GAUGE DEPENDENCE OF THE ONE-LOOP EFFECTIVE ACTION IN GRAVITY

D. JOHNSTON

Laboratoire de Physique Théorique et Hautes Energies [1], *Université de Paris XI, Bâtiment 211, F-91405 Orsay, France*

Received 10 November 1986

We rederive a result of Cohler and Chodos concerning the equivalence of the one-loop effective action in various gauges in the context of Nielsen identities. We also note that one of the conditions they impose to obtain equivalence must hold to all orders.

In this short note we reexamine the proof of Cohler and Chodos that different choices of gauge will yield the same effective action to one-loop order provided that certain conditions are satisfied [1]. As in their work we have been prompted by investigations of self-consistent dimensional reduction in Kaluza-Klein theories but the result is also valid for the effective potential of a spontaneously broken Yang–Mills theory coupled to scalars. We shall use the framework of Nielsen identities [2,3] and the method of extended BRS [4,5] transformations to rederive their result and show how the two conditions that they impose arise in this context. Although quantum gravity is apparently perturbatively non-renormalizable one may still write down a path integral in a manner similar to other gauge theories. Just as in a Yang–Mills theory we must introduce a gauge fixing term and a corresponding compensating ghost term in order to factor out the volume of the gauge group in the path integral. We thus write

$$Z[J,b] = \int \mathrm{D}g\,\mathrm{D}C\,\mathrm{D}C^*\mathrm{D}B \exp\left[\frac{\mathrm{i}}{\hbar}\left(S_{\mathrm{Cl}}+S_{\mathrm{GF}}+S_{\mathrm{Gh}} + \int J^{\mu\nu}g_{\mu\nu} + \int \eta^*_\nu C^\nu + \int \eta^\mu C^*_\mu + \int J^\mu_{\mathrm{B}} B_\mu\right)\right], \quad (1)$$

where the classical Einstein–Hilbert action is given by [6]

$$S_{\mathrm{Cl}} = \frac{1}{2K^2}\int \sqrt{g}(R+\Lambda), \quad (2)$$

with R being the scalar curvature, Λ the cosmological constant, g the determinant of the metric and $K(=\sqrt{8\pi G})$ the gravitational coupling. In order to facilitate the derivation of the Nielsen identities we use an auxiliary field from the gauge-fixing, which reduces to the standard term upon elimination of the auxiliary field B^μ

$$S_{\mathrm{GF}} = \frac{1}{2K^2}\int \left(\tfrac{1}{2}\alpha B_\mu B_\nu b^{\mu\nu} + B_\mu F^\mu\right). \quad (3)$$

In (3) $b^{\mu\nu}$ is some background metric and F^μ is the gauge-fixing term. We choose F^μ to be of the form [7]

$$F^\mu = b^{1/4}(\nabla^\nu g^\mu{}_\nu - \tfrac{1}{2}\beta\nabla^\mu g^s{}_s), \quad (4)$$

where $b=\det b^{\mu\nu}$ and both the covariant derivatives ∇ and raising and lowering indices employ the background metric $b^{\mu\nu}$. In the above both α and β are gauge parameters.

The corresponding ghost term for such a gauge-fixing is given, in the usual manner, by

$$S_{\mathrm{Gh}} = \int C^*_\mu \frac{\delta F^\mu}{\delta\rho^\nu} C^\nu, \quad (5)$$

where C^*_μ and C^ν are the antighost and ghost, respectively, and $\delta F^\mu/\delta\rho^\nu$ denotes the variation of the gauge fixing term under an infinitesimal general coordinate transformation.

The action we have written down is invariant under

[1] Laboratoire associé au Centre National de la Recherche Scientifique.

