Lattice SUSY and the DiSSEP

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1. Summary

We investigate whether the **dynamical lattice supersymmetry** discussed for various Hamiltonians, including one-dimensional quantum spin chains, by Fendley et.al. [1, 2, 3] and Hagendorf et.al. [4, 5, 6] might also exist for the Markov matrices of any one-dimensional exclusion processes.

We find that the **DiSSEP (Dissipative Symmetric Simple Exclusion Process)**, introduced by in [7, 8], provides one such example for suitably chosen parameters. The DiSSEP Markov matrix admits the supersymmetry because it is conjugate to a spin chain Hamiltonian which also possess the supersymmetry.

The length-changing supersymmetry relation for the DiSSEP Markov matrix M^L and the supercharge $Q^{L\dagger}$ for L sites, $M^L Q^{L\dagger} = Q^{L\dagger} M^{L-1}$, is reminiscent of a "transfer matrix" symmetry that has been observed in other exclusion processes and we comment on a possible link.

2. Length Changing SUSY

A **dynamical, exact lattice supersymmetry** in one dimensional lattice fermion systems and spin chains was first observed by Fendley et.al. [1, 2, 3]. A lattice Hamiltonian for L sites with such a supersymmetry can be written as

$$H^L = Q^{L\dagger} Q^L + Q^{L+1} Q^{L+1\dagger}$$

where H^L acts on the vector space $V^{\otimes L}$, with $V \simeq \mathbb{C}^2$.

The lattice supercharges $Q^L, Q^{L\dagger}$ act on chains of length L and $L-1$ respectively as

$$Q^L : V^{\otimes L} \to V^{\otimes (L-1)}$$

and

$$Q^{L\dagger} : V^{\otimes (L-1)} \to V^{\otimes L} .$$

Hamiltonians for chains of different lengths are related by

$$H^{L-1} Q^L = Q^L H^L, \quad H^L Q^{L\dagger} = Q^{L\dagger} H^{L-1} .$$

These relations are global, but the supercharges may also be formulated locally, as described below.

3. Local Supercharges

For an open chain, the supercharges may be expressed in terms of **local** supercharges as

$$Q^L = \sum_{k=1}^{L-1} (-1)^{k+1} \mathfrak{q}_{k,k+1}, \quad Q^{L\dagger} = \sum_{k=1}^{L} (-1)^{k+1} \mathfrak{q}_k^\dagger$$

where $\mathfrak{q} : V \otimes V \to V$ and $\mathfrak{q}^\dagger : V \to V \otimes V$. The subscripts denote the lattice sites on which the operators act.

Satisfying the standard nilpotency conditions for the global supercharges

$$Q^{L-1} Q^L = 0, \quad Q^{L+1\dagger} Q^{L\dagger} = 0,$$

gives the following associativity condition on the local supercharge $\mathfrak{q}$ for open chains

$$\mathfrak{q}(\mathfrak{q} \otimes \mathbb{I}) = \mathfrak{q}(\mathbb{I} \otimes \mathfrak{q}) .$$

The bulk Hamiltonian density is given in terms of $\mathfrak{q}, \mathfrak{q}^\dagger$ by

$$h = -(\mathbb{I} \otimes \mathfrak{q})(\mathfrak{q}^\dagger \otimes \mathbb{I}) - (\mathfrak{q} \otimes \mathbb{I})(\mathbb{I} \otimes \mathfrak{q}^\dagger) + \mathfrak{q}^\dagger \mathfrak{q} + \frac{1}{2}\left(\mathfrak{q}\mathfrak{q}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{q}\mathfrak{q}^\dagger\right) ,$$

with all non-nearest-neighbour terms cancelling, and the boundaries by $(1/2)\mathfrak{q}\mathfrak{q}^\dagger$.

4. The Open DiSSEP

The **DiSSEP** was presented in [7, 8] as an integrable deformation of the Symmetric Simple Exclusion Process (SSEP) which still allowed a solution via the matrix product ansatz.

A concise way to describe the dynamics is to use Dirac braket notation to describe the state. For an open system with L sites, introduce an indicator variable $n_i \in \{0, 1\}$ at each site i to denote the presence or absence of a particle and denote the probability of finding a configuration $n_1 \ldots, n_L$ at time t by $P_t(n_1, \ldots, n_L)$. The evolution of the ket vector $|P_t\rangle$

$$|P_t\rangle = \sum_{n_1,\ldots,n_L \in \{0,1\}} P_t(n_1, \ldots, n_L) \, |n_1 \ldots n_L\rangle ,$$

where $|n_1 \ldots n_L\rangle = |n_1\rangle \otimes \cdots \otimes |n_L\rangle$ and the basis vectors are $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is given by master equation

$$\frac{d|P_t\rangle}{dt} = M \, |P_t\rangle .$$

The Markov matrix M appearing in the master equation is given for the DiSSEP by

$$M(\lambda^2) = B_1 + \sum_{k=1}^{L-1} m_{k,k+1} + \overline{B}_L$$

with boundary transition matrices B, $\overline{B}$ and bulk transition matrix m

$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \quad , \quad \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

