A mountain pass solution in cylinder buckling

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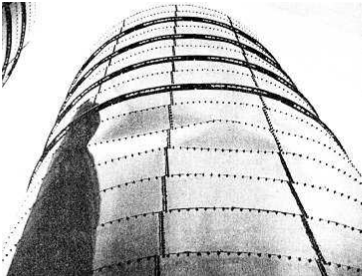
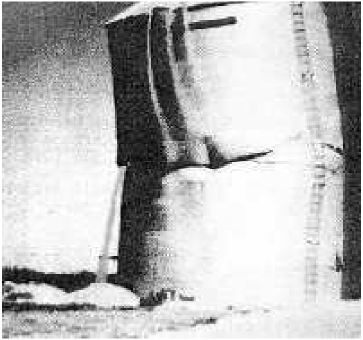
Mark A. Peletier, Technische Universiteit Eindhoven, The Netherlands





From: Rhodes & Walker '80 Thin walled structures

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Part 1 Experimental Evidence

- Axially localized solution
- Buckle/failure load

Part 2 Model

Part 3 Post-Buckle

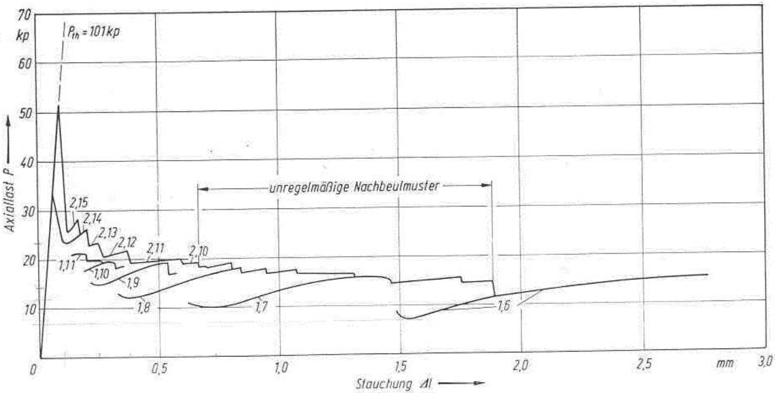
- Homoclinic solution (with G. Hunt (Bath) and A. Champneys (Bristol))

Part 4 Failure load for cylinder

- Mountain pass

Part I : Experimental results

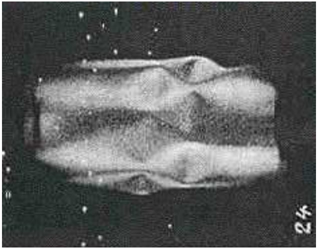
Typical end shortening *vs* load plot:



(Esslinger)

▷ Post-buckle minimum load
▷ Post-buckle plateau in load

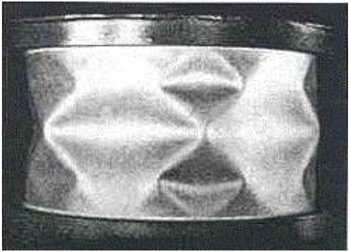
▷ Localized buckled solution



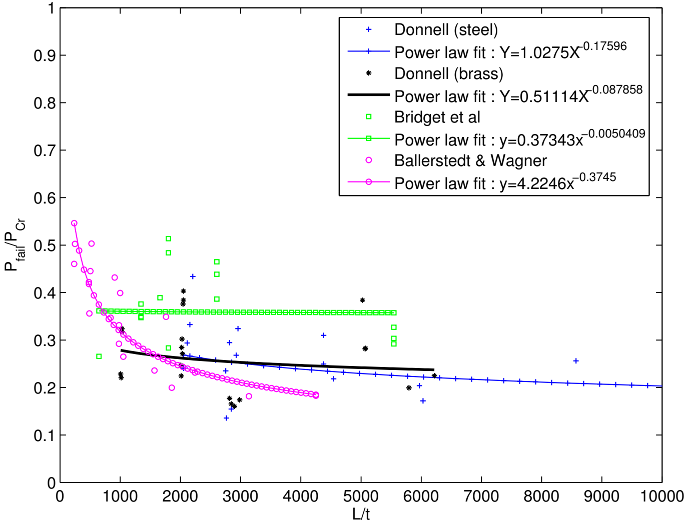
▷ Translation invariant

▷ Well defined circumferential wave number s

▷ 2 forms of solution : Symmetric & Cross Symmetric.



