Spread of information/infection on networks

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Models for the spread of infection/information

homogeneous mixing, closed population (size n)

Epidemics: SIR model $\quad (X(t) + Y(t) + Z(t) = n\,)$

$$\begin{array}{ccccc} X(t) & & Y(t) & & Z(t) \\ \text{susceptible} & \rightarrow & \text{infective} & \rightarrow & \text{removed} \\ & \lambda XY/n & & \delta Y & \end{array}$$

Rumours:

$$\begin{array}{ccccc} \text{ignorant} & \rightarrow & \text{spreader} & \rightarrow & \text{stifler} \\ & \lambda XY/n & & ?? & \end{array}$$

Spreaders in contact with spreaders or stiflers become stiflers

(Kendall, ‘57; Daley & Kendall ‘64,’65; Daley & Gani, ‘99)

Some stifling options

Undirected contacts:

spreader – spreader contacts $\Longrightarrow$ both stifled

Directed contacts: Maki-Thompson (1973) model

only initiating spreader is stifled spreader $\rightarrow$ stifler

$$\lambda(Y+Z)Y/n$$

or, spreaders become stifled with prob. p $\lambda p(Y+Z)Y/n$

add "forgetting" at rate δ $\lambda p(Y+Z)Y/n+\delta Y$

cf control via contact tracing (Borgs et al, 2010)

Rumour model becomes an SIR model if $p = 0$

Threshold behaviour

Branching process approximation

p=0 (SIR)

Starting from single infective, probability of extinction is

δ/λ for $\delta<\lambda$ (certain for $\delta \geq \lambda$)

U-shaped *final size* distribution (J-shaped for $\delta \geq \lambda$)

Reproduction ratio: $R_0 = \lambda/\delta$

p>0 (rumour model)

Exactly as SIR case: extinction is certain if $\delta \geq \lambda$, and has probability zero if $\delta = 0$, regardless of value of ***p*.**

Rumours on random networks

Look at properties of this rumour model superposed on a network structure (application: social interaction networks).

Rumour spreads along the edges of the graph.

Spreaders make contact with *each* neighbour at rate λ
i.e. contact rate is λk if node has degree k

Homogeneous mixing = completely connected graph

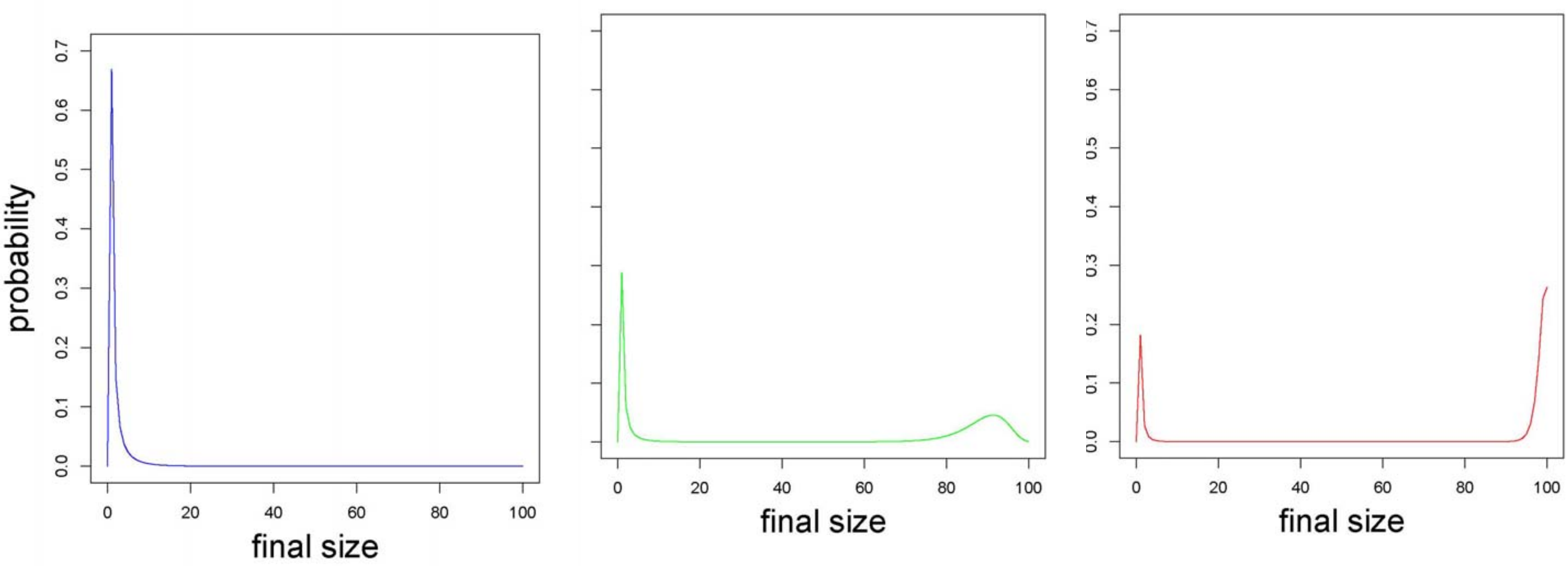
Questions….

- What is the threshold for the general rumour model on a network?
- How does the threshold depend on n?
- What is the effect of network structure?

What is a threshold for finite n?

Nåsell (1995): SIR model $(p = 0)$

threshold defined as value of R_0 for which the final size dn changes from J-shape to U-shape



Conjecture: $R_0 \sim 1 + \rho / n^{1/3}$

Insights from an approximate model

Notation:

n_k nodes of degree k

X_k, Y_k, Z_k ignorants, spreaders, stiflers of degree k

and network degree-degree correlation function

$p_{j\,k}$ = P(neighbour node has degree k | index node has degree j)

Nekovee et al (Physica A, 2007) used approximation re dependence, so that the influence of the network is encapsulated in the $p_{j\,k}$ matrix.

