Numerical Computation of ε–Entropy for Parabolic Equations

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Abstract

We report on numerical experiments computing the ε–entropy for parabolic partial differential equations. The ε–entropy is a measure of the spatial density of complexity for the dynamics on an invariant set in function space and has been studied analytically by a number of authors. The ε–entropy only requires solutions of the equation to the accuracy of the parameter ε and the resulting number is (asymptotically) independent of domain size. We consider the complex Ginzburg–Landau equation as an example where a number of analytic results exist and the Kuramoto-Sivashinsky equation where the accompanying theory has yet to be fully developed. Our numerical results for the Kuramoto-Sivashinsky equation do not contradict the conjectured linear scaling of the dimension with domain size.

Key words: ε–entropy, dimension, attractor, Gevrey regularity
MCS 35K99, 35B65, 37L30, 65M99

1 Introduction

We are interested in examining numerically the long time behaviour of parabolic equations of the following type

$$\partial_t u = \partial_x^2 u + F(u) , \qquad x \in \mathbb{R} . \tag{1.1}$$

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To this end we define the attractor $\mathcal{A}$

$$\mathcal{A} = \bigcap_{t>0} \Phi^t(B) ,$$

where Φ^t is the solution semi-flow generated by Eq.(1.1) and B is a large ball of initial data in $C(\mathbb{R})$, the space of uniformly continuous functions. The attractor captures the asymptotic dynamics of Eq.(1.1) at large times and we are interested in measuring the complexity of $\mathcal{A}$ as a set in a function space. For a review of analytical results on the long time dynamics on large and unbounded domains see for example [1,16].

One possible measure of complexity is the "box counting" dimension d (see [20]). However, it has been shown [11] that d scales with the domain size L and is therefore infinite on unbounded domains. Moreover, even in bounded but large domains, an estimation of this number requires a large number of high-precision computations. A seemingly more accessible quantity, called the (Kolmogorov) ε–entropy has been considered in a growing number of papers: for reaction-diffusion equations see for example [4–6,21,23–25] and [17] for numerical schemes. In this paper we do not intend to review in detail theory on obtaining bounds on the complexity, rather we wish to estimate this numerically, which we believe is the first attempt to estimate this quantity directly from measured data. Drawing on ideas from information theory [15], the ε–entropy is defined as

$$H_\varepsilon = \lim_{\ell\to\infty} \frac{\log \mathcal{N}(\varepsilon, \ell)}{2\ell} , \tag{1.2}$$

where $\mathcal{N}(\varepsilon, \ell)$ is the (minimum) number of balls of radius ε in the topology of $L^\infty([-\ell, \ell])$ that are needed to cover the attractor $\mathcal{A}$. A closely related quantity is the upper density of dimension

$$\mathrm{d_{up}} = \limsup_{\varepsilon\to 0} \frac{H_\varepsilon}{\log(1/\varepsilon)}. \tag{1.3}$$

The number $\mathrm{d_{up}}$ can be interpreted as a spatial density of dimension (see for example [6,5]). Given an approximation of the attractor these numbers cannot be expected to be easy to compute, as a rough order of magnitude calculation shows. Using the formula

$$\mathcal{N}(\varepsilon, \ell) \approx e^{C\ell \log(1/\varepsilon)} ,$$

assuming $C = 1$, $\ell = 20$, and $\varepsilon = 1/2$, gives an estimate of $\mathcal{N} \approx 10^7$, that is approximately 10^7 balls of radius $1/2$ needed to cover the attractor.

To explore the ε-entropy numerically, we must first build an approximation of $\mathcal{A}$. Therefore we restrict the domain of the equation to a bounded set $[-\pi L, \pi L]$ and impose periodic boundary conditions. The parameter L will be chosen as large as feasible to define a sufficiently large subset of the solutions of Eq.(1.1). We thus consider equations of the following type:

$$\partial_t u = \partial_x^2 u + F(u) , \qquad x \in [-\pi L, \pi L] . \tag{1.4}$$

This is equivalent, by a rescaling, to looking at small diffusion on a fixed domain size since the rescaled function $v(y,t) = u(Ly,t)$ satisfies the following equation

$$\partial_t v \;=\; L^{-2}\partial_y^2 v + F(v)\;, \qquad y \,\in\, [-\pi,\pi]\;,$$

with periodic boundary conditions. The attractor $\mathcal{A}(L) \subset \mathcal{A}$ is defined in a natural way:

$$\mathcal{A}(L) \;=\; \bigcap_{t>0} \Phi_L^t(B(L))\;,$$

where Φ_L^t is the solution semi-flow generated by Eq.(1.4) with $B(L)$ a large ball of initial data in $C_{\rm per}([-\pi L, \pi L])$, the space of continuous periodic functions. The aim of the ε-entropy is to describe the local complexity(in space) by introducing a localised metric on $\mathcal{A}(L)$. It only requires computing the solutions of Eq.(1.4) to a low accuracy (the parameter ε) and, moreover, the resulting number is expected to be asymptotically independent of L.

