p^{2(-2-r+m)} + \Delta t^2 N_w^{2(-\gamma+1+m)} \\ &\quad + N_w^{2(-1-\gamma+m)} + \int_0^T \mathbf{E}\Big[\sup_{0<t_k\le t} \|u(t_k)-u^N(t_k)-q^N(t_k)\|^2\Big]\, dt\Big). \quad \blacksquare\end{aligned}$$

Note $N^{2(-2-r)} \le N^{2(-2)}$ and then Gronwall's Lemma provides

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le t\le t'} \|u(t)-u^N(t)-q^N(t)\|_m^2\Big] \le & K\Big(\Delta t^2 + N^{2(-2)} + \mathbf{1}_{N\le N_w} N^{2(-1-\gamma)} + \\ & N_p^{2(-2-r+m)} + \Delta t^2 N_w^{2(-\gamma+1+m)} + N_w^{2(-\gamma-1+m)}\Big). \blacksquare\end{aligned}$$

This completes the proof of Theorem 3.2.

**Appendix A. Lemmas.** We collect two elementary lemmas used in the proof of the main theorem.

LEMMA A.1. *For $r \le \gamma+1$,*

$$\mathbf{E} \sup_{0\le t\le T} \|u(t)\|_r^2 \le K(1+\|u_0\|_r^2).$$

*Proof.* Examining the nonlinear term in (3.1) under (3.9):

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le t\le t'} \sum_n \Big|(1+n^2)^{r/2} \int_0^t e^{-(t-s)n^2} F_n\big(u(s)\big)\, ds\Big|^2\Big] \\ \le K \int_0^{t'} \Big(1+\mathbf{E}\Big[\sup_{0\le s\le t} \|u(s)\|_r^2\Big]\Big)\, dt\end{aligned}$$

and the noise term (modes with $n \neq 0$)

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le t\le t'}\sum_{n\neq 0}(1+n^2)^{r/2}\Big|\int_0^t e^{-(t-s)n^2}\lambda_n^{1/2}d\beta(s)\Big|^2\Big]\\
&\quad\le 4\mathbf{E}\Big[\sum_{n\neq 0}(1+n^2)^{(r-\gamma)}\Big|\int_0^{t'} e^{-2(t-s)n^2}(1+n^2)^{\gamma}\lambda_n ds\Big|\Big]\\
&\quad\le \sum_{n\neq 0}\frac{(1+n^2)^{(r-\gamma)}}{n^2}(1+n^2)^{\gamma}\lambda_n
\end{aligned}$$

using (3.14). This is finite if $r-\gamma\le 1$, so that the Gronwall Lemma completes the proof. □

Lemma A.2. *Under the assumptions of Lemma A.1,*

$$\mathbf{E}\sup_{0\le t\le T}\|q^N(t)\|^2 \le K(N^{2(-2)} + \mathbf{1}_{N\le N_w}N^{2(-1-\gamma)} + N^{2(-2-r)}).$$

*Proof.* We seek upper estimates on

$$\mathbf{E}\Big[\sup_{0\le t\le T}\|q^N(t)\|^2\Big].$$

To do this, estimate the influence of the initial data

$$\begin{aligned}
\sum_{N<|n|\le \max N_p,N_w}\mathbf{E}\Big[\sup_{0\le t_k\le T}|e^{-t_k n^2}u_n(0)|^2\Big] &= \sum_{N<|n|\le \max N_p,N_w} u_{0,n}^2\\
&\le KN^{-4}\sum_{N<|n|\le \max N_p,N_w}(1+n^2)^2 u_{0,n}^2.
\end{aligned}$$

If $u_0\in H^2(0,2\pi)$, this term is bounded by $KN^{2(-2)}$.

Now the nonlinear terms,

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le t_j\le T}\sum_{N<|n|\le N_p}\Big|\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}} e^{-(t_j-t_k)n^2}F_n(u^N(t_k))\,ds\Big|^2\Big]\\
&\le \mathbf{E}\Big[\sup_{0\le t_j\le T}\sum_{N<|n|\le N_p}(1+n^2)^r|F_n(u^N(t_k))|^2\Big|\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(1+n^2)^{-r/2}e^{-(t_j-t_k)n^2}\,ds\Big|^2\Big]\\
&\le \mathbf{E}\Big[\sup_{0\le t\le T}\|u^N(t)\|_r^2\Big]\cdot\Big|\sum_{k=0}^{\lfloor t/\Delta t\rfloor -1}\int_{t_k}^{t_{k+1}}(1+n^2)^{-r/2}e^{-(t_j-t_k)n^2}\,ds\Big|^2\\
&\le \mathbf{E}\Big[\sup_{0\le t\le T}\|u^N(t)\|_r^2\Big]\frac{(1+N^2)^{-r}}{N^4}.
\end{aligned}$$

This term is bounded by $K\,N^{2(-r-2)}$ by applying Lemma A.1. The noise term:

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le t_j\le T}\sum_{N<|n|\le N_w}\Big|\sum_{k=0}^{j-1}\Big(\int_{t_k}^{t_{k+1}} e^{-(t-t_k)n^2}\lambda_n^{1/2}\Delta B_{k,n}\Big)\Big|^2\Big]\\
=&4\sum_{N<|n|\le N_w}(1+n^2)^{\gamma}\lambda_n\int_0^T(1+n^2)^{-\gamma}e^{-2(t_j-t_k)n^2}\,ds\\
\le&4\;\mathbf{1}_{N\le N_w}\;N^{2(-1-\gamma)}\sum_{N<|n|\le N_w}(1+n^2)^{\gamma}\lambda_n.
\end{aligned}$$

This completes the proof. □

## REFERENCES


[1] C. Canuto, M. Y. Hussaini, Q. Quateroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Computational Physics, Springer, 1988. ISBN 3-540-17371-4, 0-387-17371-4.

