A mountain pass solution in cylinder buckling

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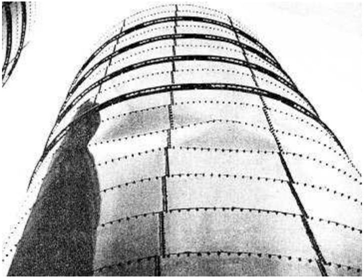

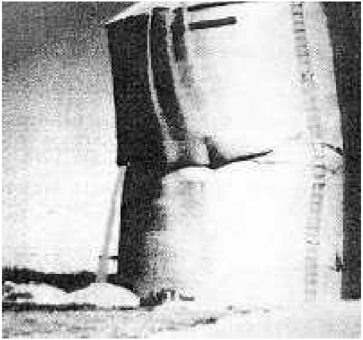

From: Rhodes & Walker '80 Thin Walled Structures

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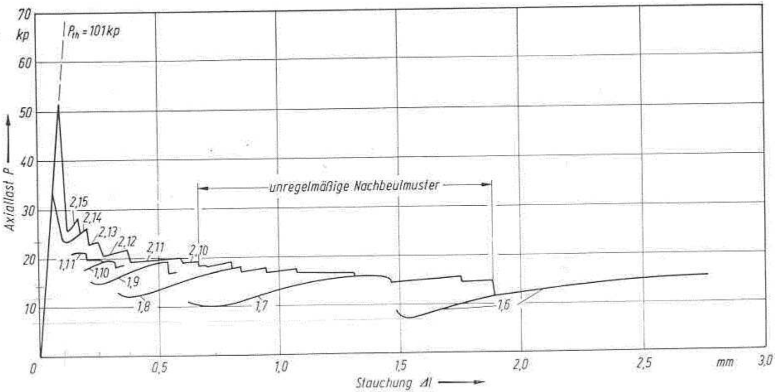
1) Experimental Evidence
2) Model
3) Mountain pass
4) Gradient flow

Part I : Experimental results

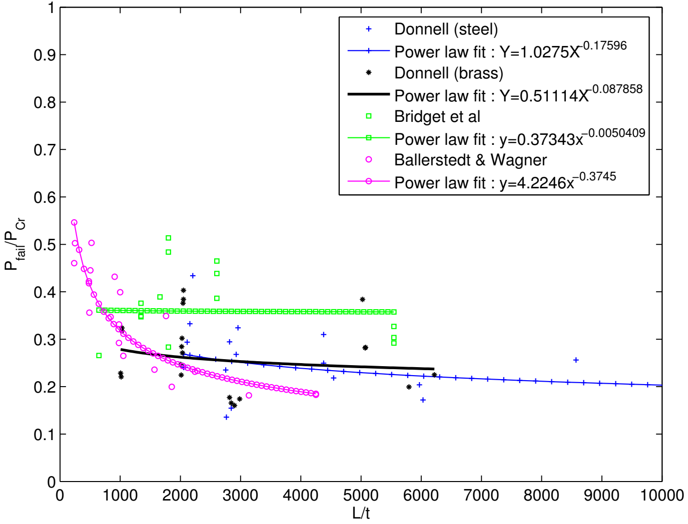
Typical end shortening *vs* load plot:



(Esslinger)

- ▷ Post-buckle minimum load
- ▷ Post-buckle plateau in load

Collection of 4 experimental results:



Linear prediction $\lambda_{fail}/\lambda_{cr} = 1$

Question :

- Can we understand the buckling load mathematically **?**
  (Can we bound the load at which cylinder buckles)

Part II : A Model

von Kármán-Donnell equations:

$$\begin{aligned} \kappa^2\Delta^2 w + \lambda w_{xx} - \rho\phi_{xx} - 2G(w,\phi) &= 0 \\ \Delta^2\phi + \rho w_{xx} + G(w,w) &= 0. \end{aligned}$$

where

$$G(u,v) = \frac{1}{2}u_{xx}v_{yy} + \frac{1}{2}u_{yy}v_{xx} - u_{xy}v_{xy}$$

$$\kappa^2 = \frac{t^2}{12(1-\nu^2)}, \qquad \lambda = \frac{P}{2\pi REt}, \qquad \rho = \frac{1}{R}$$