We specifically consider the following two model equations: the Kuramoto–Sivashinsky (KS) equation

$$\partial_t v(x,t) \;=\; -(\partial_x^4 + \partial_x^2)v(x,t) - \frac{1}{2}\partial_x v^2(x,t)\;, \quad \int v(x,t)\,\mathrm{d}x \;=\; 0\;, \tag{1.5}$$

where $v \;:\; \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$, and the complex Ginzburg–Landau (cGL) equation (see [9])

$$\partial_t u(x,t) \;=\; (1+5i)\partial_x^2 u(x,t) + (1-(1-5i)|u(x,t)|^2)u(x,t) \tag{1.6}$$

where $u \;:\; \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$. The cGL Eq.(1.6) is known to have an attractor in the Gevrey class of analytic functions both for initial data u_0 in the set of $2\pi L$-periodic functions and for u_0 in the set of bounded functions. Collet and Eckmann in [6], consider the complex Ginzburg–Landau equation posed on an infinite domain and show that the limit Eq.(1.2) exists and is bounded from above and below by $\mathcal{O}(\log(1/\varepsilon))$. This bound on the the ε–entropy H_ε suggests that the quantity $\mathrm{d_{up}}$ of (1.3) should be positive and finite. One approach for obtaining these results is to exploit the Gevrey regularity (see Section 4) of the solutions and use a classical sampling formula. For the KS Eq.(1.5), the existence of an attractor in the Gevrey class is known for $2\pi L$-periodic initial data and it is believed to be true, but not proven, for L^∞ initial data (see [2,18,3,7,8]).

For the cGL it is known [6,11] that the box counting dimension d of the attractor scales linearly with L. However analytical estimates for the KS equation are of the form [8]

$$C_1 L \;\le\; d_{KS} \;\le\; C_2 L^{51/40},$$

although this scaling is in fact conjectured to be linear and our numerical results give no indication that the conjecture may be false.

In section 2 we consider the discretisation of the equations, describe our sampling strategy of the attractor and computations of a cover. Finally we present some novel numerical results estimating the ε-entropy and $\mathrm{d_{up}}$ before concluding.

2 Numerical Computations

We describe our computations in the case of the complex Ginzburg–Landau equation Eq.(1.6) since the Kuramoto–Sivashinsky Eq.(1.5) is completely identical. We choose L large, to approximate the $L = \infty$ situation. We emphasize here that there are two different points of view one can follow: the first considers the attractors $\mathcal{A}(L)$ and asks about the limit $L \to \infty$. The other point of view, which is the one we chose here, works with the single attractor $\mathcal{A}$ and considers restrictions to $[-\ell, \ell]$ (which is a larger set than $\mathcal{A}(\ell/\pi)$) and asks about the behaviour of this subset as ℓ increases. For our most accurate simulation we take $L < \infty$ as large as numerically possible and use a range of $\ell \ll L$ to approximate the $L = \infty$ situation.

There are three stages to approximating the ε–entropy, first we need to discretise the equations and then compute a sample of the long time dynamics on the attractor. Finally we need to compute a cover for the sample of the attractor. We bring together some well established algorithms with some new ideas that allows us to perform these novel computations.

2.1 Discretisation of the equations

First we consider the discretisation of the equations. We note that although Gevrey regularity is not necessary to obtain analytic bounds on the ε–entropy, solutions of the cGL are in a Gevrey class and it is believed for L^∞ initial data that solutions for the KS equation are also in a Gevrey class. Therefore we briefly consider the smoothness of solutions to our numerical discretisations as we seek numerical solutions in the same function space as solutions of the equations.

We spatially discretise Eq.(1.6) with a grid spacing of $2\pi L/(2N)$ (that is, $2N$ Fourier modes). This is a Galerkin truncation of order $2N$ and we take N at least $\mathcal{O}(L)$. This leads to the following system of equations

$$\frac{d}{dt} u_n(t) \;=\; \frac{-n^2}{L^2} u(t) + F_n(u(t)), \quad n \;=\; -N, \ldots, N-1 \;. \tag{2.1}$$

where $F_n(u)$ is the nth coefficient in the Fourier series of $F(u)$. We can re-write this system as

$$u_n(t) \;=\; e^{-tn^2/L^2} u(0) + \int\limits_0^t e^{-(t-s)n^2/L^2} F_n(u(s))\,\mathrm{d}s \;, \tag{2.2}$$

with $n \;=\; -N, \ldots, N-1$. In [17] we considered time-integration based on (2.2)

$$u_n^{\mathrm{N}}((k+1)\mathrm{h}) \;=\; e^{-k\mathrm{h}n^2/L^2}\Big(u_n^{\mathrm{N}}(k\mathrm{h}) + \mathrm{h}F_n(u^{\mathrm{N}}(k\mathrm{h}))\Big) \tag{2.3}$$

where $n \in \{-\mathrm{N}, \ldots, \mathrm{N}\}$ and the time step $h > 0$ with $t \approx k\mathrm{h}$ and is similar to that considered in [19]. For the Gevrey integrator (2.3) it was shown in [17] that solutions are in a (discrete) Gevrey class of regularity and, by exploiting this, upper