[2] G. Da Prato and A. Debussche, *Construction of stochastic inertial manifolds using backward integration*, Stochastics & Stochastics Rep., 59 (1996), pp. 305–324.

[3] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, vol. 44 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.

[4] A. M. Davie and J. G. Gaines, *Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations*, Math. Comp., 70 (2001), pp. 121–134.

[5] A. Doelman and E. Titi, *Regularity of solutions and the convergence of the Galerkin method in the complex Ginzburg–Landau equation*, Numer. Func. Opt. Anal., 14 (1993), pp. 299–321.

[6] C. Foias, O. Manley, and R. Temam, *Modelling of the interaction of small and large eddies in two-dimensional turbulent flows*, RAIRO Modél. Math. Anal. Numér., 22 (1988), pp. 93–118.

[7] B. García-Archilla, J. Novo, and E. S. Titi, *Postprocessing the Galerkin method: a novel approach to approximate inertial manifolds*, SIAM J. Numer. Anal., 35 (1998), pp. 941–972 (electronic).

[8] ———, *An approximate inertial manifolds approach to postprocessing the Galerkin method for the Navier-Stokes equations*, Math. Comp., 68 (1999), pp. 893–911.

[9] B. García-Archilla, J. Novo, and E. S. Titi, *Postprocessing Fourier spectral methods:the case of smooth solutions*, Applied Numerical Mathematics, (2002).

[10] W. Grecksch and P. Kloeden, *Time-discretized Galerkin approximations of parabolic stochastic pdes*, Bull. Australian Math. Soc., 54 (1996), pp. 79–85.

[11] I. Gyöngy, *Lattice approximations for stochastic quasi–linear parabolic partial differential equations driven by space–time white noise II*, Potential Anal., 11 (1999), pp. 1–37.

[12] I. Gyöngy and D. Nualart, *Implicit scheme for quasi–linear parabolic partial differential equations perturbed by space–time white noise*, Stochastic Processes and their Applications, 58 (1995), pp. 57–72.

[13] E. Hausenblas, *Approximation for semilinear stochastic evolution equations*, Potential Analysis, (2002).

[14] E. Hausenblas, *Numerical analysis for semilinear stochastic evolution equations in Banach spaces*, J. Comput. Appl. Math., 147 (2002), pp. 485–516.

[15] D. J. Higham, X. Mao, and A. M. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*, SIAM J. Numer. Anal., 40 (2002), pp. 1041–1063 (electronic).

[16] M. Hochbruck and A. Ostermann, *Explicit exponential Runge-Kutta methods for semilinear parabolic problems*, SIAM J. Numer. Anal, 43 (2005), pp. 1069–1090.

[17] J. C. Jiminez, I. Shoji, and T. Ozaki, *Simulation of stochastic differential equations through the local linearization method. a comparitive study.*, J. of Statistical Physics., 94 (1999), pp. 587–601.

[18] P. Kloeden and S. Shott, *Linear-implicit strong schemes for Ito-Galerkin approximations of stochastic pdes*, J. Appl. Math. Stoch. Anal, (2001).

[19] D. J. Lawson, *Generalized Runge-Kutta processes for stable systems with large Lipschitz constants*, SIAM J. Numer. Anal., (1967).

[20] G. J. Lord and J. Rougemont, *A numerical scheme for stochastic PDEs with Gevrey regularity*, IMA J. Numer. Anal., 24 (2004), pp. 587–604.

[21] L. G. Margolin, E. S. Titi, and S. Wynne, *The postprocessing Galerkin and nonlinear Galerkin methods—a truncation analysis point of view*, SIAM J. Numer. Anal., 41 (2003), pp. 695–714 (electronic).

[22] B. Minchev and W. Wright, *A review of exponential integrators for first order semi-linear problems*, tech. rep., Norwegian University of Science and Technology, 2005.

[23] C. M. Mora, *Weak exponential schemes for stochastic differential equations with additive noise*, IMA J. Numer. Anal., 25 (2005), pp. 486–506. doi:10.1093/imanum/dri001.

[24] T. Mueller-Gronbach and K. Ritter, *Non-uniform time discretization and lower bounds for approximation of stochastic heat equations*, To appear, Foundations of Computational Mathematics, (2005).

[25] T. Shardlow, *Numerical methods for stochastic parabolic PDEs*, Numer. Funct. Anal. Optim., 20 (1999), pp. 121–145.

[26] ———, *Numerical simulation of stochastic PDEs for excitable media*, J. Comput. Appl. Math., 175 (2005), pp. 429–446.

[27] I. Shoji, *Approximation of continuous time stochastic processes by a local linearization method*, Math. Comp., 67 (1998), pp. 287–298.

[28] H. Yinnian and R. M. M. Mattheij, *Stability and convergence for the reform postprocessing Galerkin method*, Nonlinear Anal. Real World Appl., 1 (2000), pp. 517–533.

[29] Y. Yubin, *Semidiscrete Galerkin approximation for a linear stochastic parabolic partial differential equation driven by an additive noise*, BIT Numerical Mathematics, 44 (2004), pp. 829–847. DOI:10.1007/s10543-004-3755-5.