Topological and ε–entropy for Large Volume Limits of Discretised Parabolic Equations

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Abstract

We consider semi-discrete and fully discrete approximations of nonlinear parabolic equations in the limit of unbounded domains, which by a scaling argument is equivalent to the limit of vanishing viscosity. We define the spatial density of ε–entropy, topological entropy and dimension for the attractors and show that these quantities are bounded. We also provide practical means of computing lower bounds on them. The proof uses the property that solutions lie in Gevrey classes of analyticity, which we define in a way that does not depend on the size of the spatial domain. As a specific example we discuss the complex Ginzburg–Landau equation.

1 Introduction

We consider the following general parabolic equation,

$$\partial_t u \;=\; \nu\Delta u + \gamma u + F(u)\;, \qquad x\;\in\;[-L\pi, L\pi]^d\;, \qquad t\;\geq\;0\;, \tag{1.1}$$

for a complex valued function $u = u(x,t)$ and bounded continuous initial condition $u(x,0) = u_0(x)$. We restrict ourselves to $L\in\mathbb{N}$ for convenience. The coefficients of (1.1) satisfy

$$\nu\;\in\;\mathbb{C}\;, \qquad \mathrm{Re}(\nu)\;>\;0\;, \quad \gamma\;\in\;\mathbb{R}\;,$$

and we assume that $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$ are real analytic functions of $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$.

We are interested in the large volume limit ($L\to\infty$) of the long time dynamics (in particular the attractor) of (1.1) and its approximation by numerical schemes. In the latter case we are interested in the limit when the mesh size of our discretisation is kept constant while taking the limit $L\to\infty$, thereby obtaining an infinite-dimensional, but still discrete system (see Section 6 for results of upper semicontinuity of the attractors in terms of the different parameters of the problem).

We remark that by a scaling transformation, the large volume limit can be interpreted as a small viscosity limit. The rescaled function $v(y,t) = u(Ly,t)$ with $y\in[-\pi,\pi]^d$ satisfies the following equation

$$\partial_t v \;=\; \frac{\nu}{L^2}\Delta v + \gamma v + F(v)\;,$$

with periodic boundary conditions on $[-\pi,\pi]^d$. It is however easier to work with (1.1) (with periodic boundary conditions) and take $L\to\infty$. Indeed, since the problem on the full space $\mathbb{R}^d$ is well-posed, we have a priori bounds for all $L<\infty$. In fact, we view the periodic boundary conditions on $[-L\pi, L\pi]^d$ for large L as an approximation of the infinite volume.

For each fixed $L<\infty$, (1.1) generates a semi-flow Φ^t_L. We discretise this time-evolution spatially by truncating to a finite number of (Fourier) modes. We make this truncation by multiplying by a smooth function in Fourier space (rather than a sharp indicator function), to have a better control as $L\to\infty$ (when the spectrum becomes dense). We then discretise in time using an explicit scheme inspired by [26]. This scheme is amenable to analysis and also proves to be an efficient numerical scheme for smooth initial conditions.

It is not the purpose of this paper to prove the existence of global attractors for (1.1) or for the discretisations, this has been considered in different setups in a large number of publications (see for example [27, 24, 3, 1, 29]). Instead, we assume the existence a semi-flow and of a family of global attractors, $\widehat{\mathcal{A}}(L)$, for the continuous and discrete problems (see Definition 3.2).

We compute bounds on statistical quantities that are valid both for the discrete and continuous systems. The first of these statistical quantities is the (Kolmogorov) ε–entropy

$$H_\varepsilon \;:=\; \limsup_{L\to\infty} \frac{\log \mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L))}{(2L\pi)^d}\;,$$

where $\mathcal{N}$ is the minimum number of balls of radius ε in the topology of L^∞ that are needed to cover the attractor $\widehat{\mathcal{A}}(L)$ (see Definition 3.3). We prove that H_ε is a finite number in Theorem 4.3. We thereby get a bound on the upper density of dimension

$$d_{\text{up}} \;=\; \limsup_{\varepsilon\to 0} \frac{H_\varepsilon}{\log \varepsilon^{-1}}\;.$$

This is to be compared with the results of Kolmogorov and Tikhomirov [15], where they obtain a bound of the same type for the set of all entire analytic functions of exponential type. This result follows from a sampling result for such functions (Proposition D.3), namely any of these analytic functions can be reconstructed by interpolation of a discrete set of values. Although the functions on $\widehat{\mathcal{A}}$ are not entire functions, they are still determined by a discrete sampling.

Remark that it is appropriate to take the L^∞ topology, since the diameter of $\widehat{\mathcal{A}}(L)$ does not depend on L in this topology, unlike the topology of Sobolev spaces of non-zero order. We remark that the L^∞ topology is stronger than the L^2 topology, hence our results do not follow from [9, 8, 29].

We also wish to emphasise here that the order of the limits in our definition of d_{up} is important. A more 'naive' definition would be

$$\widehat{d_{\text{up}}} \;=\; \limsup_{L\to\infty} \limsup_{\varepsilon\to 0} \frac{\log \mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L))}{(2L\pi)^d \log \varepsilon^{-1}}\;.$$

The two limits do not commute in general, see [5]. We believe our approach is more natural from an experimental/numerical point of view, in the sense that L is a parameter that can be varied in a series of measurements/simulations made at a fixed accuracy ε.

We also consider the density of topological entropy in Section 5. We show that the spatial densities satisfy the analogue of the following well known inequalities [14, 22]

$$\mathcal{V} \;\le\; h_{\text{top}} \;\le\; \lambda d_{\text{up}}\;,$$

where $\mathcal{V}$ is the volume expansion rate, λ is the largest Lyapunov exponent, h_{top} is the topological entropy and d_{up} is the upper Hausdorff dimension.

The paper is organised as follows: in the remainder of this section we introduce the notation for the paper. In Section 2 the semi-discrete and fully discrete approximations to (1.1) are presented. In Section 3, we define the density of ε–entropy, topological entropy, of upper dimension and the volume growth rate and state our assumptions on the equation and its approximations. A key result of the paper is Lemma 4.2 (proved in Appendix A), which states that the evolution has a fast local smoothing effect, a property which allows us to establish upper bounds on the

ε-entropy (Section 4). This is then applied in Section 5 to show that the topological entropy is finite. We also show that it is bounded below by the volume expansion rate (Section 5.2). We discuss the upper-semicontinuity of the attractors in Section 6. Technical proofs are given at the end of the paper: Appendix B contains a proof of analyticity for the fully discrete scheme, Appendix C contains a Lemma on analytic functions and Appendix D recalls some results on Gevrey and Bernstein classes.

1.1 Notation

We use the following conventions: $\overline{z}$ is the complex conjugate of z and $|z| = \sqrt{z\overline{z}}$ its modulus. A function $f = f_1 + if_2$ with both f_1 and f_2 real-analytic is identified with the vector-valued function $f = (f_1, f_2)$. Its analytic extension to the complex plane has the form (f_1+ig_1, f_2+ig_2) and we write $|f| = \big(|f_1|^2 + |f_2|^2 + |g_1|^2 + |g_2|^2\big)^{1/2}$ which, on the real axis, is equal to the modulus of the complex function f. The convolution of two functions f, g is denoted $f \star g(x) := \int f(x-y)g(y)dy$.

If u is a function of t (time) and x (space), then we consider it either as a function of two variables with values in $\mathbb{C}$, written $u(x,t) \in \mathbb{C}$, or as a function of time with values in the functions of x, written $u(t) \in \mathcal{C}_{\rm b}(\mathbb{R}^d)$ (the set of bounded continuous functions). A function in the set $\mathcal{C}_{\rm per}([-L\pi, L\pi]^d)$ of $2L\pi$–periodic continuous functions, will often be identified with its lift (by periodic extension) to $\mathcal{C}_{\rm b}(\mathbb{R}^d)$.

The spaces $\mathcal{C}_{\rm b}(\mathbb{R}^d)$ and $\mathcal{C}_{\rm per}([-L\pi, L\pi]^d)$ are Banach spaces with the sup norm $\|\cdot\|_\infty$ and may be viewed as subspaces of $\big(L^\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$ and $\big(L^\infty([-L\pi, L\pi]^d), \|\cdot\|_\infty\big)$ respectively. We also make extensive use of the Gevrey class $\mathcal{G}_\alpha(C)$ and the Bernstein class $\mathcal{B}_\sigma(C)$. These are both discussed in Appendix D. If $\mathrm{Re}\big(f\big)$ and $\mathrm{Im}\big(f\big)$ belong to the Gevrey class $\mathcal{G}_\alpha(C)$, we use the notation $f \in [\mathcal{G}_\alpha(R)]^2$ (similarly for $\mathcal{B}_\sigma(C)$).