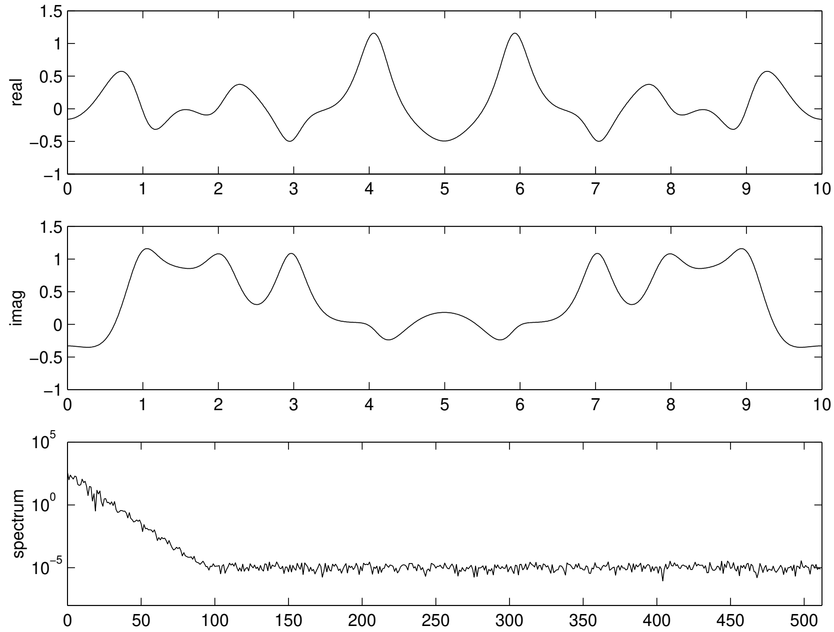

Fig. 1. Real and imaginary parts of solution to the cGL equation at $t = 10$ from the scheme (2.3). For this scheme it has been proved that solutions lie in a (discrete) Gevrey class.

and lower bounds on the ε-entropy were proved along with upper semi-continuity of the global attractors. If we implement the scheme this regularity can be observed numerically, and in figure 1 we have plotted the real and imaginary parts of the solution of the cGL equation at $t = 10$ along with the power spectrum on a semilog scale. We see from this exponential decay of the Fourier modes (in approximately the first 100 modes).

This scheme is however only of low order and of fixed time step and is unlikely to be used in most practical situations (for a comparison of methods see for example [10]). As an alternative scheme we use a popular adaptive Runge–Kutta method of order 5, the Radau5 routine from [13], to integrate (2.1). In figure 2 we have plotted at $t = 10$ the real and imaginary parts of the solution from Radau5 from the same initial condition as for figure 1 and again with the power spectrum. The exponential decay (in approximately the first 100 modes) of the spectrum indicates that solutions from this scheme also lie in a Gevrey class of regularity, although, as far as we are aware, there is no proof of this. We present results below based on both the Radau5 integrator [13] and the Gevrey integrator (2.3).

2.2 Sampling the attractor

Computations of any properties of the attractor usually assume some form of ergodicity, namely that a single (long) orbit will sample a large part of the attractor and will thus suffice to get a good approximation of it. We have found, however, that taking many shorter orbits gives better results (in the sense that the dimension of the attractor is increased). This can be interpreted as saying that if there is any

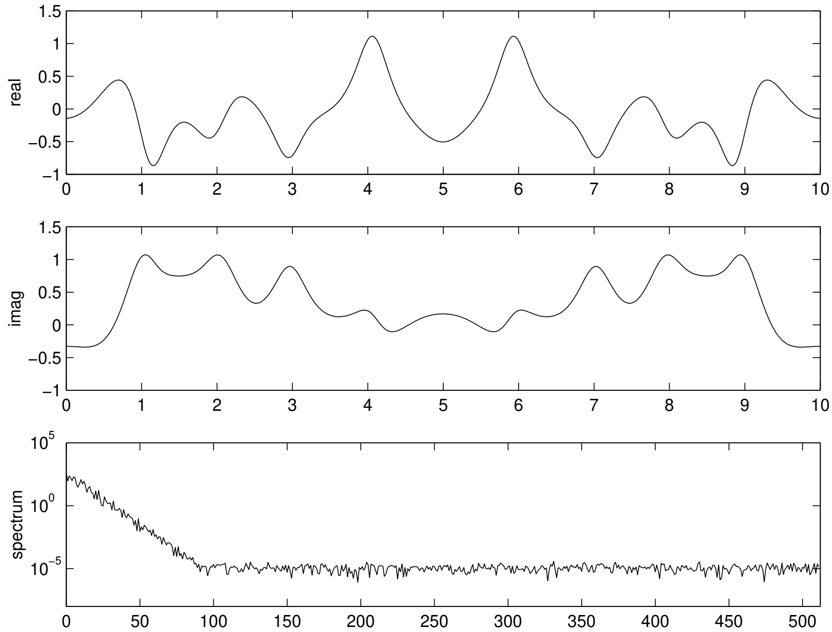

Fig. 2. Real and imaginary parts of solution to the cGL equation at $t = 10$ integrated using Radau5. For this scheme it has been not been proved that solutions lie in a (discrete) Gevrey class, however numerical results indicate that this is probable.

ergodicity in these systems, the return time is extremely large. This is not surprising since the dimension of the attractor is very large.