Specifically, the total rate of "infection" of degree k ignorants is

$$\lambda k X_k \sum_j p_{kj} \mathrm{P}(\text{node is spreader} \mid \text{degree } j \text{ and neighbour of ignorant node of degree } k).$$

$$\simeq \lambda k X_k \sum_j p_{k\,j} Y_j / n_j$$

Similarly, the total rate of stifling of degree k spreaders is

$$\simeq Y_k \{ \delta + \lambda p k \sum_j p_{k\,j} [n_j - X_j]/n_j \}$$

i.e metapopulation mixing structure (the "approximate" model)

regular network (fixed k) and no stifling ($p = 0$)

$\Longrightarrow$ SIR Model

Nekovee et al's results (deterministic, $n \to \infty$ *)*

Regular case (all nodes have same degree k) :

final size eqn $z = 1 - e^{-Rz}$ where

$$R = \lambda k(1+p)/(\delta + \lambda kp) = (1+p)/(\psi + p) \qquad (\psi = \delta/(\lambda k))$$

For rumour to spread (non-zero z) need $R > 1$

If $\delta = 0$ (no forgetting) then $R > 1$ (no threshold)

If $\delta > 0$ then need $\delta < \lambda k$ $(\psi < 1)$ regardless of p (*cf* SIR)

"Uncorrelated" case ($p_{j\,k} \propto kp_k$, where p_k is marginal degree dn)

If $\delta > 0$ then, *to leading order in* p, need

$$\delta < \lambda \left(\mu_K + \sigma_K^2/\mu_K\right) \qquad \text{(size-biased mean)}$$

Approximate model – stochastic case

- Spreading rate $\lambda k X_k \sum_j p_{k\,j} Y_j / n_j$
- Stifling rate $Y_k \{ \delta + \lambda p k \sum_j p_{k\,j} [n_j - X_j]/n_j \}$

Analytic derivation of final size distribution for regular networks (fixed k, or small number of possible degrees) allows investigation of thresholds

Threshold – in δ for fixed p and λ – at which final size distribution changes from bimodal to unimodal (increasing control)

Otherwise via simulation

Final size distribution - embedded Markov chain - regular network (fixed degree)

For $x \geq 0, y \geq 1 \text{ and } x + y \leq n$

$$(x, y) \to (x - 1, y + 1) \text{ with prob } \phi_x = \tfrac{x}{(\psi+p)n+(1-p)x}$$

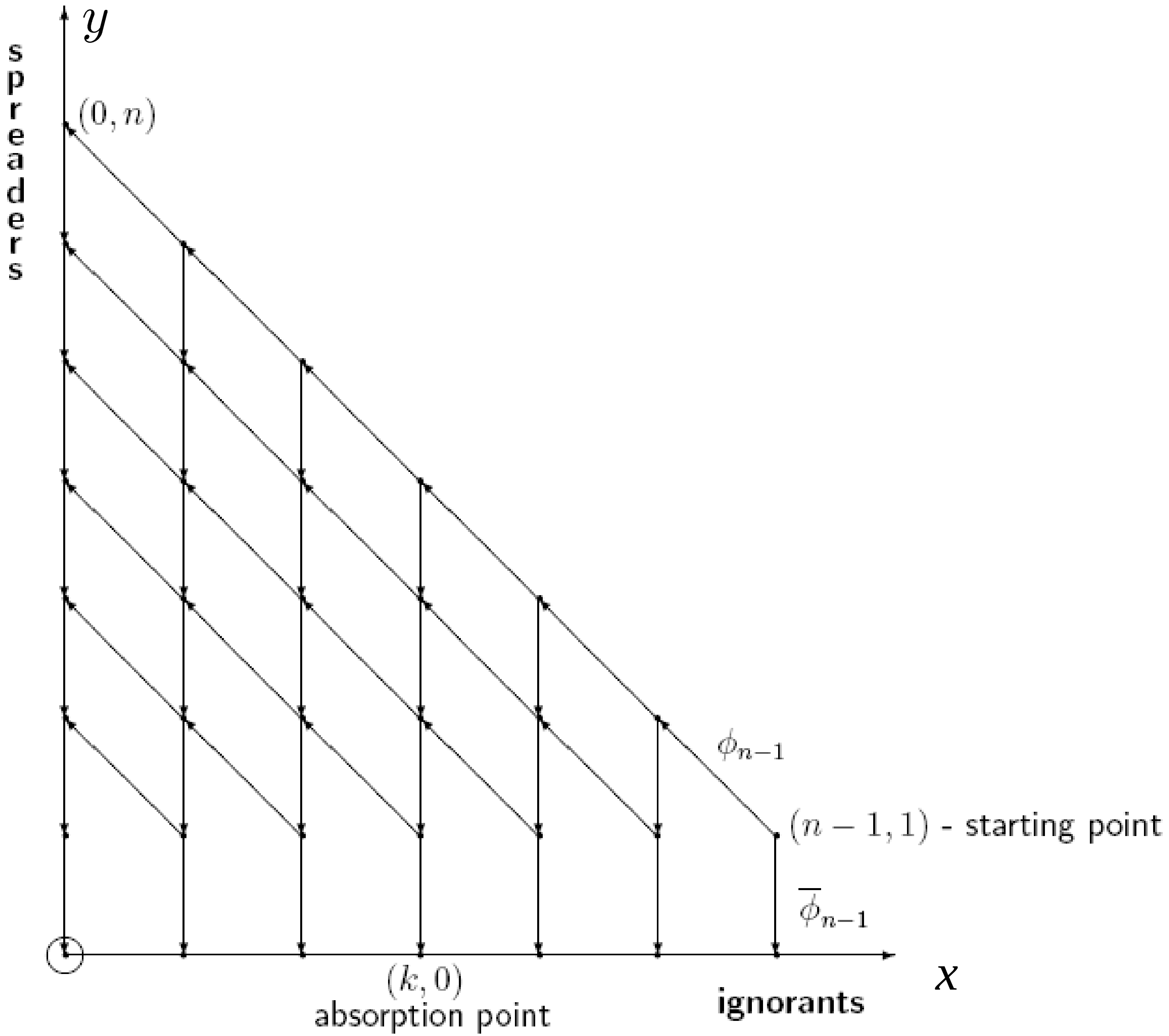
$$(x, y) \to (x, y - 1) \text{ with prob } 1 - \phi_x = \overline{\phi}_x$$

(Note that these probs are independent of y)

$x(0) = n - 1, y(0) = 1$ and states $(x, 0)$ are absorbing

Absorption in state $(s, 0)$ $\Longrightarrow$ final size is $n - 1 - s$

Time in state (x, y) : $T_{x\,y} \sim \exp\{u_{x\,y} = (\delta + \lambda pk)y + \lambda(1 - p)kxy/n\}$



y
spreaders
$(0, n)$
ϕ_{n-1}
$(n-1, 1)$ - starting point
$\overline{\phi}_{n-1}$
$(k, 0)$
absorption point
ignorants
x