Collection of experimental results:



Linear prediction $\lambda_{fail}/\lambda_{cr} = 1$

Questions :

1. Can we compute post-buckle solution and loads ?

2. Can we predict the load at which cylinder buckles ?

Part II : A Model

von Kármán-Donnell equations:

$$\kappa^2 \Delta^2 w + \lambda w_{xx} - \rho \phi_{xx} - 2G(w, \phi) = 0$$
$$\Delta^2 \phi + \rho w_{xx} + G(w, w) = 0.$$

where

$$G(u, v) = \frac{1}{2} u_{xx} v_{yy} + \frac{1}{2} u_{yy} v_{xx} - u_{xy} v_{xy}$$

$$\kappa^2 = \frac{t^2}{12(1 - \nu^2)}, \qquad \lambda = \frac{P}{2\pi R E t}, \qquad \rho = \frac{1}{R}$$

$(x, y) \in \Omega = [-L, L] \times [0, 2\pi R)$.

Assumptions:
- Thin, isotropic shell
- Elastic buckle and curvature not too large
- No pre-buckle
- Normals stay normal, plane stress and small angle approximation for strain tensor.

$$\kappa^2\Delta^2 w + \lambda w_{xx} - \rho\phi_{xx} - 2G(w,\phi) = 0$$
$$\Delta^2\phi + \rho w_{xx} + G(w,w) = 0.$$

Stored energy:

$$E(w) = \frac{Et}{2}\int_\Omega \left[\kappa^2 \Delta w^2 + \Delta\phi^2\right]\, dxdy,$$

Constraint is the average axial end-shortening associated with deflection w

$$S(w) = \frac{1}{4\pi R}\int_\Omega w_x^2\, dxdy.$$

- Solutions of vKD equations are stationary points of

  Total Potential $\quad F_\lambda(w) = E(w) - \lambda S(w).$

- Solutions also stationary points of $E(w)$ under constant $S(w)$.

Part III : Post-buckle paths:

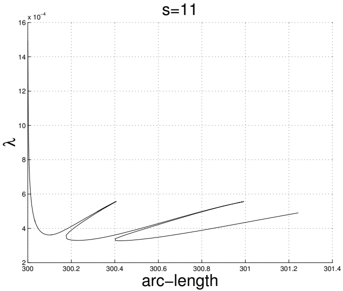
- Dynamic Analogy :
  - Seek localized buckle solutions as homoclinic solution
  - BCs ($L = \infty$): w, ϕ + derivatives $\longrightarrow 0$ as $x \longrightarrow \pm\infty$.
  - Seek solution in subspace of circumferential wave number.
  - Discretize by Galerkin circumferentially have large system of ODEs in axial direction.
- Use numerical continuation

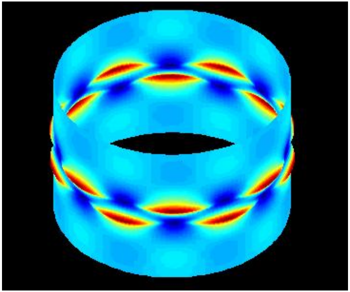
Test Results for Yamaki Shell :

$$L = 160.9(mm) \quad R = 100(mm) \quad t = 0.247(mm)$$

$$E = 5.56(GPa) \quad \nu = 0.3$$

For this shell: $L/2\pi R \approx 0.25$ ... not very long
Number of circumferential waves $s = 11$.