Let $u_i \in C_{\rm per}([-\pi L, \pi L])$, $i = 1, \ldots, I$ be randomly chosen functions and let $u_i(x,t)$ be their orbits under Eq.(1.6). We fix a time-step h and a space-step δ to sample the orbits u_i, $i = 1, \ldots, I$ at. Note that h and δ are much bigger than the actual grid spacings used in the numerical integration, and δ may not even be a multiple of $2\pi L/N$, in which case we use linear interpolation. We denote by $u_i^{h,\delta}(k)$ the vector

$$u_i^{h,\delta}(k) \;=\; \{u_i(j\delta, kh) \;:\; j = 0, \ldots, J-1\}\,,$$

where $J = [2\pi L/N]$. We fix a initial time T_0 and a final time T such that $T - T_0 = Kh$. We obtain a sample of the approximate attractor of Eq.(1.6) defined by

$$\mathcal{A}_0^{h,\delta} \;=\; \bigcup_{i=1,\ldots,I;\, k=1,\ldots,K} u_i^{h,\delta}(k) \;\subset\; (\mathbb{C}^J)^{IK}\,.$$

To obtain a larger sample without computing more points we exploit the property that the space translation operator T_x commutes with the time evolution operator Φ^t, hence we may consider the set of translates

$$\begin{aligned} \mathcal{A} \quad &= \quad \bigcup_{n=0,\ldots,J-1} T_{n\delta}\mathcal{A}_0 = \Big\{\{u_i((j+n)\delta \bmod 2\pi L, kh) \;:\; j = 0, \ldots, J-1\} \\ &\qquad\qquad :\; i = 1, \ldots, I,\; k = 1, \ldots, K,\; n = 0, \ldots, J-1\Big\} \\ &\qquad \subset (\mathbb{C}^J)^{IJK}\,. \end{aligned}$$

Let $\ell = q\delta < 2\pi L$. The restriction of $\mathcal{A}$ to the interval $[0, \ell)$ is the subset of $(\mathbb{C}^q)^{IJK}$ defined by

$$\mathcal{A}(\ell) = \Big\{ \{u_i((j+n)\delta \bmod 2\pi L, kh) \ : \ j = 0, \ldots, q-1\} \\ \ : \ i = 1, \ldots, I, \ k = 1, \ldots, K, \ n = 0, \ldots, J-1 \Big\} \, .$$

As long as we keep ℓ much smaller than the fixed L, this approximates a finite volume restriction of the infinite volume, translation-invariant attractor $\mathcal{A}(\infty)$.

2.3 *Computing a cover*

Let $\varepsilon > 0$ be given. We are interested in the cardinality $\mathcal{N}(\varepsilon, \ell)$ of a subset $\mathcal{B}$ of $\mathcal{A}(\ell)$ satisfying: for all x, y in $\mathcal{B}$, $\|x - y\| > \varepsilon$, where the norm is the taken in $\ell^\infty(\mathbb{C}^q)$ (this is an ε–distinguishable subset of $\mathcal{A}(L)$, but not a maximal one, see [15] and Section 4). We implemented a version of the box assisted sorting algorithm proposed in [12]. As far as we are aware this is the first attempt to perform this algorithm on a numerical sample of an attractor of a PDE. We start with a single list $U_1 = \{x_{1,1}, x_{1,2}, \ldots, x_{1,M}\}$ containing all the points in $\mathcal{A}(\ell)$ (in arbitrary order). We then compute the distance between $x_{1,1}$ and $x_{1,k}$, for $k > 1$. If this distance is larger than ε, then $x_{1,k}$ is put into the list U_2 (which is created if it does not exist yet), namely $x_{2,\mathrm{card}U_2+1} = x_{1,k}$, and $x_{1,k}$ is removed from U_1. When the end of list U_1 is reached, the same procedure is applied to U_2, then to U_3, etc. If no new list appears then we have a cover of $\mathcal{A}(\ell)$ with $U_1, \ldots, U_{\mathcal{N}}$. These are balls of radius less than ε centered at $x_{k,1}$, $k = 1, \ldots, \mathcal{N}$. No attempt is made to optimise over the choice of these centres. Having obtained a list for ε and ℓ we then use these lists to then sort for smaller values of ε and larger values of ℓ.