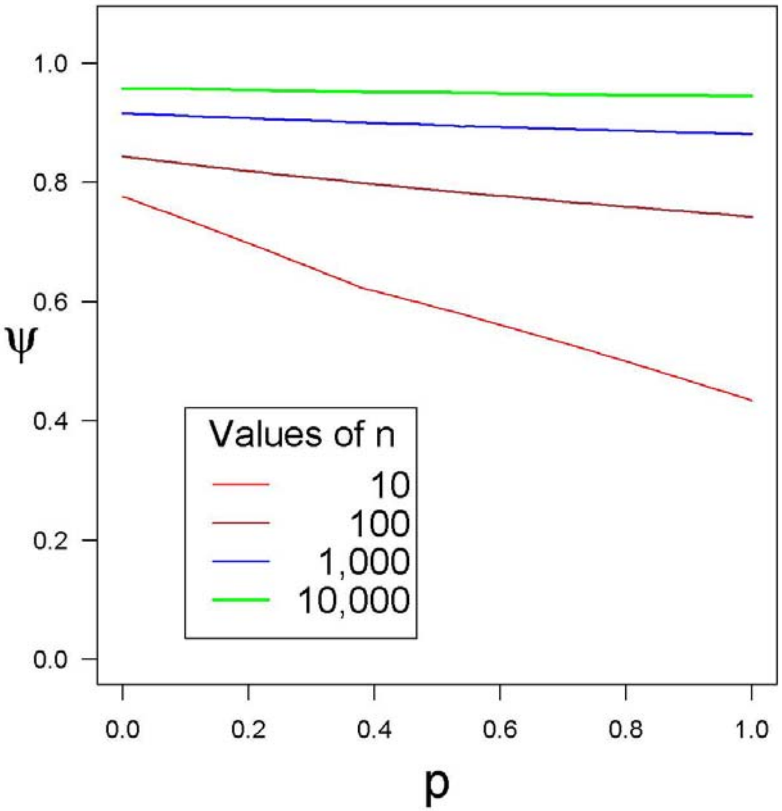
Let $\pi_{x\,y} = \mathrm{P}(\text{ever reach } (x,y))$ then

$$\begin{aligned}
\pi_{x\,y} &= \pi_{x+1\;y-1}\phi_{x+1} + \pi_{x\;y+1}\,(1-\phi_x) \text{ for } x \geq 0, y \geq 2, x+y \leq n-1. \\
\pi_{x\,y} &= \pi_{x+1\;y-1}\phi_{x+1} \qquad \text{for } 0 \leq x \leq n-2, y = n-x \\
\pi_{x\,1} &= \pi_{x\,2}\,(1-\phi_x) \qquad \text{for } 0 \leq x \leq n-2, \\
\pi_{x\,0} &= \pi_{x\,1}\,(1-\phi_x) \qquad \text{for } 0 \leq x \leq n-1,
\end{aligned}$$

with initial condition $\pi_{n-1\;1} = 1$

the equations can be solved iteratively to give $\pi_{n-1-s\;0}$.

Thresholds for stochastic case (approximate model)

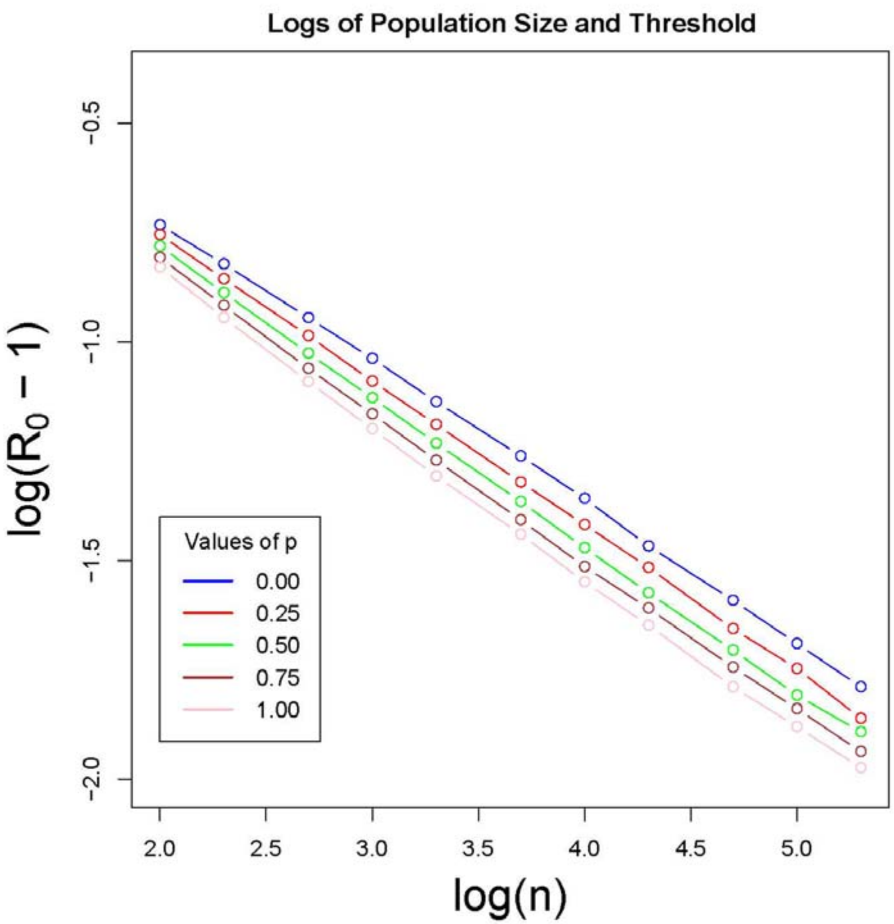


rumours: $\psi = \delta/(\lambda k)$ vs p (p=0 for SIR)

$$R_0 = (1+p)/(\psi + p)$$

Threshold becomes independent of p as n increases

Thresholds for stochastic case (approximate model) ctd



p	A	B
0.00	-0.056	0.327
0.25	-0.089	0.333
0.50	-0.127	0.335
0.75	-0.167	0.335
1.00	-0.190	0.338

Estimates of coefficients *A* and *B* in the log-linear form

$$\log(R_0 - 1) = A - B \log n$$

for a range of values of p

where $R_0 = (1+p)/(\psi + p)$

(fitting to $10^3 \leq n \leq 2 \times 10^5$)

Full network model

- What can be said about the full network?
- The thresholds in delta for the approximate model provide upper bounds for those for full network, but ignore the correlation of node states given their degrees and hence many effects of network structure
- Compare three basic networks

  - *simple random graph (Erdõs-Rényi)*

  - scale free graph (Barábasi-Albert)

  - random geometric graph

  with a fixed mean degree (denoted D in the following slides)