$(x,y) \in \Omega = [-L,L] \times [0, 2\pi R).$

Assumptions:
- ▷ Thin, isotropic shell
- ▷ Elastic buckle and curvature not too large
- ▷ No pre-buckle
- ▷ Normals stay normal, plane stress and small angle approximation for strain tensor.

von Kármán-Donnell equations:

$$\kappa^2\Delta^2 w + \lambda w_{xx} - \rho\phi_{xx} - 2G(w,\phi) = 0$$
$$\Delta^2\phi + \rho w_{xx} + G(w,w) = 0.$$

Euler–Lagrange equation with

▶Stored energy:

$$E(w) = \frac{Et}{2}\int_\Omega \left[\kappa^2\Delta w^2 + \Delta\phi^2\right]\,dxdy.$$

▶Constraint : average axial end-shortening associated with w

$$S(w) = \frac{1}{4\pi R}\int_\Omega w_x^2\,dxdy.$$

- Solutions of vKD equations are stationary points of

  Total Potential : $\qquad F_\lambda(w) = E(w) - \lambda S(w).$

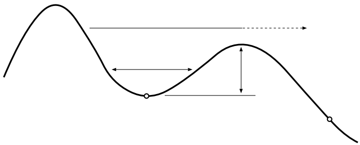
- Solutions also stationary points of $E(w)$ under constant $S(w)$.

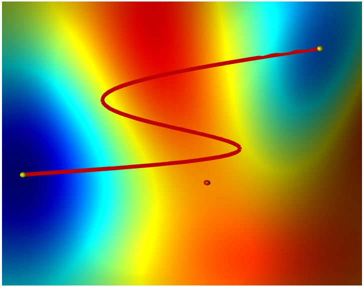
Part III : Mountain Pass Solution

Let $w_1 \neq w_2$ be two vectors in a space X. Define

$$\Gamma = \{\gamma \in C([0,1], X) \,|\, \gamma(0) = w_1, \gamma(1) = w_2\},$$
$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)).$$

If $c > \max\{F(w_1), F(w_2)\}$ and F satisfies $(\mathrm{PS})_c$, then c is a critical value of F.





Mountain Pass [Ambrosetti and Rabinowitz]

MP1. We show $w_1 = 0$ is a local minimizer: there are $\varrho, \alpha > 0$ such that $F_\lambda(w) \geq \alpha$ for all w with $\|w\|_X = \varrho$;

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Based on Yoshimura diamond pattern.

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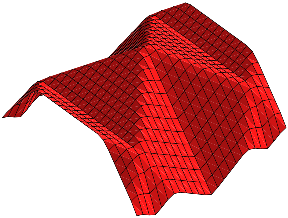

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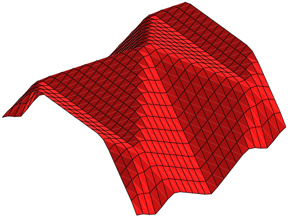


MP3. Given sequence of paths γ_n that approximates inf in defn.
Extract a (Palais-Smale) sequence of points $w_n \in \gamma_n$, each close to the maximum along γ_n.
Show this sequence converges in an appropriate manner.

In fact use the "Struwe monotonicity trick" ('90) to get a.e. λ

Mountain Pass Alg. : Y. S. Choi, P. J. McKenna (1993)

1) — Initial discrete path. Take P points:

$$z_j = w_1 + \frac{j}{P}(w_2 - w_1),\ j \in \{0, 1, \ldots, P\}$$

2) — Main loop:

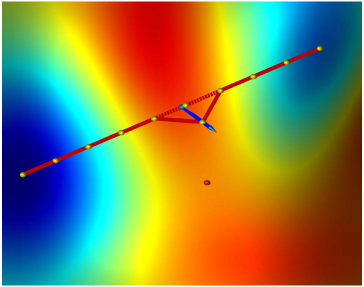
(a) find m: $\forall j\ F(z_m) \geq F(z_j)$, interpolate,