2.4 *Numerical results*

We conducted numerical experiments with the following values:

$$\delta = 0.7 \, , \ I = 15 \, , \ h = 1 \, , T_0 = 200 \, , \ K = 3000 \, ,$$

with $L = 20$, $L = 40$, $L = 60$ and $N = 2^{10}$ for both KS and cGL equations, and more accurate runs with $N = 2^{11}$, $L = 30$ for the cGL and $L = 50$ for the KS equation. We denote by

$$\widehat{H_\varepsilon} = \frac{\log \mathcal{N}(\varepsilon, \ell)}{2\ell} \, , \qquad \widehat{\mathrm{d_{up}}} = \lim_{\ell \to \infty} \frac{\widehat{H_\varepsilon}}{\log 1/\varepsilon}$$

our approximations to the H_ε and $\mathrm{d_{up}}$ respectively based on the cover computed as above.

For the cGL equation and $L = 30$, $\mathrm{card}(\mathcal{A})$ would be of the order 8×10^6 while it is 22×10^6 for the KS equation and $L = 50$. Computations have been performed on an 1.4GHz Athlon processor with 1Gb of RAM and 1Gb of swap space. The full

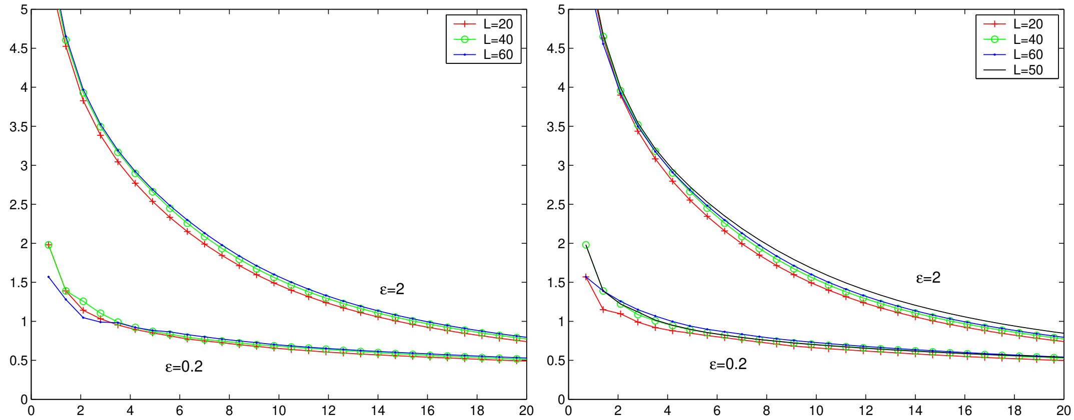

Fig. 3. Result of $\widehat{H_\varepsilon}$ against ℓ for the KS equation. Samples based on the Gevrey integrator (2.3) with $L = 20, 40, 60$ and $N = 2^{10}$ (left) and (right) the Radau5 integrator [13] with $L = 20, 40, 60$ and $N = 2^{10}$, and $L = 50$ $N = 2^{11}$. Note that results do not depend greatly on either the choice of integrator or size of L.

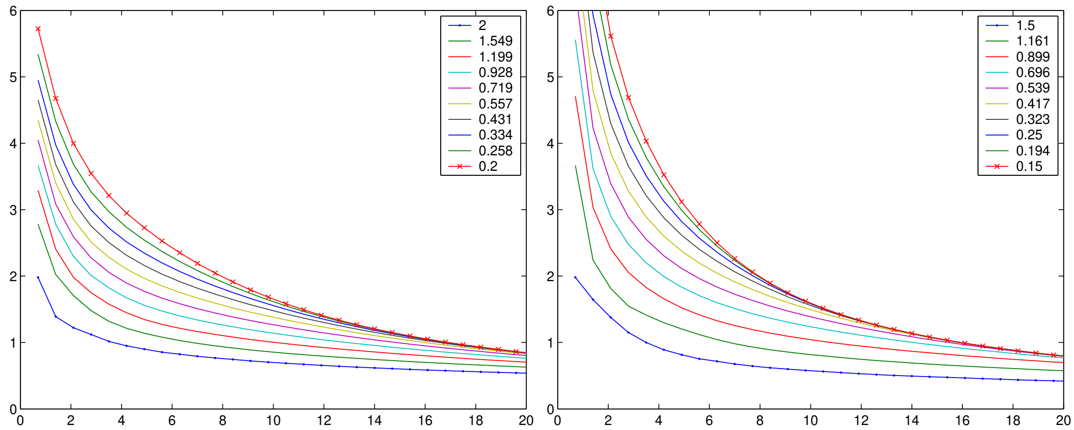

Fig. 4. Result of $\widehat{H_\varepsilon}$ against ℓ for the KS equation (left) and the cGL equation (right) based on data from Radau5 with $L = 50$ (left) and $L = 30$ (right). Curves are shown for successive fixed values of ε.

calculations for one equation took approximately 12 hours. Practical limitations on sample size were primarily due to memory size rather than processor speed. To maximize our use of memory we noted that the values of ε used are rather large and so we do not need to store the vectors $u_i^{h,\delta}(k)$ with high precision. Namely, although the numerical integration is performed in double precision arithmetic, the computation of the ε–entropy is performed with 4 digits numbers.