Network properties

Cluster coefficient: P($i \sim j$ | $i \sim k$ and $j \sim k$),

extent to which triangles appear

Correlation: corr(D_i, D_j | $i \sim j$)

- *Random networks*

  - *simple random graph (Erdõs-Rényi): n nodes, each pair of nodes is connected by an edge with prob π (iid)*

  $\Longrightarrow$ node *degree* (D, ie no of nodes to which it is connected) ~Bin $(n-1, \pi)$ (approx Poisson for large n)

  the graph is *uncorrelated,* clustering is *asymptotically negligible*

  - uncorrelated graph with an arbitrary degree distribution (iid over nodes) constructed by "pairing arms at random" (Molloy-Reed algorithm)
    e.g. regular network (fixed degree)

- Barábasi-Albert (1999) model of network growth: scale-free degree distribution (power-law tail)

$$f(d) \propto d^{-3} \quad (d \to \infty)$$

characterised by the presence of "hubs" – nodes with very high degrees

asymptotically uncorrelated graph and unclustered

- random geometric graph: start from a spatial Poisson process and define nodes to be connected if within some fixed distance. Poisson degree distribution.

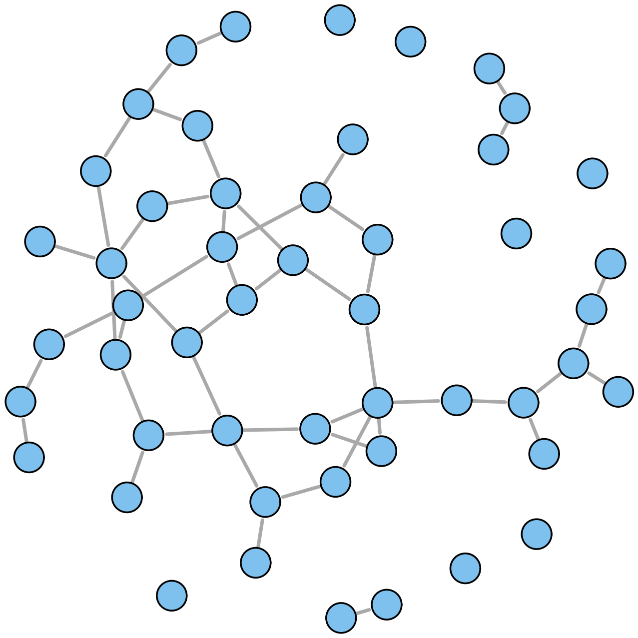
both correlation and cluster coefficient are the average (over *c*) area of overlap of two unit discs with centres *c* apart

$$\mathbf{1 - 3\sqrt{3}/(4\pi) = 0.59}$$

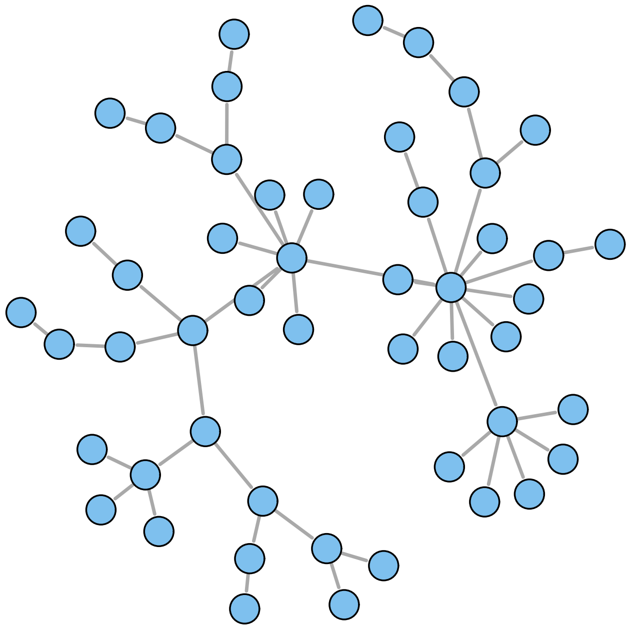
Simple random and Barábasi-Albert graphs:

n=50, mean degree = 1.96

SRG



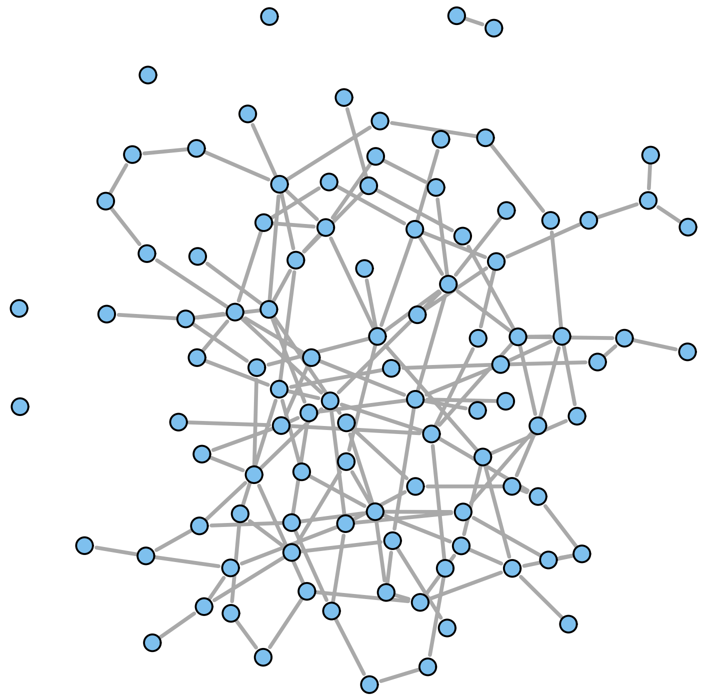
B-A



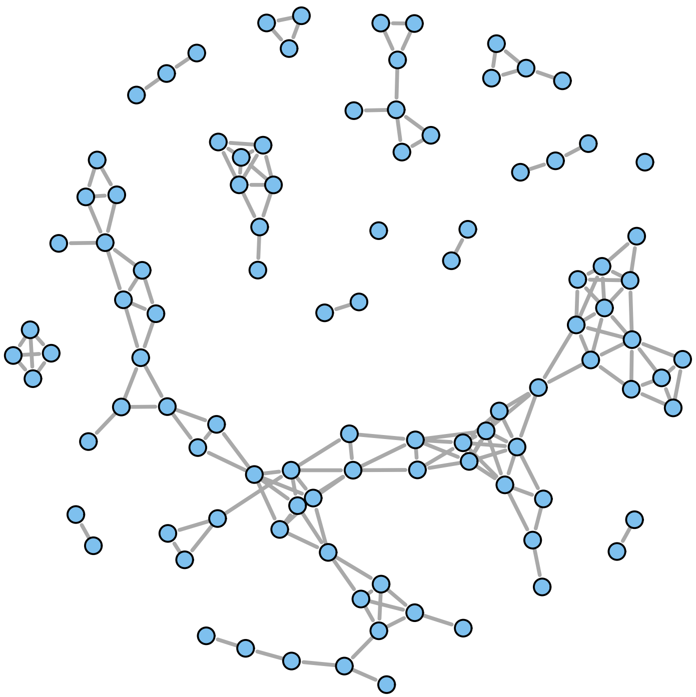
Simple random and random geometric graphs:

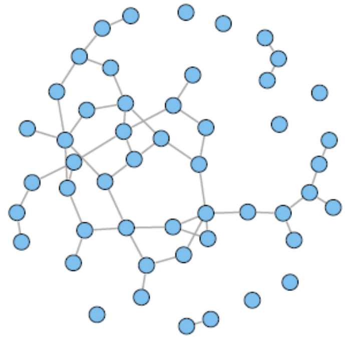
n=100, mean degree = 3

SRG

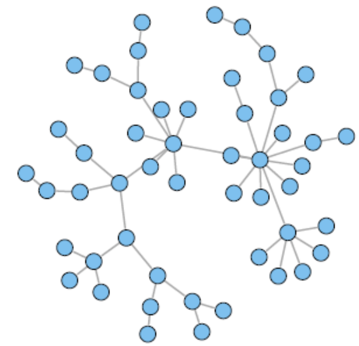


RGG

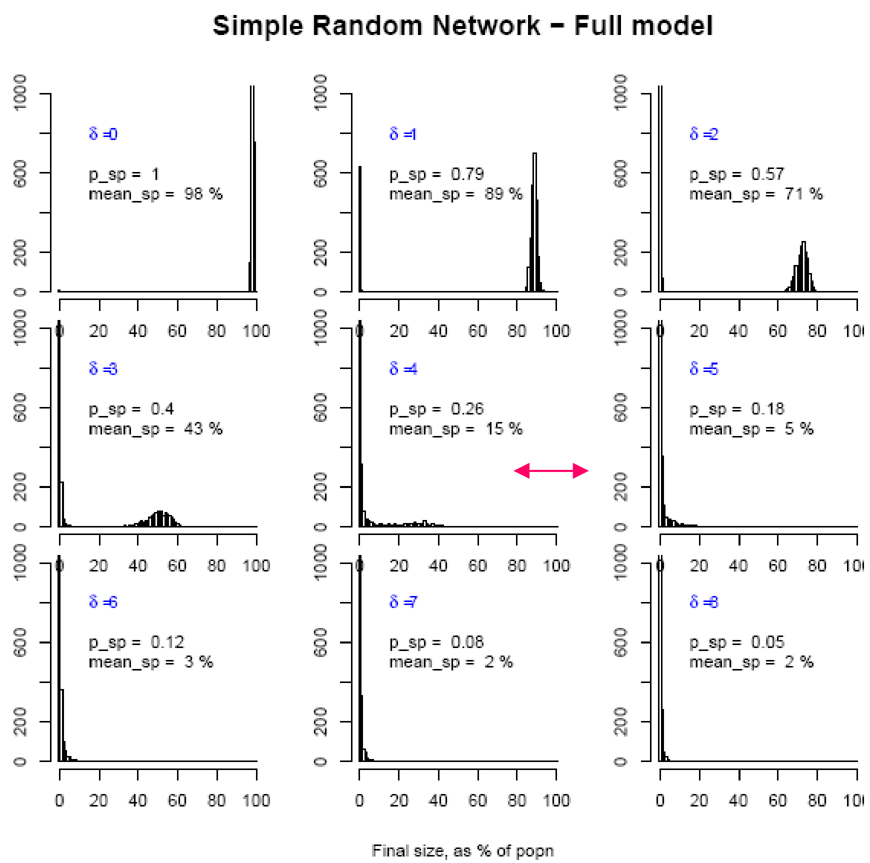


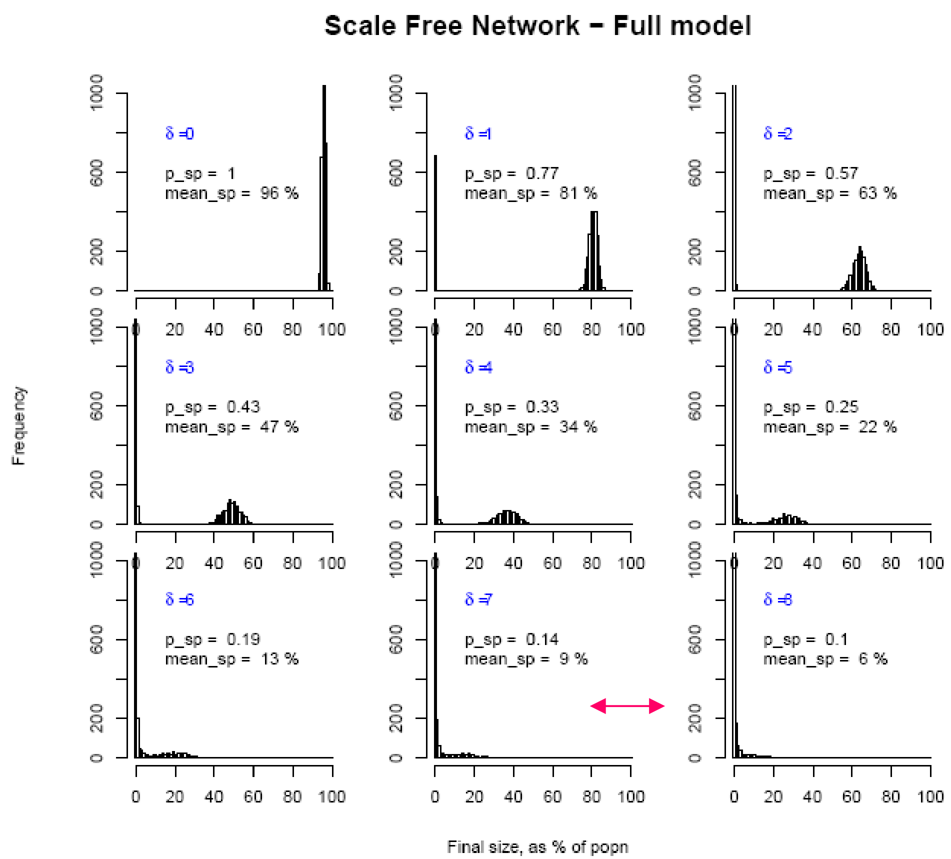