0370-2693/87/$ 03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

a set of BRS transformations reminiscent of those in a Yang–Mills theory [8]. If we denote a BRS variation of a field by δ we have

$$\delta g_{\mu\nu} = \epsilon \pounds_C g_{\mu\nu},$$

$$\delta C^*_\mu = \epsilon B_\mu, \quad \delta C^\nu = \tfrac{1}{2}\epsilon \pounds_C C^\nu, \tag{6}$$

where ϵ is the global anticommuting BRS parameter and $\pounds_C$ is the Lie derivative. By virtue of our use of an auxiliary field gauge fixing, δ^2 acting on all the field gives zero. To derive the Nielsen identities we introduce a further BRS transformation acting on the gauge parameter α [4,5]

$$\delta\alpha = \epsilon\chi, \tag{7}$$

where χ is a global anticommuting object. In order to maintain the invariance of the action we must include the extra term

$$S_\chi = \frac{1}{4K^2}\int \chi C^*_\mu B_\nu b^{\mu\nu}. \tag{8}$$

Finally, in order to linearize the identities which we shall derive, we introduce extra source terms into the generating functional which couple to the BRS variations of the fields

$$S_Q = \int (Q^{\mu\nu}\delta g_{\mu\nu} + Q_{C\mu}\delta C^\mu). \tag{9}$$

We note that $\delta S_Q = 0$ because $\delta^2 g$ and $\delta^2 C$ are zero and the sources are assumed to be BRS invariant.

If we now consider the result of the BRS transformations (6) and (7) acting on our augmented generating functional

$$Z_{\mathrm{AUG}}[J, Q, b]$$

$$= \int \mathrm{D}g\, \mathrm{D}C\, \mathrm{D}C^* \mathrm{D}B \exp\left[\frac{\mathrm{i}}{\hbar}\left(S_{\mathrm{Cl}} + S_{\mathrm{GF}} + S_\chi + S_Q + \int J^{\mu\nu} g_{\mu\nu} + \int \eta^*_\nu C^\nu + \int \eta^\mu C^*_\mu + \int J^\mu_{\mathrm{b}} B_\mu\right)\right], \tag{10}$$

we find the following identity on Z:

$$\int \left(J^{\mu\nu}\frac{\delta}{\delta Q^{\mu\nu}} - \eta^*_\mu \frac{\delta}{\delta Q_{C\mu}} - \frac{\delta}{\delta J^\mu_{\mathrm{b}}}\eta^\mu\right) Z_{\mathrm{AUG}} - \chi\frac{\partial Z_{\mathrm{AUG}}}{\partial\alpha} = 0. \tag{11}$$

To transform this into an identity on the effective action Γ we use the definition of the connected generating functional W

$$\exp\{(\mathrm{i}/\hbar) W[J, Q, b]\} = Z_{\mathrm{AUG}}[J, Q, b], \tag{12}$$

and the Legendre transform connecting Γ and W,

$$\Gamma[\bar g, \bar C, \bar C^*, \bar B, b] = W[J, Q, b] - \int J^{\mu\nu}\bar g_{\mu\nu} - \int \eta^*_\mu \bar C^\mu - \int \eta^\nu \bar C^*_\nu - \int J^\nu_{\mathrm{b}} \bar B_\nu, \tag{13}$$

where the bars denote semiclassical fields which are defined by

$$\bar g_{\mu\nu} = \delta W/\delta J^{\mu\nu} \tag{14}$$

and so on. The resulting identity on Γ is given by

$$\int \left(-\frac{\delta\Gamma}{\delta \bar g_{\mu\nu}}\frac{\delta\Gamma}{\delta Q^{\mu\nu}} + \frac{\delta\Gamma}{\delta\bar C^\mu}\frac{\delta\Gamma}{\delta Q_{C\mu}} - \bar B^\nu \frac{\delta\Gamma}{\delta\bar C^*_\nu}\right) - \chi\frac{\partial\Gamma}{\partial\alpha} = 0. \tag{15}$$

To obtain the effective action precursor of the Nielsen identity we differentiate (15) with respect to χ and then set $\chi = 0$. This gives

$$\int \left(\frac{\delta\Gamma(P)}{\delta\bar g_{\mu\nu}}\frac{\delta\Gamma}{\delta Q^{\mu\nu}} + \frac{\delta\Gamma}{\delta\bar g_{\mu\nu}}\frac{\delta\Gamma(P)}{\delta Q^{\mu\nu}} - \frac{\delta\Gamma(P)}{\delta\bar C^\mu}\frac{\delta\Gamma}{\delta Q_{C\mu}} - \frac{\delta\Gamma}{\delta\bar C^\mu}\frac{\delta\Gamma(P)}{\delta Q_{C\mu}} + \bar B^\nu(P)\frac{\delta\Gamma}{\delta\bar C^*_\nu} + \bar B^\nu\frac{\delta\Gamma(P)}{\delta\bar C^*_\nu}\right) - \frac{\partial\Gamma}{\partial\alpha} = 0, \tag{16}$$