We denote by $\mathcal{T}$ the standard Fourier transform operator

$$\big(\mathcal{T}f\big)(k) \;:=\; \frac{1}{(2\pi)^d}\int e^{ik\cdot x} f(x)\,dx\,, \qquad \big(\mathcal{T}^{-1}f\big)(x) \;:=\; \int e^{-ik\cdot x} f(x)\,dk\,.$$

The Fourier series operator for $2L\pi$–periodic functions is denoted with the same symbol:

$$\big(\mathcal{T}f\big)_n \;:=\; \frac{1}{(2L\pi)^d}\int_{|x|\le L\pi} e^{in\cdot x/L} f(x)\,dx\,, \qquad \big(\mathcal{T}^{-1}f\big)(x) \;:=\; \sum_{n\in\mathbb{Z}^d} e^{-in\cdot x/L} f_n\,. \tag{1.2}$$

We introduce two different smooth cutoff functions. The first of these, φ, acts in real space and serves as a weight in L^p norms, in order to get bounds that do not depend on L.

Definition 1.1 *Let φ be a real-space cutoff function satisfying*

$$\varphi(x) > 0 \quad \forall x \in \mathbb{R}^d, \quad \varphi(-x) = \varphi(x) , \quad \int \varphi(x)\, dx = 1 , \quad \left\| \frac{\nabla\varphi}{\varphi} \right\|_\infty < \infty ,$$

and, moreover, φ^{-1} is a tempered distribution ($\int \varphi^{-1} f < \infty$ for any Schwartz function f).

Examples. The function

$$\varphi(x) = \frac{1}{(1 + |C_\varphi x|^2)^{d/2+1}}$$

satisfies all of our requirements (here, C_φ is a normalisation constant determined by the equation $\int \varphi = 1$, similarly for C_ψ below). However, the function

$$\psi(x) = \frac{1}{\cosh(C_\psi x_1) \cdots \cosh(C_\psi x_d)} ,$$

which has a sharper decay at infinity, cannot be used because it fails the last property, namely $\cosh(x)$ is not a tempered distribution. The importance of this may be seen in Lemma 4.2.

Note that for (1.1) the function ψ could be used, and would provide sharper bounds in our proofs. This does not work however with the truncation to a finite number of modes (such as given by the semi-discrete system (2.3) or fully discrete system (2.5)).

Our second cutoff function, ξ_K, is defined in terms of its Fourier transform. It smoothly truncates to a finite set of Fourier modes hence produces a finite dimensional problem.

Definition 1.2 *Let $K > 1$ and let $\widehat{\xi}_K$ be a $\mathcal{C}^\infty$ function taking the following values:*

$$\widehat{\xi}_K(k) = \begin{cases} 1 & \text{if } |k| \le K - 1 , \\ 0 & \text{if } |k| \ge K . \end{cases}$$

Its inverse Fourier transform $\xi_K = \mathcal{F}^{-1}(\widehat{\xi}_K)$ is an (entire) Schwartz function.

Note that if f is a Schwartz function, then $\xi_K \star f$ is a Schwartz function whose Fourier transform has support in $[-K, K]^d$, hence it belongs to $\mathcal{B}_K(C)$ for some C, see [23].

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2 Semi-Discrete and Fully-Discrete Approximations

In this section, we propose a spatial discretisation of (1.1) and a fully discrete scheme.

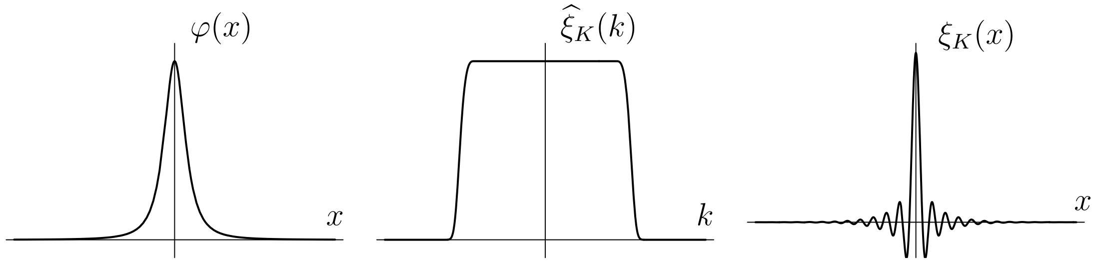

Figure 1: The cutoff functions of Definitions 1.1–1.2

2.1 Galerkin Scheme

The semi-discretisation we describe here is a spectral method. Let $N \in \mathbb{N}$, then we use the Fourier cutoff ξ_K of Definition 1.2 with $K = N$ to define the operators P^N and $\mathrm{Q}^N f := f - \mathrm{P}^N f$ where

$$\mathrm{P}^N f \;:=\; \xi_N \star f \;=\; \mathcal{T}^{-1}\Big(\widehat{\xi}_N \mathcal{T}(f)\Big) \;, \tag{2.1}$$

i.e.,

$$\mathrm{P}^N \sum_{n\in\mathbb{Z}^d} f_n e^{-in\cdot x/L} \;:=\; \sum_{|n|\le NL} \widehat{\xi}_N(n/L) f_n e^{-in\cdot x/L} \;,$$

Notice that P^N truncates to $(2NL)^d$ modes, not $(2N)^d$. In this way $\mathcal{T}\big(\mathrm{P}^N f\big)$ has support contained in $[-N,N]^d$ for all L. The operator P^N is not a projector since $\mathrm{P}^N\mathrm{P}^N \neq \mathrm{P}^N$.

The Galerkin approximation is defined as follows: the solution $u(x,t)$ to (1.1) is replaced by a finite Fourier series

$$u^N(x,t) \;:=\; \sum_{|n|\le NL} u_n(t) e^{-in\cdot x/L} \;. \tag{2.2}$$

The evolution equation is obtained by applying P^N to the nonlinear term of (1.1) and to the initial condition u_0:

$$\partial_t u^N \;=\; \big(\gamma + \nu\Delta\big)u^N + \mathrm{P}^N F(u^N) \;, \qquad u^N(x,0) \;=\; \big(\mathrm{P}^N u_0\big)(x) \;. \tag{2.3}$$

2.2 Fully Discrete Scheme

The time-discretisation is an exact exponential integrator for the linear part and a simple (order 1) quadrature for the nonlinear term appearing in the variation of constants formula. It is similar to that considered in [26], although they need a different definition of discrete Gevrey space, which depends on the time step. The full discretisation is obtained by applying this time-discretisation to the Galerkin scheme (2.3). We use this particular scheme because it makes it straightforward to prove that solutions are (Gevrey) analytic functions (uniformly in the parameters of the scheme, see Appendix B), a fact that we rely on heavily in the next sections.

Let $\mathcal{L} = \gamma + \nu\Delta$ and $\mathcal{K}(x,t)$ be the convolution kernel associated with the operator $\exp(t\mathcal{L})$:

$$\mathcal{K}(x,t) \;=\; \frac{1}{(2\pi)^d} \int e^{-ik\cdot x + (\gamma - \nu|k|^2)t}\, dk \,. \tag{2.4}$$

Note that the operator P^N commutes with $\mathcal{K} \star \cdot$ since both are convolution operators. Let $h > 0$ denote the time-step. Then the fully discrete approximation to $u(x,t)$ is defined iteratively by

$$u^N\big((n+1)h\big) \;=\; \mathcal{K}(h) \star \big(u^N(nh) + h\mathrm{P}^N F\big(u^N(nh)\big)\big) \,. \tag{2.5}$$

In terms of the Fourier coefficients, (2.2), we get

$$\begin{aligned} u^N_m\big((n+1)h\big) \;&=\; e^{h\lambda_m}\,\big(u^N_m(nh) + h\mathrm{P}^N \mathcal{T} F\big(\mathcal{T}^{-1} u^N(nh)\big)_m\big) \\ &=\; e^{h\lambda_m}\,\Big(u^N_m(nh) + h\widehat{\xi}_N(m/L)\mathcal{T} F\big(\mathcal{T}^{-1} u^N(nh)\big)_m\Big) \;, \end{aligned}$$

where $\{\lambda_m\}_{m\in\mathbb{Z}^d}$ are the eigenvalues of $\mathcal{L}$, namely $\lambda_m = \gamma - \nu|m|^2/L^2$, n is the time index, m is the Fourier index, and $\mathcal{T}$ is the Fourier transform (1.2).