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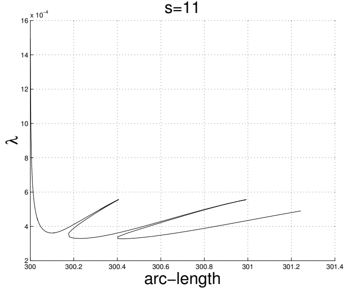
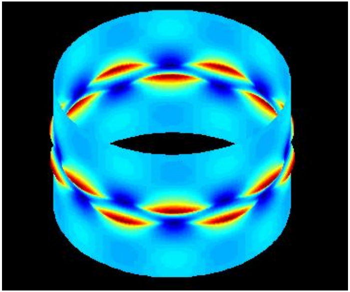
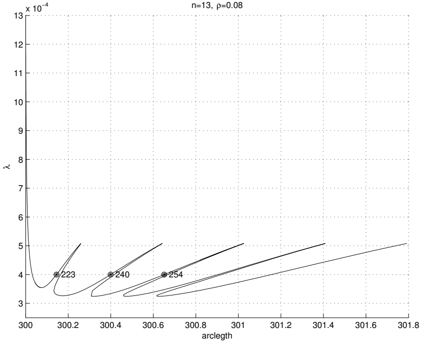


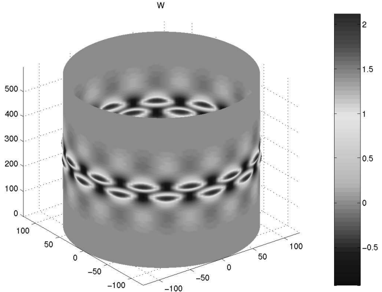


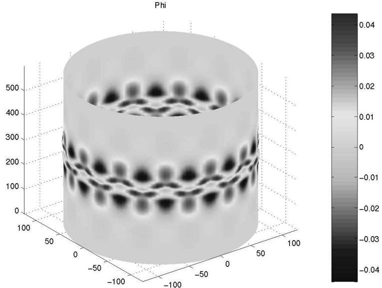
s=11	$\frac{\lambda_{\min}}{\lambda_{cl}}$	$W_{\min}$	$W_{\max}$
Experiment	0.24	-0.9	1.9
Homoclinic	0.242	-0.866	1.966

Rel. Error : ≈ 0.05

Cellular buckling







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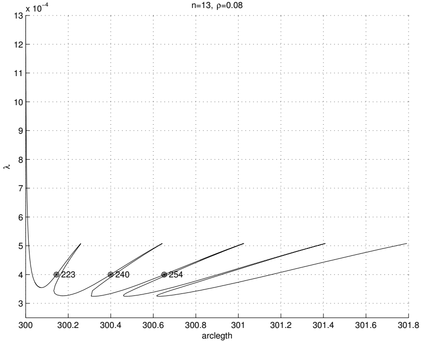

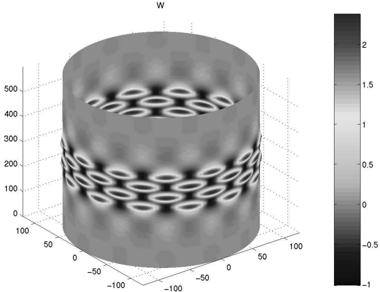

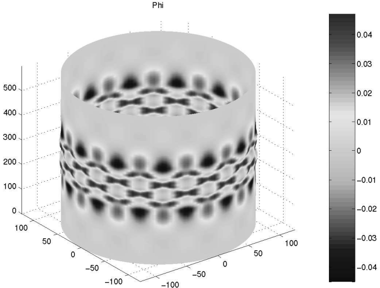

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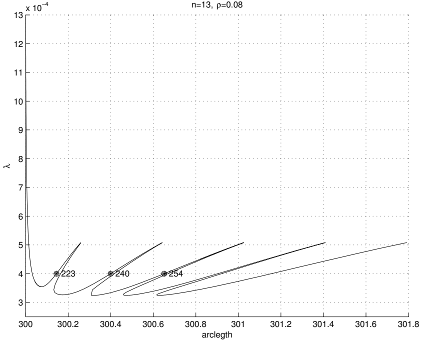
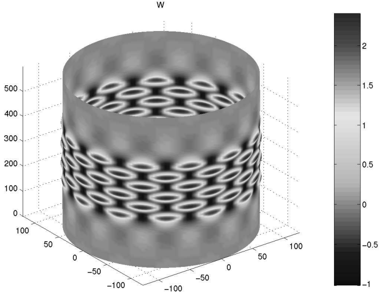
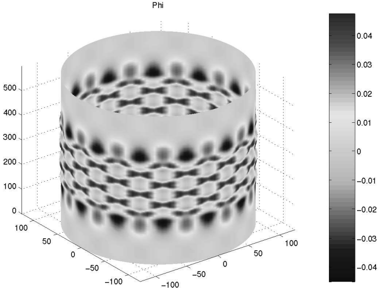