First we examine the effect of the computational length L for the KS equation and the effect of the different integrators. In figure 3 we have plotted for $\varepsilon = 2$ and $\varepsilon = 0.2$ the approximate ε–entropy $\widehat{H_\varepsilon}$ as a function of ℓ for the Gevrey (2.3) and the adaptive Radau5 [13] integrators. We see that the numbers obtained are robust to changing the size of the computational domain L, in particular for small ε. This figure also illustrates that the choice of integrator is not crucial and infact the difference in the number obtained for $\widehat{H_\varepsilon}$ is small.

We now focus on our most accurate simulations using the higher order Radau5 scheme. We comment on the accuracy of the computation and examine the data to extract a computed bound on $\mathrm{d_{up}}$. In figure 4 we have plotted $\widehat{H_\varepsilon}$ as computed directly from the data, different curves on the same plot correspond to values of $\varepsilon \in \{1.5, 1.16, 0.90, 0.70, 0.54, 0.42, 0.32, 0.25, 0.19, 0.15\}$ and $\varepsilon \in \{2, 1.55, 1.2, 0.99, 0.72, 0.56, 0.43, 0.33, 0.26, 0.2\}$ where the values here have been rounded to 2 decimal places. Recall that from the definition (1.2) of H_ε we expect exponential growth as ℓ is increased. This gives a check on the range of ℓ for which an estimate is valid and this can be used to extrapolate values for $\mathrm{d_{up}}$. Figures 5 and 6 show the results for the cGL equation (1.6) and the KS equation (1.5) respectively. Remark that the number of balls $\mathcal{N}(\varepsilon, \ell)$ is monotonically decreasing in ε and monotonically increasing in ℓ). The top left plot shows the size of the cover $\mathcal{N}$ (computed as in section 2.3), as a function of ℓ. It can be seen that although for small ℓ the behaviour looks exponential, there is a rapid saturation due to the finite size of the sample. This is better seen on the top right, we show $\log \mathcal{N}$ as a function of ℓ, with linear fits near the origin. In the bottom left plot, where we show $\widehat{H_\varepsilon}$ as a function of ℓ, with the same fits as before, the fitted lines should be compared to plot from the raw data in figure 4. The fits should tend to a postive constant, as $\ell \to \infty$ and this limit is displayed as a function of $1/\varepsilon$ in the last plot. It is reassuring to see that there appears to be a positive uniform lower bound to the curve.

3 Conclusion

Although (Gevrey) smoothness of solutions for adaptive methods (such as Radau5 [13]) has not been proved, numerically these solutions are observed to be smooth. We see that for the KS equation the choice of integrator that has spatial smoothness is not a crucial factor in estimating the ε–entropy. The observation that computing a number of different orbits rather than one single orbit improved the sampling the attractor has previously been noted for ODEs, see example [22]. To approximate the attractor $\mathcal{A}(\infty)$ we computed on a finite domain of size L considered restrictions to $[-\ell, \ell]$ and examined the behaviour of this subset as ℓ increases. We further improved the sampling of the attractor by exploiting spatial translations. Computations based on the KS equation seem to suggest that we have convergence in L.

The computations performed here were limited not by the speed of the processor but by memory. With current memory access and speeds reliable computation of the ε–entropy H_ε and upper density of dimension $\mathrm{d_{up}}$ is difficult. We have used a numerical fit to obtain asymptotic estimates, but perhaps with more data it may be possible to use an extrapolation method. The computations we have performed indicate a lower bound on these quantities, that is an underestimation, due to the finite size of the sample. It is reassuring that the numerical values estimated are positive and differ in the two examples. Finally we note that the results for the KS equation give no evidence that contradicts a linear scaling of the dimension with domain size.

4 Appendix

We provide here a few definitions referred to in the main text.

Gevrey class [14]

The Gevrey class $\mathcal{G}_\alpha(C)$ is the set of all real functions f admitting an analytic extension to a strip of width 2α around the real axis and which are uniformly bounded in this strip:

$$|f(x+iy)| \leq C , \forall (x,y) \in \mathbb{R} \times [-\alpha, \alpha] .$$

For periodic functions this is, by the Paley–Wiener–Schwartz Theorem, equivalent to

$$\sum_{n \in \mathbb{Z}} e^{2\alpha|n|} |f_n|^2 < \infty ,$$

where $\{f_n\}_{n\in\mathbb{Z}}$ are the the Fourier coefficients of f. By extension, a complex function g is said to belong to $\mathcal{G}_\alpha(C)$ if both $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ are in $\mathcal{G}_\alpha(C)$.

Nets and covers [15]

Let Y be a compact subset of a metric space X.