where we have denoted an insertion of the operator $(1/4K^2)\int C^*_\mu B_\nu b^{\mu\nu}$ in Γ by $\Gamma(P)$. If we now drop terms which vanish because of the conservation of ghost number we find

$$\frac{\partial\Gamma}{\partial\alpha} - \int \frac{\delta\Gamma}{\delta\bar g_{\mu\nu}}\frac{\delta\Gamma(P)}{\delta Q^{\mu\nu}} - \int \bar B^\nu \frac{\delta\Gamma(P)}{\delta\bar C^*_\nu} = 0, \tag{17}$$

which is the effective action precursor of the Nielsen identity. Taking the constant field limit, we find

$$\frac{\partial V}{\partial\alpha} - \int \frac{\delta\Gamma(P)}{\delta Q^{\mu\nu}}\frac{\partial V}{\partial\bar g_{\mu\nu}} + \frac{1}{\Omega}\int \bar B^\nu \frac{\delta\Gamma(P)}{\delta\bar C^*_\nu} = 0, \tag{18}$$

where we have defined the effective potential V by

$$V\Omega = -\Gamma\,|_{\text{constant field}}, \tag{19}$$

with Ω being the spacetime volume. The preceding derivation is entirely analogous to that of the Nielsen identity in a spontaneously broken Yang–Mills theory coupled to scalars. We now observe that for the concept of an effective potential to make any sense the last term in (18) must be zero [3,5]. Briefly, this is because we need to derive an equation of the form

$$\frac{\partial V}{\partial \alpha} + C^{\mu\nu} \frac{\partial V}{\partial \bar{g}^{\mu\nu}} = 0, \tag{20}$$

which states that, under a change in the gauge parameter $\alpha \to \alpha + \delta\alpha$, (here δ is *not* a BRS variation) $\bar{g}_{\mu\nu}$ undergoes a compensating change $\bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu} + C_{\mu\nu}\delta\alpha$. The explicit gauge parameter dependence (via α) and the implicit gauge parameter dependence (via $\bar{g}_{\mu\nu}$, which is calculated using a gauge fixed action) thus cancel out and the gauge independence of physical quantities is assured. The last term in (18) would spoil this so we must have $\bar{B}=0$. In other words, eliminating B using its equation of motion, the semiclassical value of the gauge fixing should be zero. This is precisely the second condition derived by Cohler and Chodos in their paper for the equivalence of one-loop effective actions. We note it must apply to all orders otherwise the gauge independence of physical quantities will be lost. In the language of Cohler and Chodos

$$Q_\mu{}^{s\sigma}(g_{s\sigma} - b_{s\sigma}) = b^{1/2}(\nabla^\nu g_{\mu\nu} - \tfrac{1}{2}\beta \nabla_\mu g^s{}_s) \tag{21}$$

is the gauge fixing (the different factor of $b^{1/4}$ comes from our use of auxiliary field gauge fixing and does not affect the result). Thus, demanding $Q_\mu{}^{s\sigma} \times (\bar{g}_{s\sigma} - b_{s\sigma}) = 0$ gives

$$b^{1/2}(\nabla^\nu \bar{g}_{\mu\nu} - \tfrac{1}{2}\beta \nabla_\mu \bar{g}^s{}_s) = 0, \tag{22}$$

whereas B_μ (from (3) and (4) with $\chi = 0$) is given by

$$-(1/\alpha) b^{1/4}(\nabla^\nu g_{\mu\nu} - \tfrac{1}{2}\beta \nabla_\mu g^s{}_s). \tag{23}$$

We can see from (23) that $\bar{B}_\mu = 0$ is equivalent to (22). In order that the one-loop effective potentials should be equivalent in different gauges, Cohler and Chodos also imposed a further condition.