***Exact models**: simple vs Barábasi-Albert random graph*

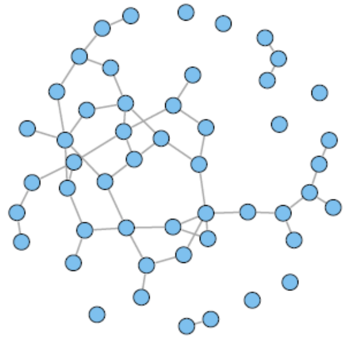


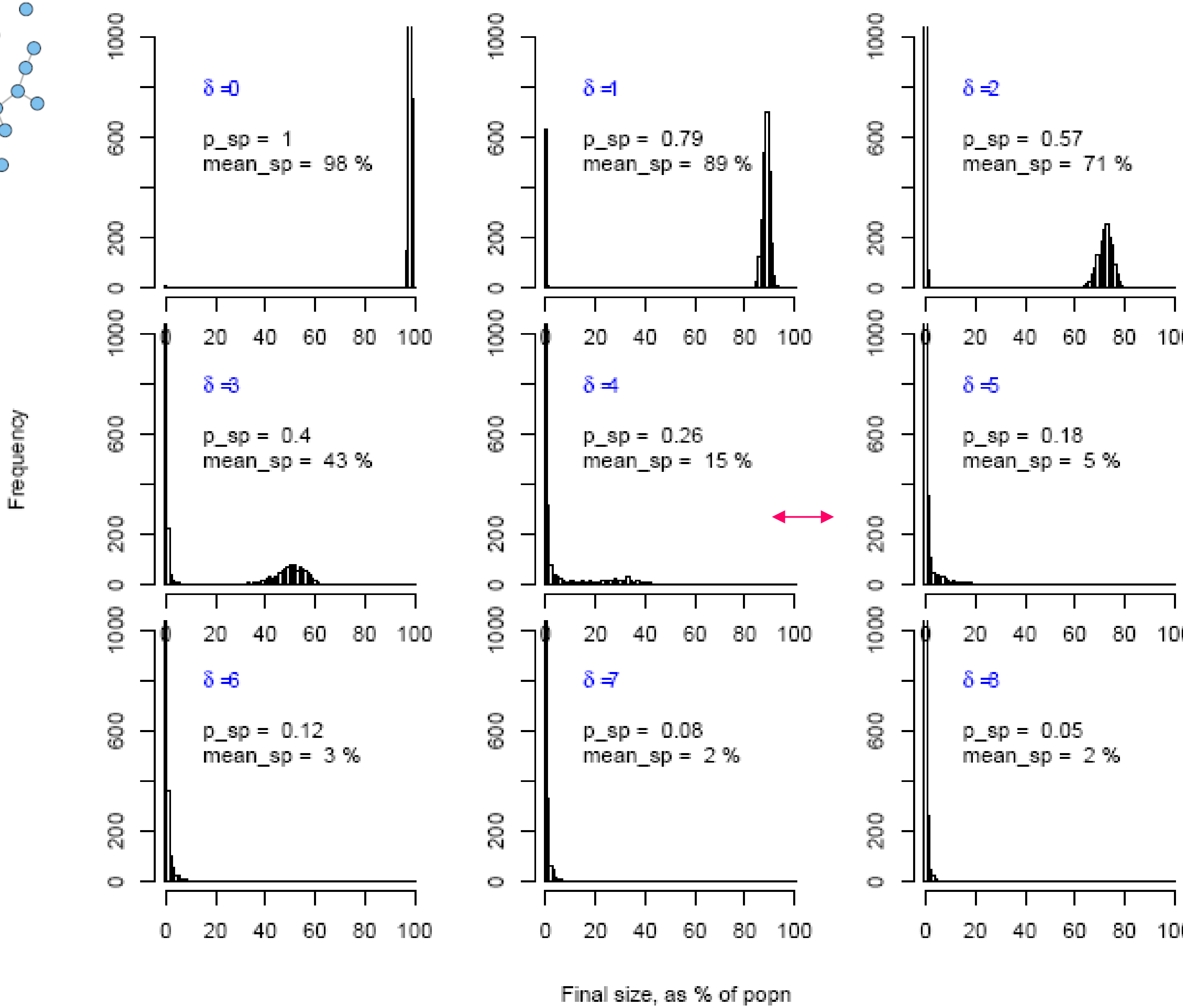
SRG n=10^3, λ=1, E(D)=6, p=0.1, ψ = δ/[λE(D)] = 0,…,1.33 B-A