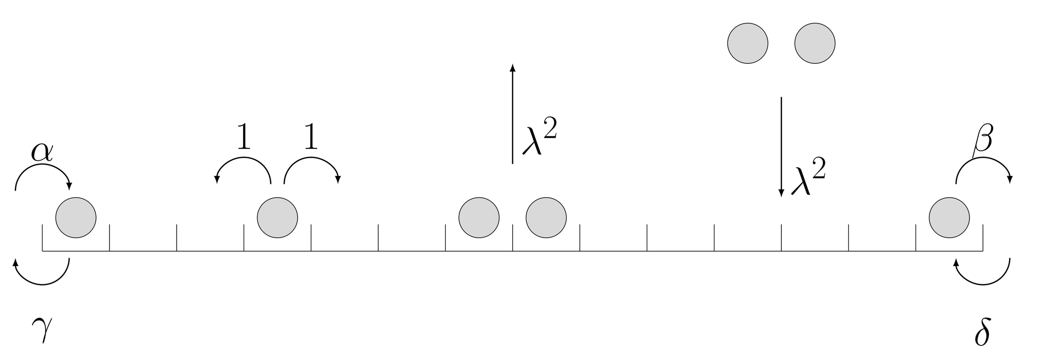
$$m = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix} .$$

The bulk Markov matrix m acts between nearest neighbour sites, giving forward and backward hops and pair addition and annihilation in the bulk, while the boundary matrices $B, \overline{B}$ allow the addition and removal of particles at both ends of the system.

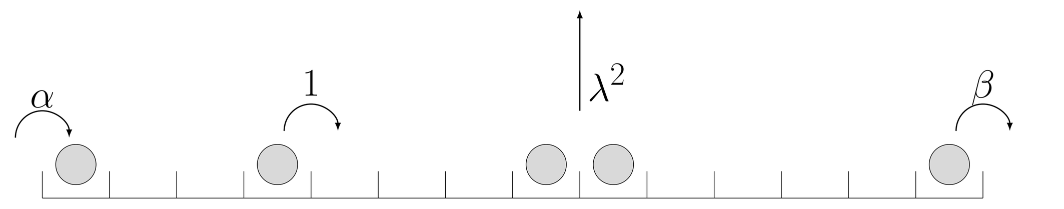
The **stochastic** nature of the model is evident from the column sums of the various matrices being zero, since they describe rates. This is the distinguishing feature of this class of models.

5. The DiSSEP and ASAP

The allowed moves and rates for the DiSSEP are shown below:



A closely related model is the ASAP (Asymmetric Annihilation process). For this only rightward hops and annihilation of adjacent pairs of particles are allowed.



This has been shown to possess a **global "transfer matrix symmetry"**

$$M^L T^{L\dagger} = T^{L\dagger} M^{L-1}$$

which is clearly similar in form to the length-changing SUSY.

6. SUSY and the DiSSEP

The one-parameter family of supercharges $\mathfrak{q}(y)$

$$\mathfrak{q}(y) = x \begin{bmatrix} 1 & y & y & 1 \\ y & 1 & 1 & y \end{bmatrix}$$

where $x = (1+|y|^2)^{-1/2}$, generates the bulk DiSSEP Markov matrix $M(\lambda^2)$ for $\lambda^2 = 1$ (independent of y) when inserted into the expression for the Hamiltonian density/Markov matrix in terms of the local supercharges.

This DiSSEP Markov matrix be written as

$$H_X(y) = -\sum_{k=1}^{L-1} \left(\sigma_k^x \sigma_{k+1}^x - \mathbb{I}\right) + B_1(y) + \bar{B}_L(y) ,$$

and the boundary terms $(1/2)\mathfrak{q}(y)\mathfrak{q}^\dagger(y)$ also become stochastic when $y = -1$

$$B_1(-1) = \bar{B}_L(-1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

corresponding to an open DiSSEP with $\lambda^2 = 1$ and $\alpha = \beta = \gamma = \delta = 1$.

7. Conjugation, SUSY and the DiSSEP

The DiSSEP Markov matrix $M(\lambda^2)$ is conjugate to an XXZ spin chain Hamiltonian $H_{XXZ}(\lambda^2)$

$$H_{XXZ}(\lambda^2) = -U_1 U_2 \ldots U_L \, M(\lambda^2) \, U_1^{-1} U_2^{-1} \ldots U_L^{-1}$$

where

$$U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$H_{XXZ}(\lambda^2) = -(\alpha - \gamma)\sigma_1^+ + \frac{\alpha + \gamma}{2}(\sigma_1^z + \mathbb{I}) \; - \; (\delta - \beta)\sigma_L^+ + \frac{\delta + \beta}{2}(\sigma_L^z + \mathbb{I})$$
$$+ \frac{\lambda^2 - 1}{2} \sum_{k=1}^{L-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \frac{\lambda^2 + 1}{\lambda^2 - 1}(\sigma_k^z \sigma_{k+1}^z - \mathbb{I})\right) .$$

When $\lambda^2 = 1$, $\alpha = \beta = \gamma = \delta = 1$ this simplifies to an Ising Hamiltonian

$$H_Z(y) = -\sum_{k=1}^{L-1} \left(\sigma_k^z \sigma_{k+1}^z - \mathbb{I}\right) + B_{c,1}(y) + \bar{B}_{c,L}(y) .$$

with diagonal boundary conditions. This is also supersymmetric with supercharge

$$\mathfrak{q}_c(y) = x \begin{bmatrix} y-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y+1 \end{bmatrix}$$

conjugate to $\mathfrak{q}(y)$ via $\mathfrak{q}_c(y) = U \, \mathfrak{q}(y) \, U^{-1} \otimes U^{-1}$.

The DiSSEP (for a suitable choice of parameters) thus displays the local lattice SUSY as does its conjugate Ising Hamltonian.

Curiously, both the ASAP and ASEP (Asymmetric Exclusion Process) display the global length changing transfer matrix symmetry but do **not** appear to possess this local SUSY – or at least, we have been unable to write down a local supercharge which generates their Markov matrices.

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