$$\frac{\delta S_{\text{cl}}[\bar{g}]}{\delta \bar{g}_{\mu\nu}} = 0. \tag{24}$$

To derive this we expand (20) order by order in $\hbar$, noting that, because of the insertion of the operator P, $C^{\mu\nu}$ receives its first perturbative contribution at the one-loop level [1,3]. Using superscripts to denote the order in $\hbar$ we find that the lowest non-trivial order of (20) is given by

$$\frac{\partial V^{(1)}}{\partial \alpha} + C^{(1)}_{\mu\nu} \frac{\partial V^{(0)}}{\partial \bar{g}_{\mu\nu}} = 0. \tag{25}$$

We now observe that if we want the one-loop effective potential to be gauge parameter independent we must have

$$\frac{\partial V^{(0)}}{\partial \bar{g}_{\mu\nu}} = 0. \tag{26}$$

However, $V^{(0)}$ is simply the classical action evaluated with the appropriate semiclassical field argument so we have obtained the first condition of Cohler and Chodos.

Our proof is not restricted to a particular class of gauge, such as the background field gauge we have used as an example. One may use a gauge-fixing term which is parametrized to interpolate between two different gauge conditions and introduce the extra BRS transformations and compensating terms for these parameters. For instance, in their paper Cohler and Chodos mention the equivalence of the one-loop effective potential calculated in the two gauges

$$\partial_5 g^{\mu 5} = 0, \tag{27}$$

and

$$\partial_\mu g^{\mu\nu} = 0, \tag{28}$$

where the space–time is five-dimensional and the greek indices run over all five dimensions. To use our proof we would consider a gauge-fixing of the form

$$B_\mu(\alpha \partial_i g^{\mu i} + \partial_5 g^{\mu 5}) + \tfrac{1}{2}\beta B^2 + \chi C^*_\mu \partial_i g^{\mu i}, \tag{29}$$

where the i index runs from 1 to 4 only and we have introduced an extra BRS transform on α (but not β)

$$\delta\alpha = \epsilon\chi. \tag{30}$$

$\alpha = 0$ corresponds to (27) and $\alpha = 1$ corresponds to (28). One thus obtains an equation of the form

$$\frac{\partial V}{\partial \alpha} - \int \frac{\delta \Gamma(C_\mu^* \partial_\iota g^{\mu\iota})}{\delta Q^{\mu\nu}} \frac{\partial V}{\partial \bar{g}_{\mu\nu}} = 0, \tag{31}$$

provided that, once again, $\bar{B}_\mu = 0$. At the one-loop level the second term vanishes as (26) is satisfied and we have

$$\frac{\partial V^{(1)}}{\partial \alpha} = 0 \tag{32}$$

so $V^{(1)}$ is independent of α and hence either (27) or (28) gives an equivalent one-loop effective potential.

We have seen that the result of Cohler and Chodos may be derived naturally in the context of Nielsen identities. In a recent preprint on self-consistent dimensional reduction in a Kaluza–Klein gravity model Kunstatter and Leivo [9] work with a gauge which does not satisfy (24) and show that the effective potential (and hence the radius of the compactified dimension and other "physical" quantities) is apparently gauge parameter dependent. Although they note that the Nielsen identities solve a similar problem in spontaneously broken Yang–Mills theories they suggest that this is not so in gravity, citing its non-renormalizability. However, the identities are a consequence of the BRS invariance of the theory and we see in (6) that it is possible to write down BRS transformations for the bare theory. It is also possible to renormalize the theory at the one-loop level (which is as seriously as one takes quantum gravity in self-consistent dimensional reduction) so it seems strange that the Nielsen identities are not applicable. It should be possible to derive eq. (25) explicitly to see whether the formal arguments break down and work is in progress on this at the moment. The result is of some interest because self-consistent dimensional reduction does take the quantum nature of the compactification process seriously and any problems encountered in the $M^4 \times S^1$ model of Kunstatter and Leivo might be generic.

This work was supported by a Royal Society /CNRS European Science Exchange Research Program Fellowship. I would also like to thank the staff of LPTHE where this work was carried out for their hospitality and tolerance of my French.

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