(b) compute $\nabla F(z_m)$,

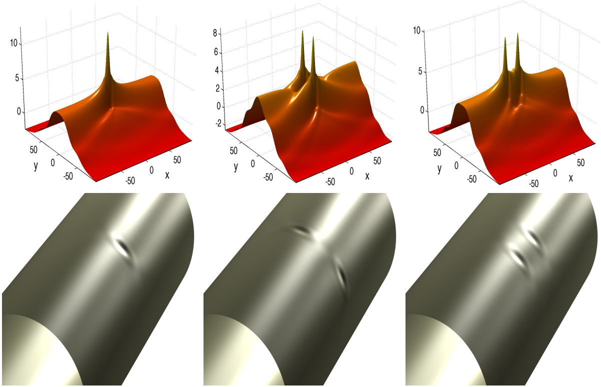
(c) deform the path: $\delta > 0$ (small) $z_m^{\mathrm{new}} = z_m - \delta \nabla F(z_m)$,

(d) STOP when F increases.

3) — Infinite loop: re-distribute points on path



Numerical Solutions



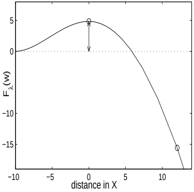
$\Omega = (-100, 100) \times (-100, 100)$, $\Delta x = \Delta y = 0.5$, $\lambda = 1.1$

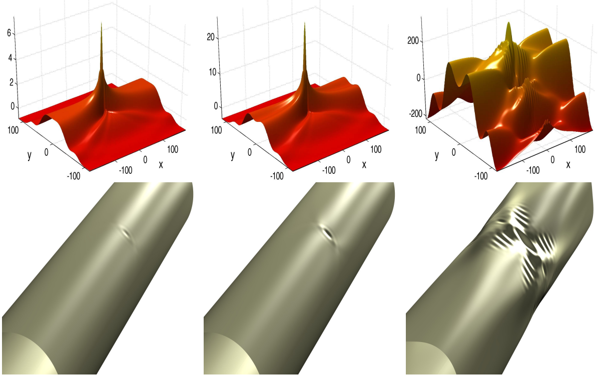
Found using different choices of w_2.

Min energy solution $\equiv$ Single Dimple

Steepest Descent:

(a) $F_\lambda \approx 5$ (b) $F_\lambda \approx -15$ (c) $F_\lambda \approx -5 \times 10^4$

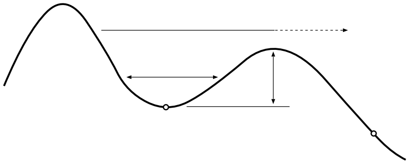




$$\Omega = (-200, 200) \times (-115, 115),\ \Delta x = \Delta y = 0.5,\ \lambda = 1.1$$

Interpretation of MP ?

- Have found the mountain pass energy for the perfect cylinder
  –
  how does this give a handle on an imperfect "real" cylinder ?
- Consider the minimum mountain–pass energy: $V = \inf_{w_2} F_\lambda$.

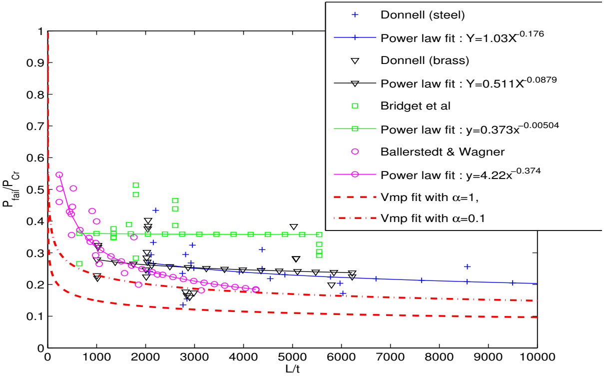


- In order to leave the basin of attraction of w_1, the surplus energy should exceed $V(\lambda)$