For the purposes of analysis, it is useful to consider this scheme in terms of piecewise solutions of a linear differential equation. Indeed, $u^N(x,(n+1)h)$ is the solution at time $t = h$ of

$$\partial_t u(x,t) \;=\; \nu\Delta u(x,t) + \gamma u(x,t) \tag{2.6}$$

with initial condition $u^N(x,nh) + h\mathrm{P}^N F\big(u^N(x,nh)\big)$ at $t = 0$.

Remark. We could apply our techniques to other numerical schemes. We only require the numerical approximation to belong to the Gevrey class $\mathcal{G}_\alpha(C)$ of bounded real analytic functions for some $\alpha > 0$, $C > 0$ (see Appendix D). There exists many wavelet and finite element schemes satisfying this requirement, see [7, 18]. In particular, Propositions D.2 and D.3 provide a natural example of a different basis of analytic functions on which our problem can be decomposed and then a truncation applied: this basis consists of the functions

$$\Psi_{j,k}(x) \;=\; \frac{3e^{ik\cdot x}\sin\big(2x - \frac{1}{3}j\pi\big)\sin\big(6x - j\pi\big)}{(6x - j\pi)^2} \;,$$

for $j,k \in \mathbb{Z}^d$. These functions have the advantage of being localised both in real space and in Fourier space although the numerical implementation is more involved.

3 Definitions and Assumptions

Since we are interested in the large volume limit we specify this dependence in the definitions below.

Assumption 3.1 *For initial data $u_0 \in \mathcal{C}_{\text{per}}([-L\pi, L\pi]^d)$, we assume that*

- *equation (1.1) is the generator of a semi-flow $\Phi_L^t : u_0 \mapsto u(t)$;*
- *for all $N > N_0$, the semi-discrete equation (2.3) is the generator of a semi-flow $\Phi_{L,N}^t : u_0 \mapsto u(t)$;*
- *for all $N > N_0$ and $h < h_0$ the fully discrete equation (2.5) is the generator of a semi-flow $\Phi_{L,N,h}^t : u_0 \mapsto u(t)$ with $t = nh$, $n \in \mathbb{N}$.*

Furthermore we assume for each of the semi-flows above that there exists constants $\alpha > 0$ and $R > 0$, independent L and t, such that $\mathrm{Re}\big(u(t)\big)$ and $\mathrm{Im}\big(u(t)\big)$ belong to the Gevrey class $\mathcal{G}_\alpha(R)$ for all $t > T(u)$ and so $u(t) \in [\mathcal{G}_\alpha(R)]^2$. In other words, the following sets are absorbing balls for their corresponding semi-flows:

$$
\begin{aligned}
B(L) &:= \mathcal{C}_{\text{per}}([-L\pi, L\pi]^d) \bigcap [\mathcal{G}_\alpha(R)]^2 \,, \\
B_N(L) &:= \mathrm{P}^N \mathcal{C}_{\text{per}}([-L\pi, L\pi]^d) \bigcap [\mathcal{G}_\alpha(R)]^2 \,, \\
B_{N,h}(L) &:= \mathrm{P}^N \mathcal{C}_{\text{per}}([-L\pi, L\pi]^d) \bigcap [\mathcal{G}_\alpha(R)]^2 \,.
\end{aligned}
$$

Throughout the paper we use $\widehat{\Phi}^t$ to denote any of the semi-flows (with t taken appropriately) defined above and $\widehat{B}(L)$ to denote the corresponding absorbing balls.

We next define the attractors of the different evolutions introduced above.

Definition 3.2 *We define the following invariant attracting sets for the flows defined in Assumption 3.1*

$$
\begin{aligned}
\mathcal{A}(L) &:= \bigcap_{t>0} \Phi_L^t\big(B(L)\big) \,, \\
\mathcal{A}_N(L) &:= \bigcap_{t>0} \Phi_{L,N}^t\big(B_N(L)\big) \,, \\
\mathcal{A}_{N,h}(L) &:= \bigcap_{n\in\mathbb{N}} \Phi_{L,N,h}^{nh}\big(B_{N,h}(L)\big) \,.
\end{aligned}
$$

Throughout the paper we use $\widehat{\mathcal{A}}(L)$ to denote any of the above attracting sets.

Clearly finite trigonometric sums like (2.2) are entire functions. However, the assumption that there exists a strip around the real axis where u^N is bounded by the same constant for all N is not trivial. Results of this type are know for a number of parabolic partial differential equations of the form (1.1), under the assumptions that F is dissipative in an appropriate sense (see for example [1, 25]). For numerical approximations, existence of semi-flows and global attractors is a well considered problem (see for example [24]). Gevrey regularity of solutions for numerical schemes has not been so widely considered, two different approaches are in [17, 26]. Appendix B contains a sketch of how to obtain this result for the fully discrete scheme given by (2.5). The proof only relies on an a priori L^∞ bound on the solutions and the assumption that the nonlinearity F is analytic.

We next introduce the notion of ε–entropy. The proof that this is a finite quantity will be given in Section 4. From this we define the upper density of dimension.

Definition 3.3 *Let Y be a subset of a metric space X. A set $\mathcal{U} = \{U_1, \ldots, U_N\}$ of open sets in X is called a cover of Y if $\bigcup_{n=1}^N U_n \supset Y$. It is called an ε–cover if $\max_{n=1,\ldots,N} \operatorname{diam}(U_n) \leq \varepsilon$.*

Let $\widehat{\mathcal{A}}(L)$ be endowed with the metric defined by the norm $\|\cdot\|_\infty$. Let

$$\mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L)) := \inf\left\{\operatorname{card}(\mathcal{U}) : \mathcal{U} \text{ is an } \varepsilon\text{–cover of } \widehat{\mathcal{A}}(L)\right\}.$$

We define the ε–entropy H_ε as the limit

$$H_\varepsilon := \limsup_{L\to\infty} \frac{\log \mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L))}{(2L\pi)^d}.$$

The upper density of dimension d_{up} is defined by

$$d_{\mathrm{up}} := \limsup_{\varepsilon\to 0} \frac{H_\varepsilon}{\log \varepsilon^{-1}}.$$

Remark. In [4, 5, 6], H_ε was defined with a limit instead of a limit superior. The existence of the limit followed from a subadditivity argument which cannot be used here because of the boundary conditions. That is, the set $\widehat{\mathcal{A}}(L)$ we are considering here changes with L, whereas in the papers [4, 5, 6], only the topology on $\mathcal{A}$ depended on L, not the set itself. See also [31, 32] for similar results.

Another, more classical notion of entropy is the topological entropy. It serves to measure to complexity of a dynamical system. Similarly to the previous definition, we consider here the spatial density of topological entropy. See Section 5 for results on the topological entropy.

Definition 3.4 *For $\tau > 0$, we define a pseudo-metric $d_{m,\tau}$ on $\mathcal{C}_{\mathrm{per}}([-L\pi, L\pi]^d)$ by*

$$d_{m,\tau}(u, v) := \max_{k=0,\ldots,m-1} \|\widehat{\Phi}^{k\tau}(u) - \widehat{\Phi}^{k\tau}(v)\|_\infty.$$

An (m,ε)*–cover of* $\widehat{\mathcal{A}}(L)$ *is a collection of open sets whose diameter in the metric* $d_{m,\tau}$ *is at most* ε *and whose union contains* $\widehat{\mathcal{A}}(L)$*. Let* $\mathcal{M}_{m,\tau}(\varepsilon,\widehat{\mathcal{A}}(L))$ *be the cardinality of such a minimal* (m,ε)*–cover.*

The (spatial density of) topological entropy is defined as follows:

$$h_{\text{top}} := \limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{1}{(2L\pi)^d}\lim_{m\to\infty}\frac{1}{m\tau}\log\mathcal{M}_{m,\tau}(\varepsilon,\widehat{\mathcal{A}}(L))\ . \tag{3.1}$$

The existence of the first limit in (3.1) can be proved by a subadditivity argument, see [4, 6, 14]. A useful way of computing a lower bound on the topological entropy is by measuring the volume expansion rate (see Section 5.2).