A set $\mathcal{U} = \{U_1, \ldots, U_N\}$ of open sets in X is called a cover of Y if $\bigcup_{n=1}^N U_n \supset Y$. It is called an ε–cover if $\max_{n=1,\ldots,N} \mathrm{diam}(U_n) \leq 2\varepsilon$.

A subset Z of X is an ε–net for Y if $\mathrm{dist}(z,y) \leq \varepsilon$ for all $z \in Z$ and $y \in Y$.

A subset W of Y is ε–distinguishable if $\mathrm{dist}(w_1,w_2) > \varepsilon$ for all $w_1, w_2 \in W$.

If U_ε is a minimal ε–cover, Z_ε a minimal ε–net and W_ε a maximal ε–distinguishable subset, then $\mathrm{card}\, W_{2\varepsilon} \leq \mathrm{card}\, U_\varepsilon \leq \mathrm{card}\, Z_\varepsilon \leq \mathrm{card}\, W_\varepsilon$.

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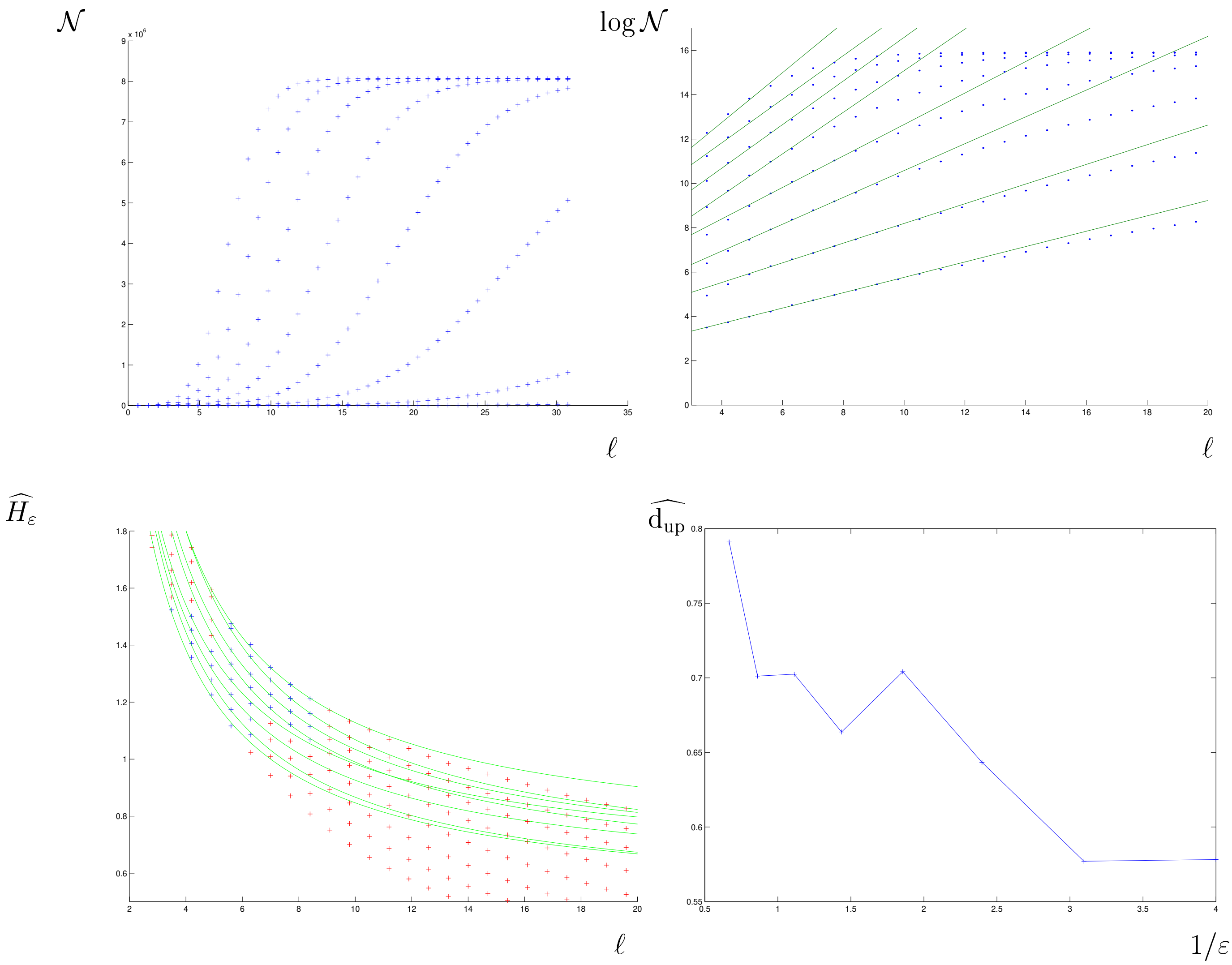

Fig. 5. CGL equation with $\mathrm{card}(\mathcal{A}) \approx 8 \times 10^6$ for values of $\varepsilon \in [1.5, 0.15]$. Top left: plot of number of the number of balls $\mathcal{N}$ in the cover as a function of domain size, we see a rapid saturation due to the domain size. Top right: on a log scale with fitting to show the (small) region of exponential growth. Bottom left: as before but scaled by $\ell \log \varepsilon^{-1}$ to form $\widehat{H_\varepsilon}$. Bottom right : estimates of the upper density of dimension $\widehat{\mathrm{d_{up}}}$ obtained from the limits of the fitted curves in the bottom left figure.