Post-buckle & Homoclinics

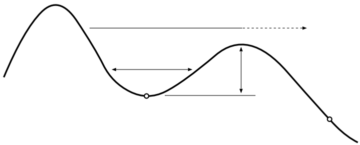
- Given circumferential wave number s get good agreement with post-buckle regime ...
- Finite shell length in experiments : but infinite homoclinic approximation works well.
- Determination of circumferential wave number next project ??

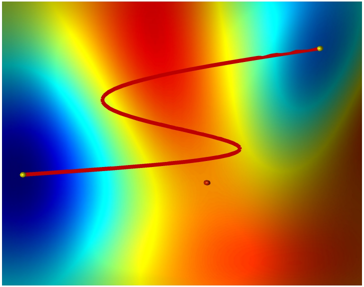
Part IV : Mountain Pass Solution

Let $w_1 \neq w_2$ be two vectors in a space X. Define

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = w_1, \gamma(1) = w_2\},$$
$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)).$$

If $c > \max\{F(w_1), F(w_2)\}$ and F satisfies $(\mathrm{PS})_c$, then c is a critical value of F.

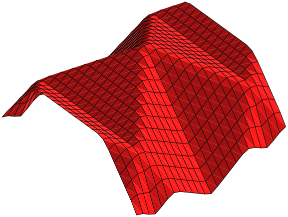




Mountain Pass

MP1. We show that $w_1 = 0$ is a local minimizer: there are $\varrho, \alpha > 0$ such that $F_\lambda(w) \geq \alpha$ for all w with $\|w\|_X = \varrho$;

MP2. If domain is large enough, then there exists w_2 with $\|w_2\|_X > \varrho$ and $F_\lambda(w_2) \leq 0$. Based on Yoshimura diamond pattern.



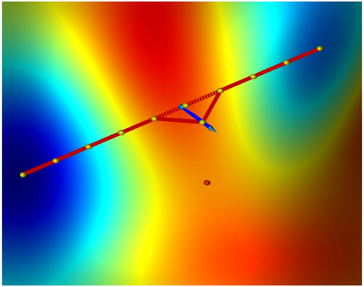
MP3. Given a sequence of paths γ_n that approximates the infimum in defn, we extract a (Palais-Smale) sequence of points $w_n \in \gamma_n$, each close to the maximum along γ_n, and show that this sequence converges in an appropriate manner.

Mountain Pass Alg. : Y. S. Choi, P. J. McKenna (1993)

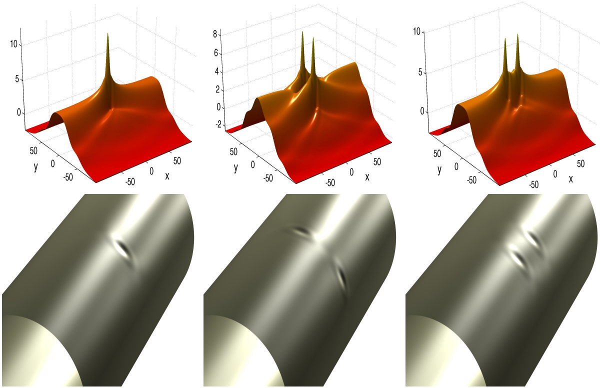
Phase 1 — Initial discrete path. Take P points:
$z_j = w_1 + \frac{j}{P}(w_2 - w_1)$, $j \in \{0, 1, \dots, P\}$
Phase 2 — Main loop:

(a) find m: $\forall j\ F(z_m) \geq F(z_j)$, interpolate,
(b) compute $\nabla F(z_m)$,
(c) deform the path: $\delta > 0$ (small) $z_m^{\mathrm{new}} = z_m - \delta \nabla F(z_m)$,
(d) STOP when F increases.