Definition 3.5 *Let* $L\mapsto\mathcal{D}(L)$ *be a family of* ℓ*–dimensional* $\mathcal{C}^\infty$ *sub-manifolds of the absorbing ball* $\widehat{B}$*. We define* $\mathcal{V}$*, the volume expansion rate, by*

$$\mathcal{V} := \limsup_{L\to\infty}\frac{1}{(2L\pi)^d}\limsup_{m\to\infty}\frac{1}{m\tau}\log\mathrm{Vol}_\ell\big(\widehat{\Phi}^{m\tau}(\mathcal{D}(L))\big)\ ,$$

where Vol_ℓ *is the* ℓ*–dimensional (Euclidean) volume.*

4 Upper Bound on the ε–Entropy

We now work towards proving our main result which is a bound on the ε–entropy. First we discuss a preliminary result on the smoothing property of the semi-flow which is proved in Appendix A.

4.1 Smoothing Property of the Semi-Flow

We consider here differences $w=u-v$ of two orbits u and v of the semi-flow $\widehat{\Phi}^t$ of Assumption 3.1. We define functions G_1 and G_2 in such a way that w satisfies

$$\partial_t w = \big(\gamma+\nu\Delta\big)w+\mathrm{P}^N\big(G_1(u,v)w+G_2(u,v)\overline{w}\big) \tag{4.1}$$

for continuous time and

$$w\big((n+1)h\big) = \mathcal{K}(h)\star\big(w(nh)+h\mathrm{P}^N\big(G_1(nh)w(nh)+G_2(nh)\overline{w}(nh)\big)\big) \tag{4.2}$$

for discrete time. From now on we view G_1 and G_2 as functions of x and t (rather than of u and v) and we use the following consequence of Assumption 3.1:

Lemma 4.1 *There exists $\alpha > 0$ and $R > 0$, both independent of N, L, and t, such that $w(t)$, $G_1(t)$ and $G_2(t)$ all belong to $[\mathcal{G}_\alpha(R)]^2$ for all $t > 0$ (and $t/h \in \mathbb{N}$ for (4.2)).*

Remark. We may assume without loss of generality that the R and the α of Assumption 3.1 and Lemma 4.1 are equal, and that they are also equal for the fully continuous, semi-discrete and fully discrete equations.

We compute bounds on the weighted L^2–norm of w shifted in the complex plane over a finite time interval. Instead of taking the usual (flat) L^2 norm over $[-L\pi, L\pi]^d$, which would not behave well in the limit $L \to \infty$, we take a norm over the whole of $\mathbb{R}^d$ weighted with the function φ from Definition 1.1. Therefore, L disappears completely from our estimates. However, in Definition 3.3, we chose to work with the L^∞ topology. We therefore use the following bootstrap argument. From a bound in L^∞ at time $t = 0$, we get a bound in weighted L^2 at time $t = 0$. Using the next lemma we deduce a bound at $t = 1$ in a weighted L^2 space on a strip of the complex plane. This is in turn combined with Lemma C.1 and provides an L^∞ bound at $t = 1$.

Lemma 4.2 *There is a constant $b > 0$ such that for any $\beta \in (-\alpha, \alpha)$, any N and L, the following bound holds on w a solution of (4.1) (or (4.2)) as long as $t \leq 1$ (and $t/h \in \mathbb{N}$ in the case of a fully discrete scheme):*

$$\sup_{|y|\leq L\pi} \int \varphi(x-y)|w(x+i\beta t,t)|^2\,dx \;\leq\; e^{2bt} \sup_{|y|\leq L\pi} \int \varphi(x-y)|w(x,0)|^2\,dx\;. \tag{4.3}$$

The proof of Lemma 4.2 is given in Appendix A.

These L^2 norms shifted in the complex plane can be understood in terms of the classical Gevrey norms. Consider first $\varphi \equiv 1$. Then, using Fourier series and taking $\beta > 0$, we see that

$$\int \big(|f(x+2i\beta)|^2 + |f(x-2i\beta)|^2\big)\,dx \;=\; \Big\|\Gamma e^{\beta(-\Delta)^{1/2}} f\Big\|_2^2\;, \tag{4.4}$$

where Γ is the bounded invertible operator defined by

$$\big(\mathcal{T}(\Gamma f)\big)_n \;=\; (1+e^{-2\beta|n|/L})(\mathcal{T}f)_n\;.$$

This means that the left-hand side of (4.4) is equivalent to a Gevrey norm (similar norms have been used in [11, 12]). We apply a non-constant weight function φ to this norm in order to get estimates which are independent of L and take the sup over $|\beta| \leq \alpha$ to be able to use Lemma C.1. Similar issues have been raised in the paper [21] but our approach is different in that we never explicitly work in Fourier space. We note also that the norms used in [21] grow with the domain size (due to the embedding constant), a problem we avoid here by using the cutoff φ.

4.2 Proof of the Upper Bound

We next show that the ε–entropy H_ε (Definition 3.3) is of order $\log \varepsilon^{-1}$ at most.

Theorem 4.3 *There exists a constant $C < \infty$, independent of ε such that*

$$H_\varepsilon \le C \log \left(\frac{R}{\varepsilon} \right) ,$$

where R is the radius of the absorbing ball $\widehat{B}(L)$ in Assumption 3.1.

The proof is based the following Lemma:

Lemma 4.4 *There is a constant $C > 0$ such that for all $\varepsilon > 0$, the following holds:*

$$H_\varepsilon \le H_{2\varepsilon} + C .$$

Proof. The proof is a consequence of the smoothness result of the previous section. We give the proof for the time continuous cases (1.1), (2.3). The time discrete case (2.5) is similar, it only requires restricting t to multiples of h.

Suppose we are given a 2ε–cover $\{U_1, \dots, U_\mathcal{N}\}$ of $\widehat{\mathcal{A}}(L)$. Then by invariance of $\widehat{\mathcal{A}}$ the set

$$\{\widehat{\Phi}^t(U_1), \dots, \widehat{\Phi}^t(U_\mathcal{N})\}$$

is a cover of $\widehat{\mathcal{A}}(L)$ for all $t > 0$. Moreover, if $u, v \in U_1$, by Lemma C.1 combined with Lemma 4.2, we have

$$\sup_{|x| \le L\pi\,,\, 2|y| \le \alpha} \left| \big(\widehat{\Phi}^1(u) - \widehat{\Phi}^1(v)\big)(x + iy) \right| \le C\varepsilon .$$

That is, if we let $w = \widehat{\Phi}^1(u) - \widehat{\Phi}^1(v)$, then $w \in [\mathcal{G}_{\alpha/2}(C\varepsilon)]^2$ with C independent of L and ε.

We now use an argument due to Tikhomirov [28], discussed in [15], §8, Theorem XXII. By Proposition D.2 w can be written as

$$w(z) = \sum_{n \in \mathbb{Z}^d} e^{-\alpha |n|/2} e^{in\cdot z} w_n(z) , \tag{4.5}$$

with w_n in the Bernstein class $[\mathcal{B}_2(C'\varepsilon)]^2$ (see Appendix D for the definition of $\mathcal{B}_2$). Thus, splitting the sum in (4.5) in two, we can find a K independent of ε and L, and a $\widetilde{w} \in [\mathcal{B}_K(C'\varepsilon)]^2$, such that

$$\|w - \widetilde{w}\|_\infty \le \frac{\varepsilon}{2} .$$

If $\widetilde{w} \in [\mathcal{B}_K(C\varepsilon)]^2$, then by Proposition D.3,

$$\widetilde{w}(x) = \sum_{n\in\mathbb{Z}^d} \widetilde{w}\big(x_K(n)\big)\mathcal{F}_K\big(x - x_K(n)\big) ,$$

hence there is a $\delta > 0$ depending only on K such that $\|\widetilde{w}\|_\infty \leq \varepsilon/2$ if $|\widetilde{w}(x_K(n))| \leq \delta\varepsilon$ for all $n \in \mathbb{Z}^d$ for which $x_K(n) = (n\pi)/(3K) \in [-L\pi, L\pi]^d$. There are $c(K)(2L\pi)^d$ such points, hence at most

$$\left(\frac{C\varepsilon}{\delta\varepsilon}\right)^{c(K)(2L\pi)^d} =: C_*^{(2L\pi)^d}$$

balls of radius $\varepsilon/2$ will be needed to cover $[\mathcal{B}_K(C\varepsilon)]^2$. This covers all the functions $\widetilde{w}$ obtained from the set $\widehat{\Phi}^1(U_1)$ by the above construction. Consequently, $\widehat{\Phi}^1(U_1)$ can be covered with the same number of balls of diameter ε.