Imperfections and MP

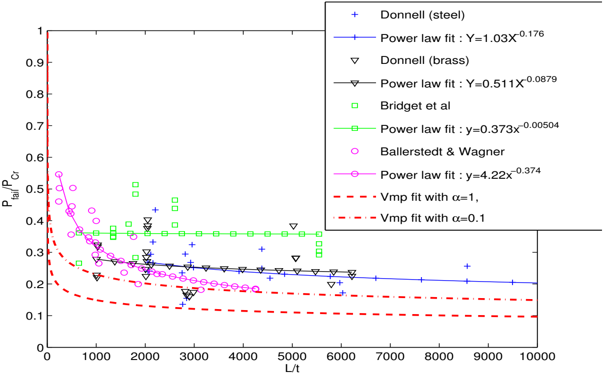
Suppose stored energy from being under load can be transfered to overcome the mountain pass.
Rescale MP energy $V(\lambda)$ by elastic strain energy stored in cylinder of length L :

$$\alpha = \frac{1}{2\pi\sqrt{3(1-\nu^2)}}\ \frac{t}{L}\ \frac{V(\lambda)}{\lambda^2}.$$

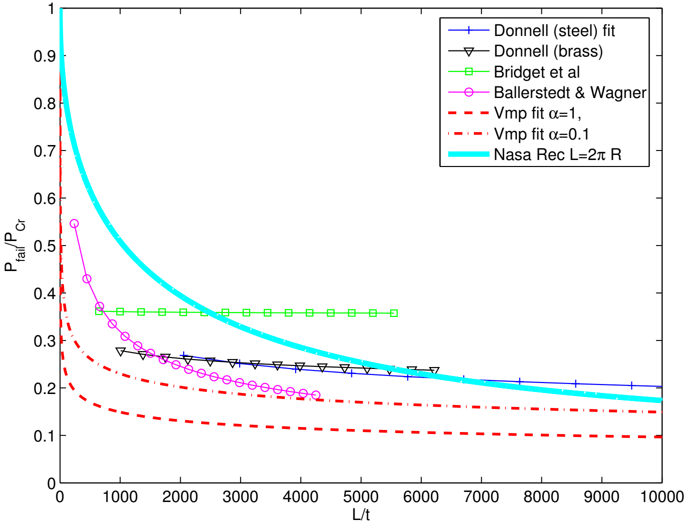


Imperfections and MP

1. The general trend of the constant-α curves is very similar to the trend of the experimental data;
2. The $\alpha = 1$ curve, which indicates the load at which the mountain-pass energy equals the stored energy in the prebuckled cylinder, appears to be a lower bound to the data.

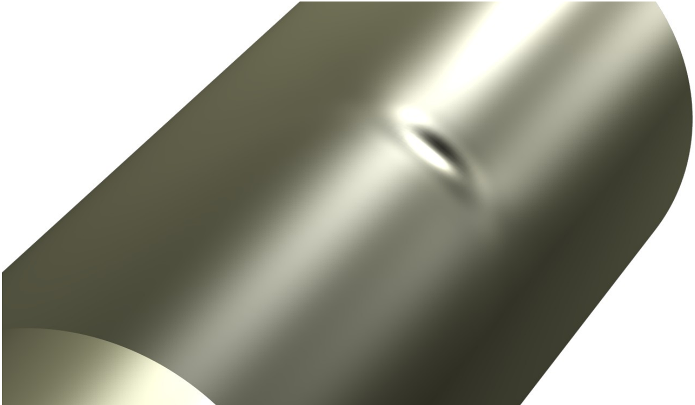


Nasa knockdown



Other single dimples ...

- Single dimples are seen in the high-speed camera images of Esslinger.
- Some "worst imperfections" by Deml and Wunderlich, Deml, Wunderlich and Albertin are single dimples.
- Single dimples are seen in finite element simulations (eg Schweizerhof)



Summary ...

- Mountain pass solutions
  - Elements for proof
  - Numerical algorithm
  - Solutions
- From MP solutions seems can get a lower bound on the buckling load.