Phase 3 — Infinite loop: re-distribute points on path



Numerical Solutions

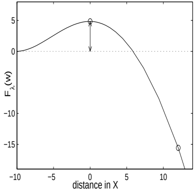


$\Omega = (-100, 100) \times (-100, 100)$, $\Delta x = \Delta y = 0.5$, $\lambda = 1.1$

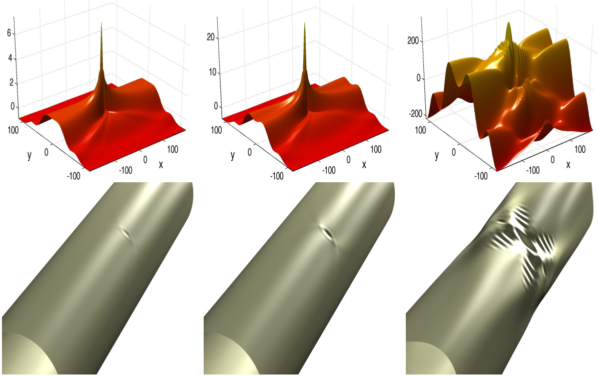
Found using different choices of w_2.

Min energy solution $\equiv$ Single Dimple

Steepest Descent:



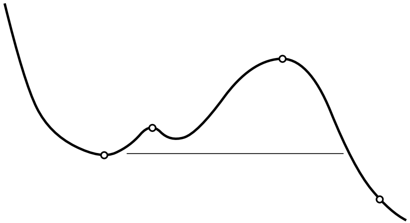
(a) $F_\lambda \approx 5$ (b) $F_\lambda \approx -15$ (c) $F_\lambda \approx -5 \times 10^4$



$$\Omega = (-200, 200) \times (-115, 115),\ \Delta x = \Delta y = 0.5,\ \lambda = 1.1$$

Interpretation of MP ?

Have found the mountain pass energy for the perfect cylinder – how does this give a handel on an imperfect "real" cylinder ?
Consider the minimum mountain–pass energy: $V = \inf_{w_2} F_\lambda$.

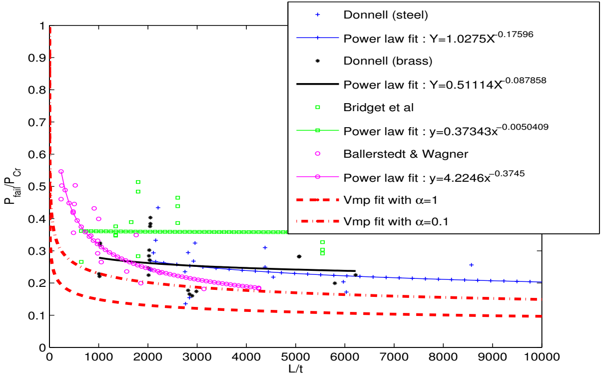


In order to leave the basin of attraction of w_1, the surplus energy should exceed $V(\lambda)$

Imperfections and MP

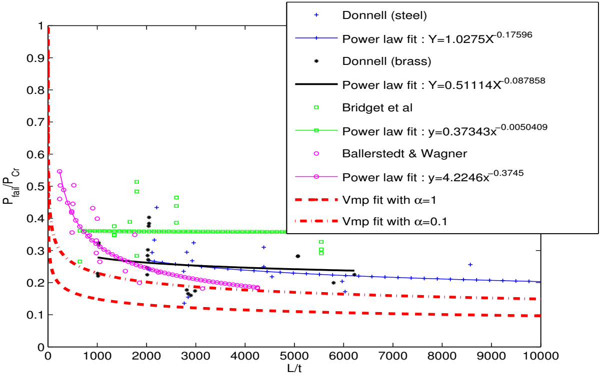
Suppose stored energy from being under load can be transfered to overcome the mountain pass.
Rescale MP energy $V(\lambda)$ by elastic strain energy stored in cylinder of length L :

$$\alpha = \frac{1}{2\pi\sqrt{3(1-\nu^2)}} \frac{t}{L} \frac{V(\lambda)}{\lambda^2}.$$