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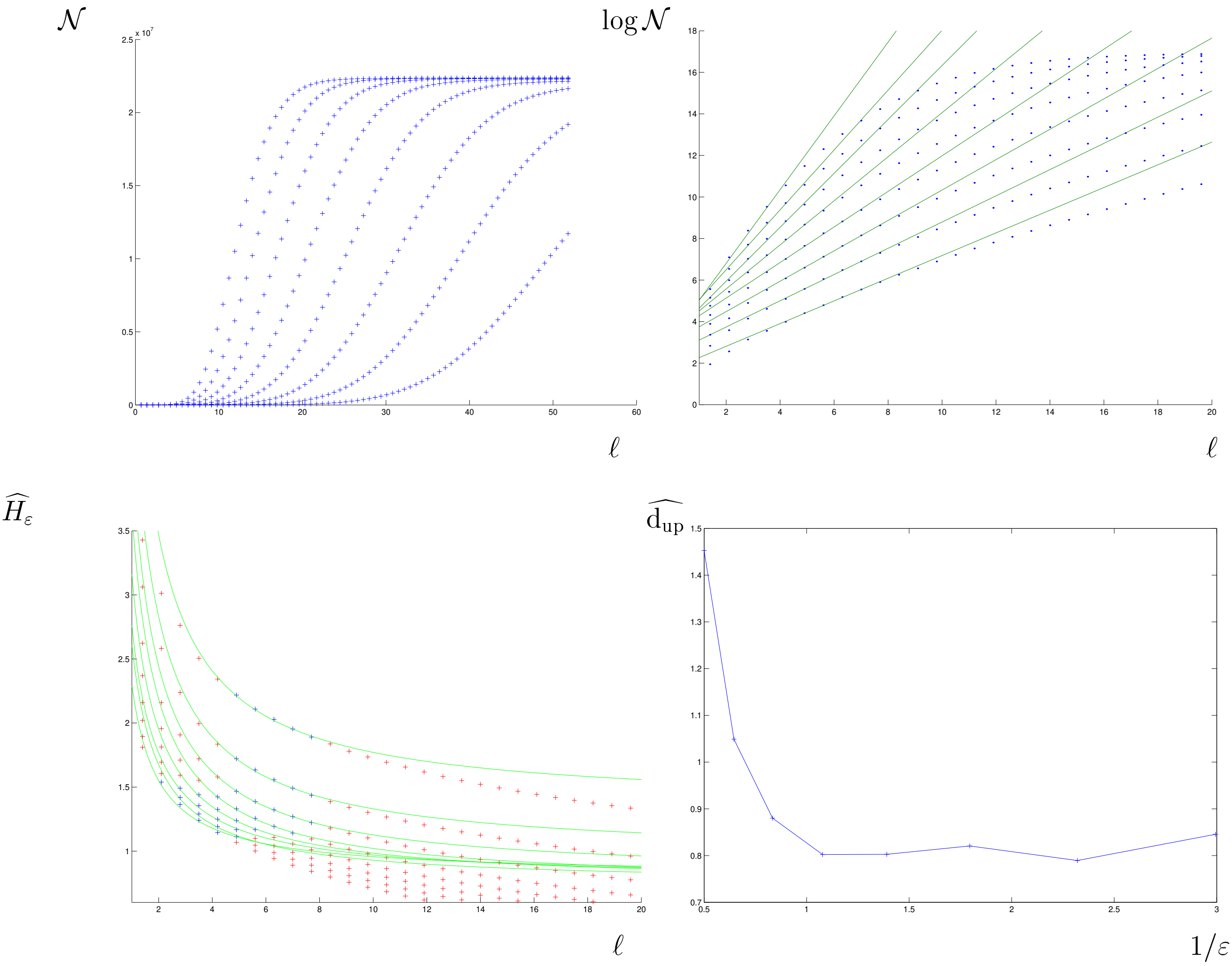

Fig. 6. KS equation, $\mathrm{card}(\mathcal{A}) \approx 22 \times 10^6$ for values of $\varepsilon \in [0.2, 2]$. Top left: plot of number of the number of balls $\mathcal{N}$ in the cover as a function of domain size, we see a rapid saturation due to the domain size. Top right: on a log scale with fitting to show the (small) region of exponential growth. Bottom left: as before but scaled by $\ell \log \varepsilon^{-1}$ to form $\widehat{H_\varepsilon}$. Bottom right : estimates of the upper density of dimension $\widehat{\mathrm{d_{up}}}$ obtained from the limits of the fitted curves in the bottom left figure.