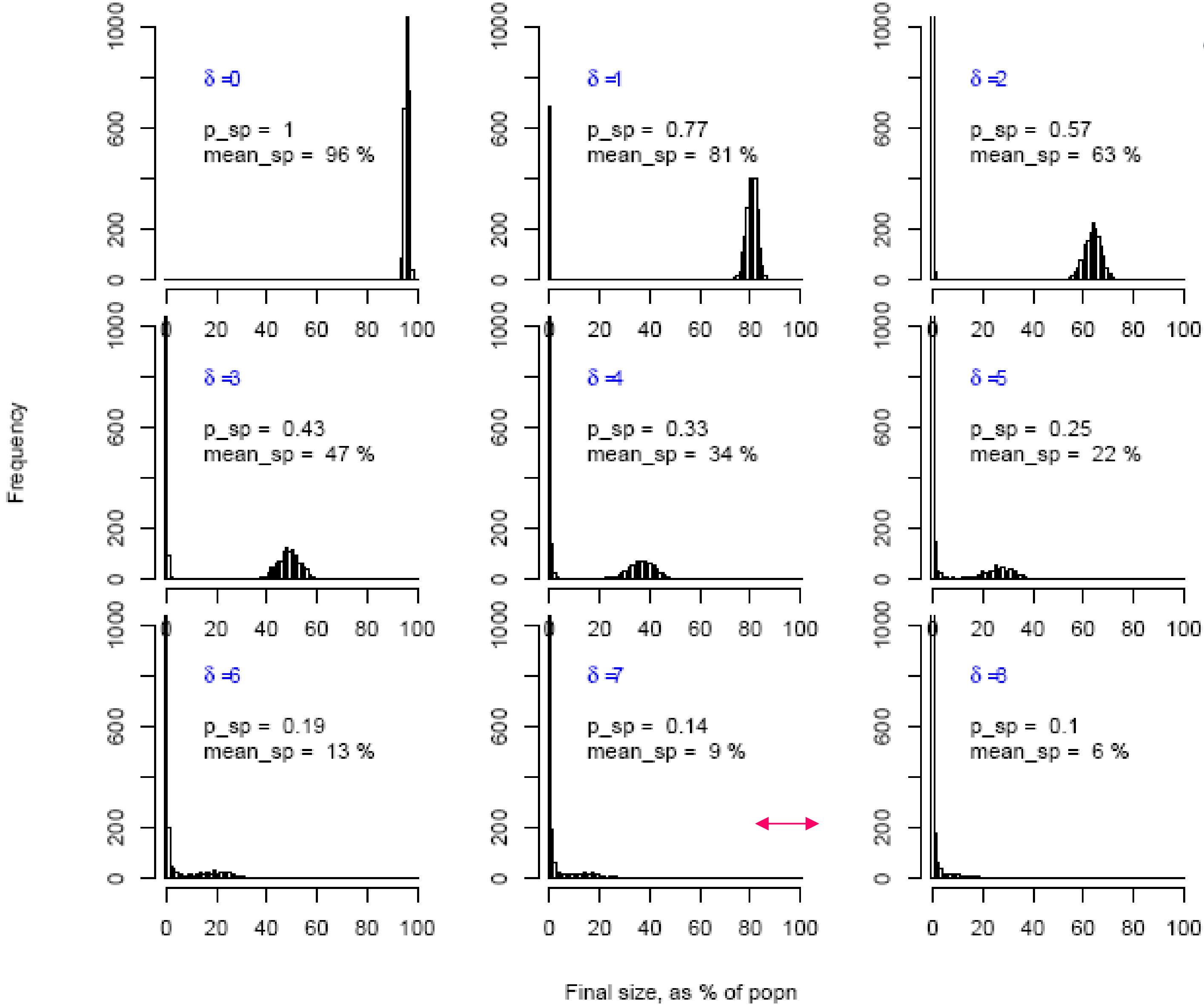


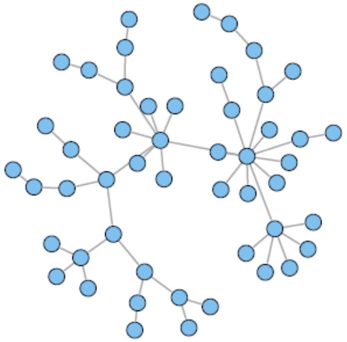
Simple Random Network – Full model

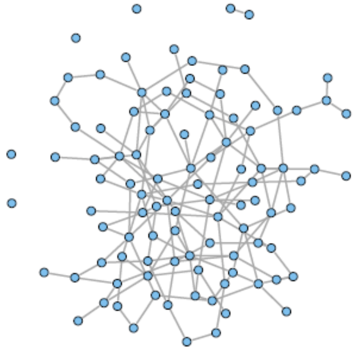




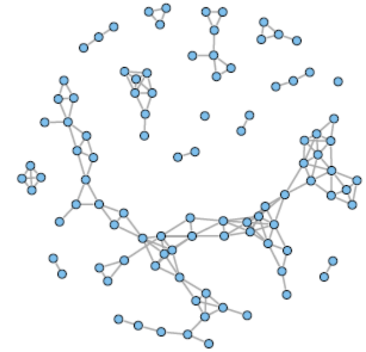
Scale Free Network – Full model



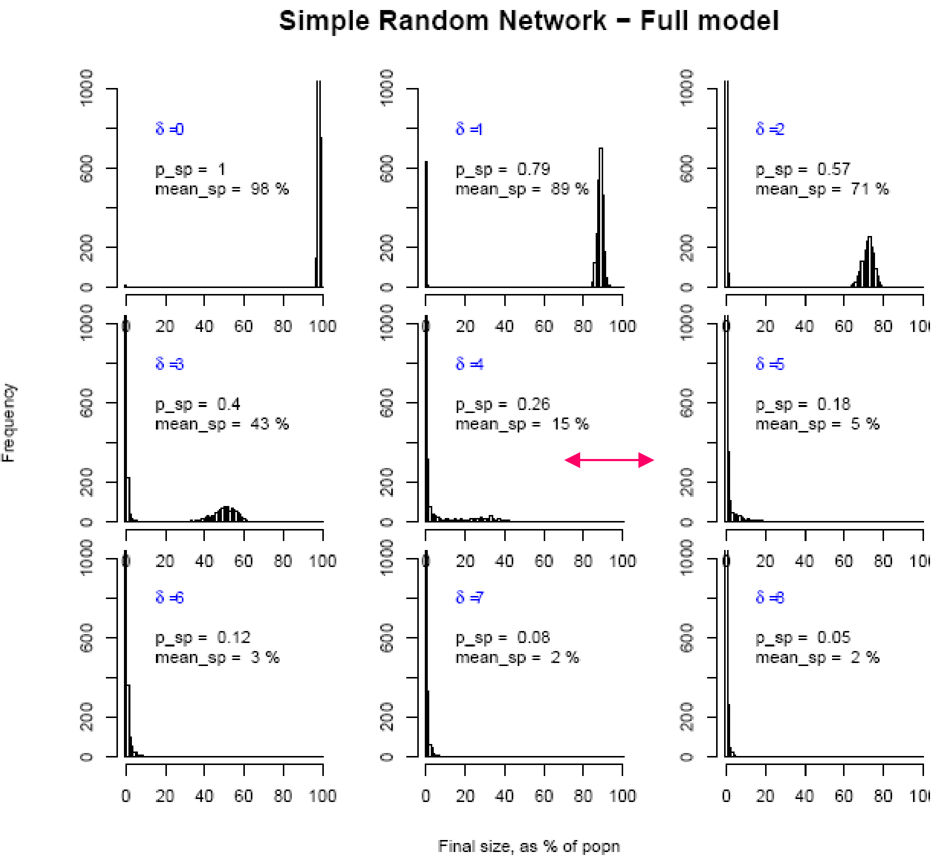