Repeating the operation with each one of the $\mathcal{N}(2\varepsilon, \widehat{\mathcal{A}}(L))$ sets of diameter 2ε of the original cover $\{U_1, \ldots, U_\mathcal{N}\}$, we obtain a cover with at most

$$\mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L)) \leq \mathcal{N}(2\varepsilon, \widehat{\mathcal{A}}(L))C_*^{(2L\pi)^d}$$

elements. Taking the logarithm, dividing by $(2L\pi)^d$ and passing to the limit $L \to \infty$, we obtain Lemma 4.4.

Proof of Theorem 4.3. It trivially holds that $H_R = 0$, because $\mathcal{N}(R, \widehat{\mathcal{A}}(L)) = 1$ by Assumption 3.1. Let k be the smallest integer larger than $\log(R/\varepsilon)/\log 2$, then by Lemma 4.4 we have

$$H_\varepsilon \leq H_{2\varepsilon} + C \leq \cdots \leq H_{2^k\varepsilon} + Ck \leq C' \log R/\varepsilon .$$

This proves Theorem 4.3.

5 The Topological Entropy

5.1 Upper Bound by the Dimension

In this section, we prove that the topological entropy of the attractors $\widehat{\mathcal{A}}$ is bounded by a multiple of the upper density of dimension, a quantity related to the ε–entropy. The corresponding inequality for finite dimensional dynamical systems is well-known, see [14].

Theorem 5.1 *There is a $b < \infty$ such that*

$$h_{\text{top}} \leq bd_{\text{up}} < \infty . \tag{5.1}$$

Proof. The right-hand inequality is a direct consequence of Theorem 4.3. The left-hand inequality follows from the arguments in [4, 14] that we summarise here. Let $\rho > 0$ be such that $H_\varepsilon \leq \big(d_{\rm up} + \rho\big) \log 1/\varepsilon$ for all $\varepsilon < \varepsilon_0$ and then let $L_0 = L_0(\varepsilon, \rho)$ be such that for all $L > L_0$,

$$\frac{\log \mathcal{N}(\varepsilon, \widehat{\mathcal{A}}(L))}{(2L\pi)^d} \;\leq\; H_\varepsilon + \rho \;\leq\; \big(d_{\rm up} + \rho\big) \log \frac{1}{\varepsilon} + \rho \;.$$

By iterating Lemma C.1 and Lemma 4.2, there is a $b > 0$ such that for all L and all (sufficiently small) $\varepsilon > 0$, if $\|u - v\|_\infty \leq \varepsilon$ then for $t > 0$,

$$\|\widehat{\Phi}^t(u) - \widehat{\Phi}^t(v)\|_\infty \;\leq\; e^{bt}\varepsilon \;.$$

Let $\varepsilon' = \exp(-bT)\varepsilon$. Let an ε'–cover of $\widehat{\mathcal{A}}(L)$ (in the sense of Definition 3.3) be given. Then it is also a $(T/\tau, \varepsilon)$–cover (in the sense of Definition 3.4), hence

$$\mathcal{M}_{T/\tau,\tau}(\varepsilon, \widehat{\mathcal{A}}(L)) \;\leq\; \mathcal{N}(\varepsilon', \widehat{\mathcal{A}}(L)) \;.$$

It follows that

$$\begin{aligned}
h_{\rm top} \;&=\; \limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \frac{1}{(2L\pi)^d} \lim_{T\to\infty} \frac{1}{T} \log \mathcal{M}_{T/\tau,\tau}(\varepsilon, \widehat{\mathcal{A}}(L)) \\
&=\; \limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \frac{1}{(2L\pi)^d} \inf_{T} \frac{1}{T} \log \mathcal{M}_{T/\tau,\tau}(\varepsilon, \widehat{\mathcal{A}}(L)) \\
&\leq\; \limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \frac{1}{T} \frac{\log \mathcal{N}(\varepsilon', \widehat{\mathcal{A}}(L))}{(2L\pi)^d} \\
&\leq\; \limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \frac{1}{T}\Big((d_{\rm up} + \rho) \log \frac{1}{\varepsilon'} + \rho\Big) \;.
\end{aligned}$$

Since $\log 1/\varepsilon' = bT + \log 1/\varepsilon$, the limit $T \to \infty$ and $\rho \to 0$ leaves only $bd_{\rm up}$ on the r.h.s. above.

5.2 Lower Bound by the Expansion Rate

We provide here a way of computing a lower bound on the topological entropy (hence on the upper dimension $d_{\rm up}$ by Theorem 5.1), based on Yomdin's Theorem [30], an account of which may be found in [22].

Theorem 5.2 *Let $h_{\rm top}$ be as in Definition 3.4. Then for all choices of $\mathcal{D}(L)$ in Definition 3.5,*

$$\mathcal{V} \;\leq\; h_{\rm top} \;.$$

Remark. The lower bound in [5] is in the same spirit. An adequate sequence of sub-manifolds is chosen (small balls around the trivial solution). The volume expansion rate of that sequence can be controlled, yielding a lower bound on the (ε–)entropy.

Proof. The proof follows from the argument by Yomdin and Gromov [10, 30]. By a Lemma of Gromov [10], there exists a $C > 0$ such that if $\widehat{\Phi}^\tau$ is $\mathcal{C}^r$, then

$$\mathrm{Vol}_\ell\big(\widehat{\Phi}^{m\tau}(\mathcal{D}(L))\big) \;\leq\; \mathcal{M}_{m,\tau}(\varepsilon, \widehat{\mathcal{A}}(L))(C\|D\widehat{\Phi}^\tau\|_\infty)^{m\ell/r} \;,$$

hence

$$\begin{aligned}
&\limsup_{L\to\infty} \frac{1}{(2L\pi)^d} \limsup_{m\to\infty} \frac{1}{m\tau} \log \mathrm{Vol}_\ell\big(\widehat{\Phi}^{m\tau}(\mathcal{D}(L))\big) \\
&\quad\leq\; \limsup_{L\to\infty} \frac{1}{(2L\pi)^d} \limsup_{m\to\infty} \frac{1}{m\tau} \log \mathcal{M}_{m,\tau}(\varepsilon, \widehat{\mathcal{A}}(L)) \\
&\qquad + \limsup_{L\to\infty} \frac{\ell/r}{(2L\pi)^d} \log\big(C^{1/\tau}\|D\widehat{\Phi}^\tau\|_\infty^{1/\tau}\big) \;.
\end{aligned}$$

Since τ can be arbitrarily large, the constant C drops out, and since $\widehat{\Phi}^\tau$ is $\mathcal{C}^\infty$, the second term is arbitrarily small by letting $r \to \infty$. The first term tends to h_{top} upon letting $\varepsilon \to 0$.

6 Upper Semicontinuity of the Infinite Volume Attractors

In this section we discuss four different invariant sets and their mutual relationship. The first two invariant sets are $\mathcal{A}_{N,h}(L)$ and $\mathcal{A}(L)$ from Definition 3.2. Then we also introduce two large volume limits:

$$\mathcal{A}_{N,h}(\infty) \;:=\; \overline{\bigcup_{L\in\mathbb{N}} \mathcal{A}_{N,h}(L)} \;, \qquad \mathcal{A}(\infty) \;:=\; \overline{\bigcup_{L\in\mathbb{N}} \mathcal{A}(L)} \;, \tag{6.1}$$

where the closure is taken in the uniformly local topology of [19]. We define the distance between a point and a set and between two sets in the standard way

$$\begin{aligned}
\mathrm{dist}\big(U, \mathcal{V}\big) \;&:=\; \inf_{V\in\mathcal{V}} \|U - V\|_{L^\infty([-L\pi,L\pi]^d)} \;, \\
\mathrm{dist}\big(\mathcal{U}, \mathcal{V}\big) \;&:=\; \sup_{U\in\mathcal{U}} \mathrm{dist}\big(U, \mathcal{V}\big) \;.
\end{aligned}$$

We claim that

$$\lim_{N\to\infty, h\to 0} \mathrm{dist}\big(\mathcal{A}_{N,h}(L), \mathcal{A}(L)\big) \;=\; 0 \;, \lim_{N\to\infty, h\to 0} \mathrm{dist}\big(\mathcal{A}_{N,h}(\infty), \mathcal{A}(\infty)\big) \;=\; 0 \;, \tag{6.2}$$

and the following relations are straightforward from (6.1):

$$\begin{aligned}
\lim_{L\to\infty} \mathrm{dist}\big(\mathcal{A}_{N,h}(L), \mathcal{A}_{N,h}(\infty)\big) \;&=\; 0 \;, \\
\lim_{L\to\infty} \mathrm{dist}\big(\mathcal{A}(L), \mathcal{A}(\infty)\big) \;&=\; 0 \;.
\end{aligned}$$

Hence we obtain the following diagram, in which each arrow represents a relation of upper semi-continuity:

$$\begin{tikzcd} \mathcal{A}_{N,h}(L) \arrow[r, "N\to\infty", "h\to 0"'] \arrow[d, "L\to\infty"'] & \mathcal{A}(L) \arrow[d, "L\to\infty"] \\ \mathcal{A}_{N,h}(\infty) \arrow[r, "N\to\infty", "h\to 0"'] & \mathcal{A}(\infty)\ . \end{tikzcd}$$

The relation (6.2) is a consequence of the following (see *e.g.* [13, 17, 19, 20]).