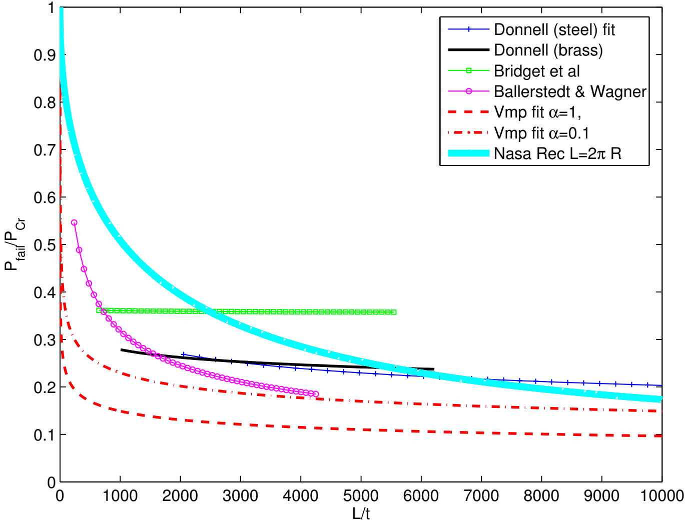


Imperfections and MP

1. The general trend of the constant-α curves is very similar to the trend of the experimental data;
2. The $\alpha = 1$ curve, which indicates the load at which the mountain-pass energy equals the stored energy in the prebuckled cylinder, appears to be a lower bound to the data.

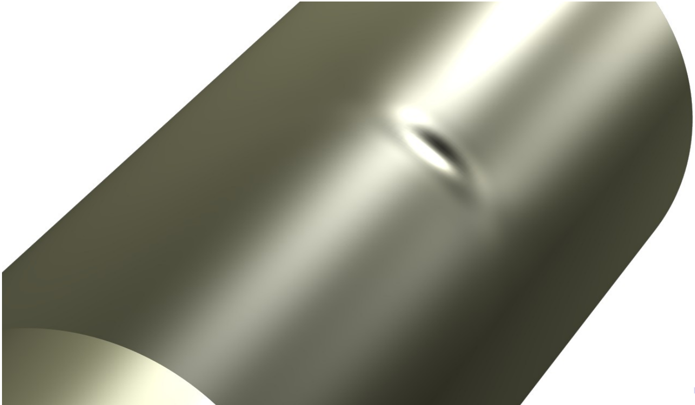


Nasa knockdown



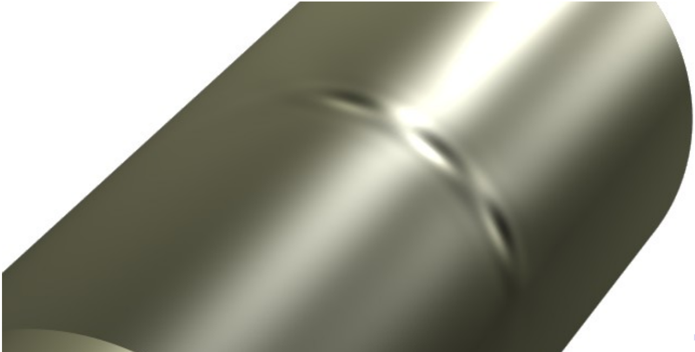
Other single dimples ...

- Single dimples are seen in the high-speed camera images of Esslinger.
- Some "worst imperfections" by Deml and Wunderlich, Deml, Wunderlich and Albertin are single dimples.
- Single dimples are seen in finite element simuations (eg Schweizerhof)



Summary ...

- Axially localized solutions : found as homoclinic orbit
- Computations of post-buckle paths and cellular buckling
- Mountain pass solutions
  - Elements for proof
  - Numerical algorithm
  - Solutions
- From MP solutions seems can get a lower bound on the buckling load.



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