Theorem 6.1 *For all $\varepsilon > 0$, there is a T_1, an h_1 and an N_1 such that if $h < h_1$ and $N > N_1$, then for all $L \in \mathbb{N}$*

$$\Phi^T_{L,N,h}\left(B_{N,h}(L)\right) \subset \mathcal{U}_\varepsilon(\mathcal{A}(L)) \qquad \forall T > T_1\ ,$$

where $\mathcal{U}_\varepsilon(\mathcal{A}(L))$ is the ε–neighbourhood of $\mathcal{A}(L)$ in L^∞.

Proof. The proof is by induction using the attracting property of the attractor and a finite time error estimate.

By the attraction property of $\mathcal{A}(L)$, there exists a T such that $\forall T > T_1$

$$\Phi^T_L\left(B(L) \cup B_{N,h}(L)\right) \subset \mathcal{U}_{\varepsilon/2}(\mathcal{A}(L))\ ,$$

for all $L \in \mathbb{N}$. Hence for any $u_0 \in B_{N,h}(L)$ we have

$$\begin{aligned} \operatorname{dist}\left(\Phi^{nh}_{L,N,h}(u_0), \mathcal{A}(L)\right) &= \inf_{u\in\mathcal{A}(L)} \|\Phi^{nh}_{L,N,h}(u_0) - u\|_\infty \\ &\le \inf_{u\in\mathcal{A}(L)} \|\Phi^{nh}_L(u_0) - u\|_\infty + \|\Phi^{nh}_{L,N,h}(u_0) - \Phi^{nh}_L(u_0)\|_\infty \\ &\le \frac{\varepsilon}{2} + \|\Phi^{nh}_{L,N,h}(u_0) - \Phi^{nh}_L(u_0)\|_\infty\ , \end{aligned} \tag{6.3}$$

provided $nh > T$.

We next show that N, h can be chosen in such a way that the second term above is smaller than $\varepsilon/2$ for all $T \in (0, 2T_1]$.

Let $v(t) = \Phi^t_L(u_0)$ and $w(nh+s) = \Phi^s_{\mathrm{Lin}}\Phi^{nh}_{L,N,h}(u_0)$ where Φ^s_{Lin} is the solution semi-flow of (2.6). We thus have for $s < h$

$$\begin{aligned} \partial_t\big(v(nh+s) - w(nh+s)\big) &= (\gamma+\nu\Delta)\big(v(nh+s) - w(nh+s)\big) + F\big(v(nh+s)\big) \\ &= (\gamma+\nu\Delta)\big(v(nh+s) - w(nh+s)\big) + \mathrm{P}^N\Big(F\big(v(nh+s)\big) - F\big(w(nh+s)\big)\Big) \\ &\quad -\mathrm{P}^N F\big(w(nh+s)\big) + \mathrm{Q}^N F\big(v(nh+s)\big)\ . \end{aligned}$$

Using Proposition D.2 we see that

$$\sup_{s<h} \|\mathrm{Q}^N F\big(v(nh+s)\big)\|_\infty \le C(R)e^{-\alpha N} .$$

It is also quite easy (using Fourier transforms) to see that

$$\| \int_0^h \big(\mathrm{P}^N F\big(w((n+1)h)\big) - \mathcal{K}(s) \star \mathrm{P}^N F\big(w((n+1)h-s)\big)\big)\, ds\|_\infty \le C(R)h .$$

Hence, using the same analysis as in the proof of Lemma 4.2, we obtain

$$\|v((n+1)h) - w((n+1)h)\|_\infty \le e^{ch}\|v(nh) - w(nh)\|_\infty + C(R)h\big(1 + e^{-\alpha N}\big) .$$

By iteration, we obtain

$$\|v(nh) - w(nh)\|_\infty \le e^{cnh}\|v(0) - w(0)\|_\infty + C(R)e^{cnh}h(1 + e^{-\alpha N}) . \tag{6.4}$$

Taking h small enough, we can make the second term of (6.3) smaller than $\varepsilon/2$ for all $T \in (0, 2T_1]$.

To complete the induction we note that the absorbing ball is forward invariant and so we can repeat the argument for $T > 2T_1$.

7 Discussion: The Complex Ginzburg–Landau Equation

An interesting example to which our results apply is the (cubic) complex Ginzburg–Landau equation in $d = 1$ space dimension

$$\partial_t u(x,t) = (1+ia)\partial_x^2 u(x,t) + u(x,t) - (1+ib)|u(x,t)|^2 u(x,t) . \tag{7.1}$$

In terms of the notations of (1.1), we have:

$$d = 1 , \quad \nu = 1 + ia , \quad \gamma = 1 , \quad F(u) = -(1+ib)|u|^2 u .$$

Remark that the equation for the difference $w = u - v$ of two solutions u and v that we use in Section 4.1 admits a simple expression:

$$\begin{aligned} \partial_t w(x,t) &= (1+ia)\partial_x^2 w(x,t) + w(x,t) \\ &\quad + \int \xi_N(x-y)\Big(G_1(y,t)w(y,t) + G_2(y,t)\overline{w}(y,t)\Big)dy , \end{aligned}$$

where

$$G_1(x,t) = -(1+ib)\big(|u(x,t)|^2 + |v(x,t|^2\big) , \qquad G_2(x,t) = -(1+ib)u(x,t)v(x,t) .$$

The CGL equation (7.1) arises as a ‘normal form’ in certain types of bifurcation with continuous spectrum, see [1, 3]. Assumption 3.1 for the continuous case follows from the works [2, 1, 25]. In particular, the following results have been proved:

Theorem 7.1 *Equation 7.1 defines a semi-flow Φ^t on $L^\infty(\mathbb{R})$ which has an absorbing ball B in $\mathcal{G}_\alpha(C)$ for some $C > 0$ and $\alpha > 0$ (see Appendix D). The attractor $\mathcal{A} = \cap_{t>0}\Phi^t(B)$ exists and is compact in $L^\infty([-L, L])$ for any $L > 0$.*

Remark that these results hold on the whole space without boundary conditions, but they obviously remain true on the set of spatially periodic solutions, which is invariant under the time evolution.

The following rigorous upper and lower bounds on the ε–entropy in unbounded volumes were obtained in [5]:

Theorem 7.2 *Let $\mathcal{A}$ be the attractor of Eq.(7.1) for general initial conditions in $L^\infty(\mathbb{R})$ and let $\mathcal{N}(\varepsilon, \mathcal{A})$ be the minimum the number of balls in an ε–cover of $\mathcal{A}$ in the topology of $L^\infty([-L, L])$. There is a $C > 0$ for which*

$$C^{-1}\log(1/\varepsilon) \;\leq\; H_\varepsilon(\mathcal{A}) \;=\; \lim_{L\to\infty} \frac{\log \mathcal{N}(\varepsilon, \mathcal{A})}{2L} \;\leq\; C\log(1/\varepsilon)\;.$$

In particular, the limit exists.

The discretisation (2.5) in the particular case of the CGL equation is

$$u_m^N\big((n+1)h\big) \;=\; e^{(1-(1+ia)m^2)nh}\Big(1 - h(1+ib)\widehat{\xi}_N(m/L)|u_m^N(nh)|^2\Big)u_m^N(nh)\;, \tag{7.2}$$

where $n = 0, 1, \ldots$ is the time index and $m = -N, \ldots, N$ is the Fourier index.

A closely related time discretisation was considered in [26]. Although there is no formal proof of existence of a semi-flow and global attractor for the modified Galerkin scheme considered here, this can be seen to be true by considering the error bound 6.4 and the results of Theorem 7.1 over a finite time interval $[0, T]$. This suffices to prove that the discretised evolution is well defined and solutions stay bounded on that time interval. Iterating over $[qT, (q+1)T]$ for all $q > 0$ we obtain the existence of a global semi-flow. The proof of existence of the absorbing balls of Assumption 3.1 is sketched in Appendix B. This implies that the following theorem holds as a special case of Theorem 4.3:

Theorem 7.3 *Consider the CGL equation 7.2. There exists a constant $C < \infty$, independent of ε such that*

$$H_\varepsilon \;\leq\; C\log\left(\frac{R}{\varepsilon}\right)\;,$$

where R is the radius of the absorbing ball B in Gevrey space for 7.2, and H_ε is defined in Definition 3.3.

A Proof of Lemma 4.2

We first consider the time-continuous case (4.1). We write the analytic extension of w as a vector valued function with components w_{r} and w_{i} (each of which is complex-valued) and its complex argument $x + iy$ is also written as a vector of reals. Namely

$$w(x+iy,t) \;=\; \big(w_{\mathrm{r}}(x,y;t), w_{\mathrm{i}}(x,y;t)\big)\;.$$

As a preparation for the proof, we estimate the following expression:

$$\begin{aligned}&\mathrm{Re}\,\nu\int \varphi(x)\Big(\overline{w}_{\mathrm{r}}(x,y;t)\Delta_x w_{\mathrm{r}}(x,y;t)+\overline{w}_{\mathrm{i}}(x,y;t)\Delta_x w_{\mathrm{i}}(x,y;t)\Big)\,dx\\ &\quad+\mathrm{Re}\,i\beta\int \varphi(x)\Big(\overline{w}_{\mathrm{r}}(x,y;t)\nabla_y w_{\mathrm{r}}(x,y;t)+\overline{w}_{\mathrm{i}}(x,y;t)\nabla_y w_{\mathrm{i}}(x,y;t)\Big)\,dx \end{aligned}\tag{A.1}$$

By using the Cauchy–Riemann equations ($|\nabla_y u_{\mathrm{r,i}}| = |\nabla_x u_{\mathrm{r,i}}|$), we obtain:

$$\begin{aligned}&\mathrm{Re}\,\nu\int\varphi\Big(\overline{w}_{\mathrm{r}}\Delta_x w_{\mathrm{r}}+\overline{w}_{\mathrm{i}}\Delta_x w_{\mathrm{i}}\Big)\,dx+\mathrm{Re}\,i\beta\int\varphi\Big(\overline{w}_{\mathrm{r}}\nabla_y w_{\mathrm{r}}+\overline{w}_{\mathrm{i}}\nabla_y w_{\mathrm{i}}\Big)\,dx\\
=\;\;& -\mathrm{Re}\,\nu\int\varphi\big(|\nabla_x w_{\mathrm{r}}|^2+|\nabla_x w_{\mathrm{i}}|^2\big)\,dx-\mathrm{Re}\,\nu\int\nabla_x\varphi\Big(\overline{w}_{\mathrm{r}}\nabla_x w_{\mathrm{r}}+\overline{w}_{\mathrm{i}}\nabla_x w_{\mathrm{i}}\Big)\,dx\\
&+\mathrm{Re}\,i\beta\int\varphi\Big(\overline{w}_{\mathrm{r}}\nabla_y w_{\mathrm{r}}+\overline{w}_{\mathrm{i}}\nabla_y w_{\mathrm{i}}\Big)\,dx\\
\le\;\;& -\mathrm{Re}\,\nu\int\varphi\big(|\nabla_x w_{\mathrm{r}}|^2+|\nabla_x w_{\mathrm{i}}|^2\big)\,dx+|\nu|\,\Big\|\frac{\nabla\varphi}{\varphi}\Big\|_\infty\int\varphi\Big(|w_{\mathrm{r}}||\nabla_x w_{\mathrm{r}}|+|w_{\mathrm{i}}||\nabla_x w_{\mathrm{i}}|\Big)\,dx\\
&+|\beta|\int\varphi\Big(|w_{\mathrm{r}}||\nabla_x w_{\mathrm{r}}|+|w_{\mathrm{i}}||\nabla_x w_{\mathrm{i}}|\Big)\,dx\\
\le\;\;& \frac{|\beta|^2+|\nu|^2\|\nabla\varphi/\varphi\|_\infty^2}{2\mathrm{Re}\nu}\int\varphi\big(|w_{\mathrm{r}}|^2+|w_{\mathrm{i}}|^2\big)\,dx\\
=:\;\;& b_0\int\varphi\big(|w_{\mathrm{r}}|^2+|w_{\mathrm{i}}|^2\big)\,dx\;. \end{aligned}\tag{A.2}$$

Define

$$\varphi_y(x)\;:=\;\varphi(x-y)\;,\qquad \xi_y^*(x)\;:=\;\xi_N(x-y)\;,$$

where φ and ξ_N are as in Definitions 1.1–1.2. We next compute the time-derivative of the left-hand side of (4.3). The expression (A.1) is the linear part of the time-derivative, hence we simply insert the bound (A.2) and compute the non-linear part:

$$\begin{aligned}&\frac12\partial_t\sup_y\int\varphi_y(x)|w(x+i\beta t,t)|^2\,dx\;\le\;\frac12\sup_y\partial_t\int\varphi_y(x)|w(x+i\beta t,t)|^2\,dx\\
&\quad\le\;\;(\gamma+b_0)\sup_y\int\varphi_y(x)|w(x+i\beta t,t)|^2\,dx\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Re}\sup_y \Big| \int \varphi_y(x)\overline{w}(x+i\beta t,t) \\
&\times \left(\int \xi_x^*(z) \Big(G_1(z+i\beta t,t)\overline{w}(z+i\beta t,t) + G_2(z+i\beta t,t) w(z+i\beta t,t) \Big) dz \right) dx \Big| \\
\leq\ & \big(\gamma + b_0\big) \sup_y \int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \\
&+ \sup_y \int \varphi_y(x) |w(x+i\beta t,t)| \\
&\times \left(\int \frac{|\xi_x^*(z)|}{\sqrt{\varphi_x(z)}} \sqrt{\varphi_x(z)} \Big(|G_1(z+i\beta t,t)| + |G_2(z+i\beta t,t)| \Big) |w(z+i\beta t,t)| dz \right) dx \, .
\end{aligned}
$$

At this point, we apply the Cauchy–Schwarz inequality to each of the two integrals on the right-hand side. Using Lemma 4.1 we know that

$$
\sup_{|\beta|\leq\alpha} \sup_{t\leq 1} \sup_{x\in\mathbb{R}^d} \big(|G_1(x+i\beta,t)| + |G_2(x+i\beta,t)| \big) \;\leq\; 2R \, .
$$

This gives

$$
\begin{aligned}
\frac{1}{2}\partial_t \sup_y \int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \;&\leq\; \big(\gamma+b_0\big) \sup_y \int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \\
&+ \sup_y \left(\int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \right)^{1/2} \left(\int \varphi(x)\, dx \int \frac{\xi_N^2(z)}{\varphi(z)} \, dz \right)^{1/2} \\
&\quad \times 2R \left(\sup_x \int \varphi_x(z) |w(z+i\beta t,t)|^2 \, dz \right)^{1/2} \\
&\leq\; \left(\gamma + b_0 + 2R \left(\int \frac{\xi_N^2}{\varphi} \right)^{1/2} \right) \sup_y \int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \\
&=:\; b \sup_y \int \varphi_y(x) |w(x+i\beta t,t)|^2 \, dx \, ,
\end{aligned}
$$

where we used that by Definition 1.1, $\int \xi_N^2/\varphi < \infty$ because ξ_N^2 is a Schwartz function and $1/\varphi$ a Schwartz distribution. Equation (4.3) now follows from Gronwall's Lemma.

In the discrete case, we solve the linear differential equation (see (2.6))

$$
\partial_t w(nh+t) \;=\; \big(\gamma + \nu\Delta\big) w(nh+t)
$$

for $t \in [0,h)$ with initial condition $w(nh) + h\xi_N \star \big(G_1(nh)w(nh) + G_2(nh)\overline{w}(nh)\big)$, and then we iterate for $n = 0$ to $n = [1/h]+1$. Over one time-step, the same calculations as in the continuous case give

$$
\begin{aligned}
&\sup_{|y|\leq L\pi} \int \varphi(x-y) |w(x+i\beta(n+1)h,(n+1)h)|^2 \, dx \\
&\qquad \leq\; e^{2bh} \sup_{|y|\leq L\pi} \int \varphi(x-y) |w(x+i\beta nh, nh)|^2 \, dx \, ,
\end{aligned}
$$

and similarly

$$\begin{aligned}
&\sup_{|y|\le L\pi}\int \varphi(x-y)|h\xi_N \star \big(G_1((n+1)h)w((n+1)h)+G_2((n+1)h)\overline{w}((n+1)h)\big)|^2\,dx\\
&\le\ (2Rh)^2\left(\int\frac{\xi_N^2}{\varphi}\right)\sup_{|y|\le L\pi}\int\varphi(x-y)|w(x+i\beta(n+1)h,(n+1)h)|^2\\
&\le\ e^{Ch}\sup_{|y|\le L\pi}\int\varphi(x-y)|w(x+i\beta(n+1)h,(n+1)h)|^2\,,
\end{aligned}$$

hence we can iterate:

$$\sup_{|y|\le L\pi}\int\varphi(x-y)|w(x+i\beta nh,nh)|^2\,dx\ \le\ e^{2bnh}\sup_{|y|\le L\pi}\int\varphi(x-y)|w(x,0)|^2\,dx\,.$$

This completes the proof of Lemma 4.2.

B Analyticity for the Fully Discrete Scheme

The full discretisation discussed in Section 2.2 is similar to that introduced in [26], where Gevrey regularity is proved. We give here another simple and direct proof that the semi-group generated by (2.5) maps into $\mathcal{G}_\alpha(C)$ (see Appendix D) for some α and C independent of N and L. Our proof is in the spirit of Collet [1] or Takáč *et al.* [25]. We assume that the solution $u(x,nh)$ of (2.5) has reached an absorbing ball in L^∞, hence there is an $R>0$ such that $\|u(nh)\|_\infty\le R$ irrespective of u_0 and n. We then use a contraction argument to show that for small T, for $nh\in[0,T]$, there is a unique solution to (2.5) in the metric space of functions satisfying $|||u|||\le R$, where

$$|||f|||\ =\ \max_{nh\in[0,T]}\sup_{|x|\le L\pi}|f(x+i\sqrt{nh},nh)|\,.$$

Remark that if $T<h$, there is nothing to prove (the solutions are entire functions anyway). The purpose of this section is to provide bounds on the radius of analyticity which are independent of h and N, hence we may assume h to be small.

We seek a solution to the equation $u(nh)=\mathcal{Y}\big(u,u_0\big)(nh)$ with $\mathcal{Y}$ defined by

$$\mathcal{Y}\big(f,f_0\big)(nh)\ =\ \mathcal{K}(nh)\star f_0+\sum_{j=0}^{n-1}h\mathcal{K}\big(h(n-j)\big)\star \mathrm{P}^N F(f(jh))\,,$$

where $\mathcal{K}$ is given by (2.4).

It is easy to see that for small $T>0$, $\mathcal{Y}(\cdot,f_0)$ is a contraction:

$$\big|\mathcal{Y}\big(f,f_0\big)(x+i\sqrt{nh},nh)-\mathcal{Y}\big(g,f_0\big)(x+i\sqrt{nh},nh)\big|$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{n-1} \int h|\mathrm{P}^N \mathcal{K}\big(y - z + i(\sqrt{nh} - \sqrt{jh}), h(n-j)\big)| \\
&\qquad \times \Big| F\big(f(z + i\sqrt{jh}, jh)\big) - F\big(g(z + i\sqrt{jh}, jh)\big) \Big| dz \\
&\leq \mathrm{Lip}(F, R)\, |\!|\!| f - g |\!|\!| \sum_{j=0}^{[T/h]} \int h|\mathrm{P}^N \mathcal{K}\big(x + i(\sqrt{nh} - \sqrt{jh}), h(n-j)\big)|\, dx
\end{aligned}
$$

(here $\mathrm{Lip}(F, R)$ is the Lipschitz constant of F in the ball of radius R) hence by taking T small enough (depending on $\mathrm{Lip}(F, R)$ only) the solution to the fixed point problem exists and is unique. Since u belongs to an absorbing ball of L^∞, the argument can be iterated indefinitely, hence u is analytic for all times thereafter.

C Uniform Bounds on Complex Analytic Functions

In this section we show that an L^p bound in a strip of the complex plane provides an L^∞ bound in a smaller strip.

Lemma C.1 *Let $p \geq 1$. There is a constant $C = C(\varphi, \delta)$ such that any function f analytic in $|\mathrm{Im}(x)| \leq \delta$ satisfies:*

$$
|f(y + iz)|^p \;\leq\; C \sup_{|\gamma| \leq \delta} \int \varphi(x - y) |f(x + i\gamma)|^p \, dx \; ,
$$

for all $y \in \mathbb{R}^d$ and $|z| \leq \delta/2$.

Proof. We take $y = 0$ and $\delta = 1$ for simplicity. The general case is obtained by translation and scaling. Since analytic functions are harmonic the following Mean Value Property holds (see [16]). Let $\mathcal{D}$ be the unit ball centred at 0 in the n–dimensional complex space, then

$$
f(0) \;=\; \frac{1}{\mathrm{Vol}(\mathcal{D})} \int_{\mathcal{D}} f(x + i\gamma)\, dx\, d\gamma \; .
$$

We apply Jensen's inequality and use that there is a C for which

$$
\inf_{|x| \leq 1} C\varphi(x) \;\geq\; 1
$$

(see Definition 1.1), to obtain

$$
|f(0)|^p \;\;\leq\;\; \frac{1}{\mathrm{Vol}(\mathcal{D})} \int_{\mathcal{D}} |f(x + i\gamma)|^p \, dx\, d\gamma
$$

$$
\begin{aligned}
&\leq \frac{1}{\mathrm{Vol}(\mathcal{D})} \sup_{|\gamma|\leq 1} \int_{|x|\leq 1} |f(x+i\gamma)|^p \, dx \\
&\leq \frac{C}{\mathrm{Vol}(\mathcal{D})} \sup_{|\gamma|\leq 1} \int \varphi(x) |f(x+i\gamma)|^p \, dx \, .
\end{aligned}
$$

D Gevrey and Bernstein Classes of Analytic Functions

We introduce here the metric spaces $\mathcal{B}_\sigma(C)$ (the Bernstein class) and $\mathcal{G}_\alpha(C)$ (the Gevrey class) and recall two properties of functions belonging to these spaces (see [7, 15, 18] for details).

Definition D.1 *The Bernstein class $\mathcal{B}_\sigma(C)$ is the set of all functions f having an analytic extension to the whole of $\mathbb{C}^d$ with exponential growth along the imaginary directions:*

$$
|f(x+iy)| \leq Ce^{\sigma|y|} \, , \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \, .
$$

The Gevrey class $\mathcal{G}_\alpha(C)$ is the set of all functions f admitting an analytic extension to a strip of width 2α around the real axes and which are uniformly bounded in this strip:

$$
|f(x+iy)| \leq C \, , \forall (x,y) \in \mathbb{R}^d \times [-\alpha, \alpha]^d \, .
$$

The first result states that any function in $\mathcal{G}_\alpha(C)$ can be written as a sum of entire functions:

Proposition D.2 *Let $f \in \mathcal{G}_\alpha(C)$. Then there exists a C' depending on C only such that*

$$
f(z) = \sum_{n\in\mathbb{Z}^d} e^{-\alpha|n|} e^{in\cdot z} f_n(z) \, ,
$$

with $f_n \in \mathcal{B}_2(C')$.

The second result is a classical sampling formula (see [7] or [15] where it is called the Cartwright formula).

Proposition D.3 *For all $f \in \mathcal{B}_\sigma(C)$, the following identity holds:*

$$
f(z) = \sum_{n\in\mathbb{Z}^d} f\big(x_\sigma(n)\big) \mathcal{F}_\sigma\big(z - x_\sigma(n)\big) \, ,
$$

where

$$
x_\sigma(n) = \frac{n\pi}{3\sigma} \, , \qquad \mathcal{F}_\sigma(x) = \frac{\sin(3\sigma x)\sin(\sigma x)}{3\sigma^2 x^2} \, .
$$

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