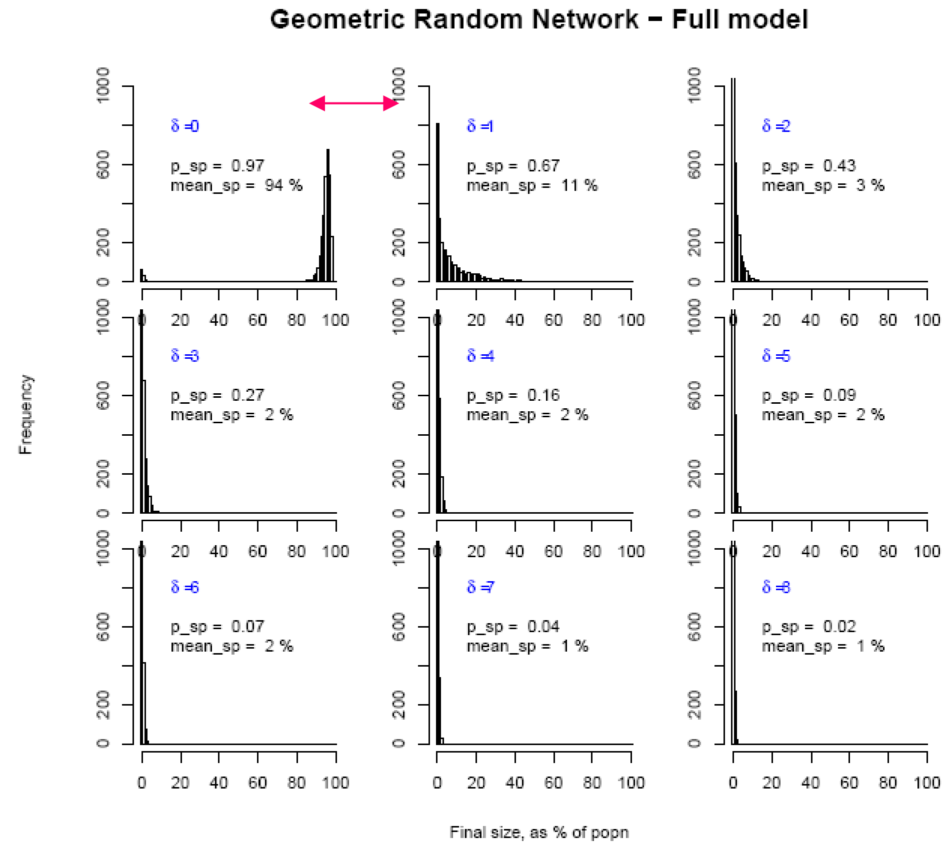


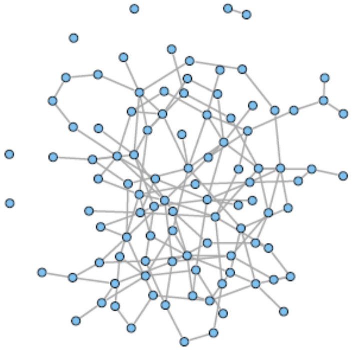
***Exact models**: simple random vs random geometric graph*



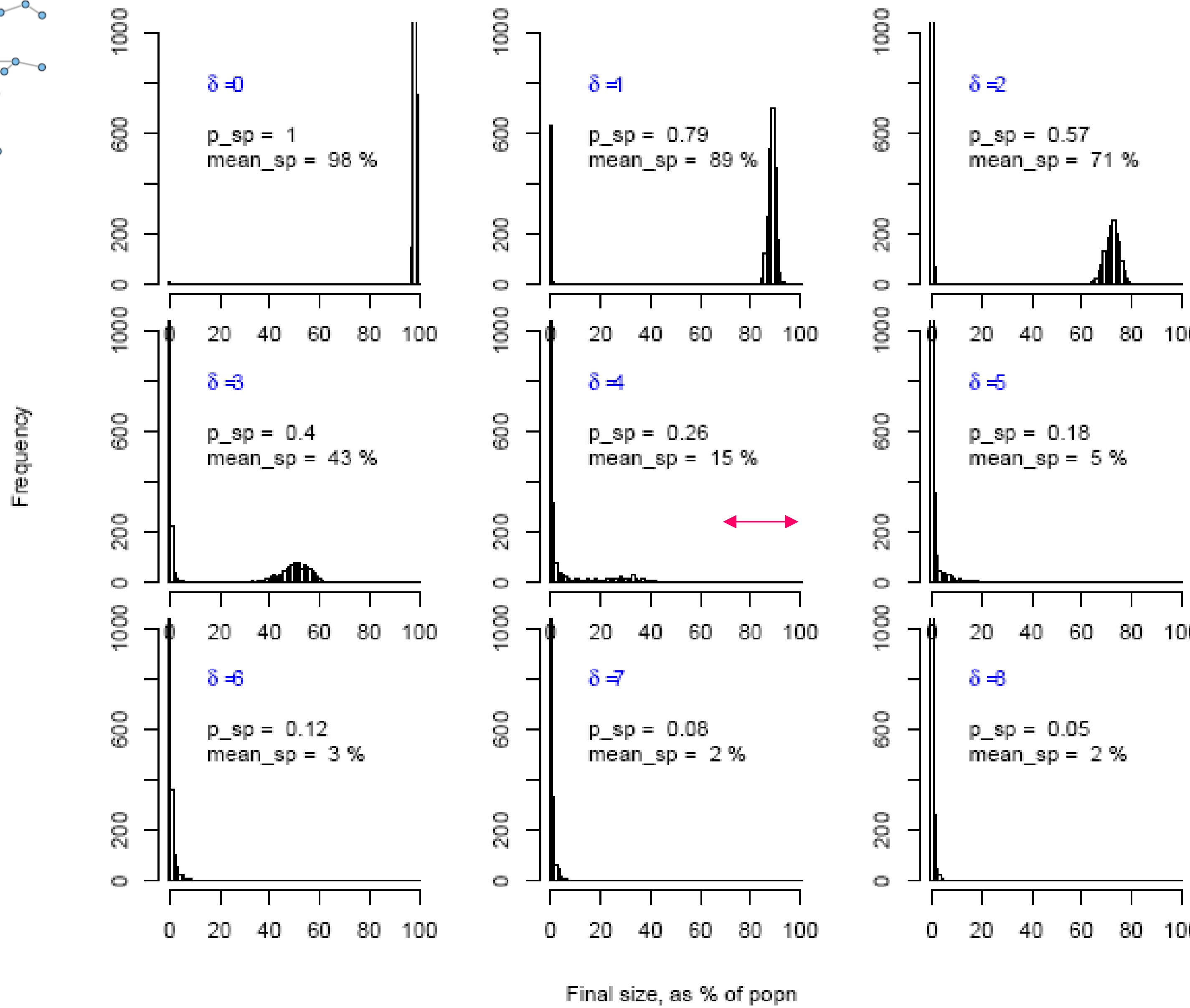
SRG $n=10^3$, $\lambda=1$, $\mathrm{E}(D)=6$, $p=0.1$, $\psi = \delta/[\lambda \mathrm{E}(D)] = 0,\ldots,1.33$ RGG



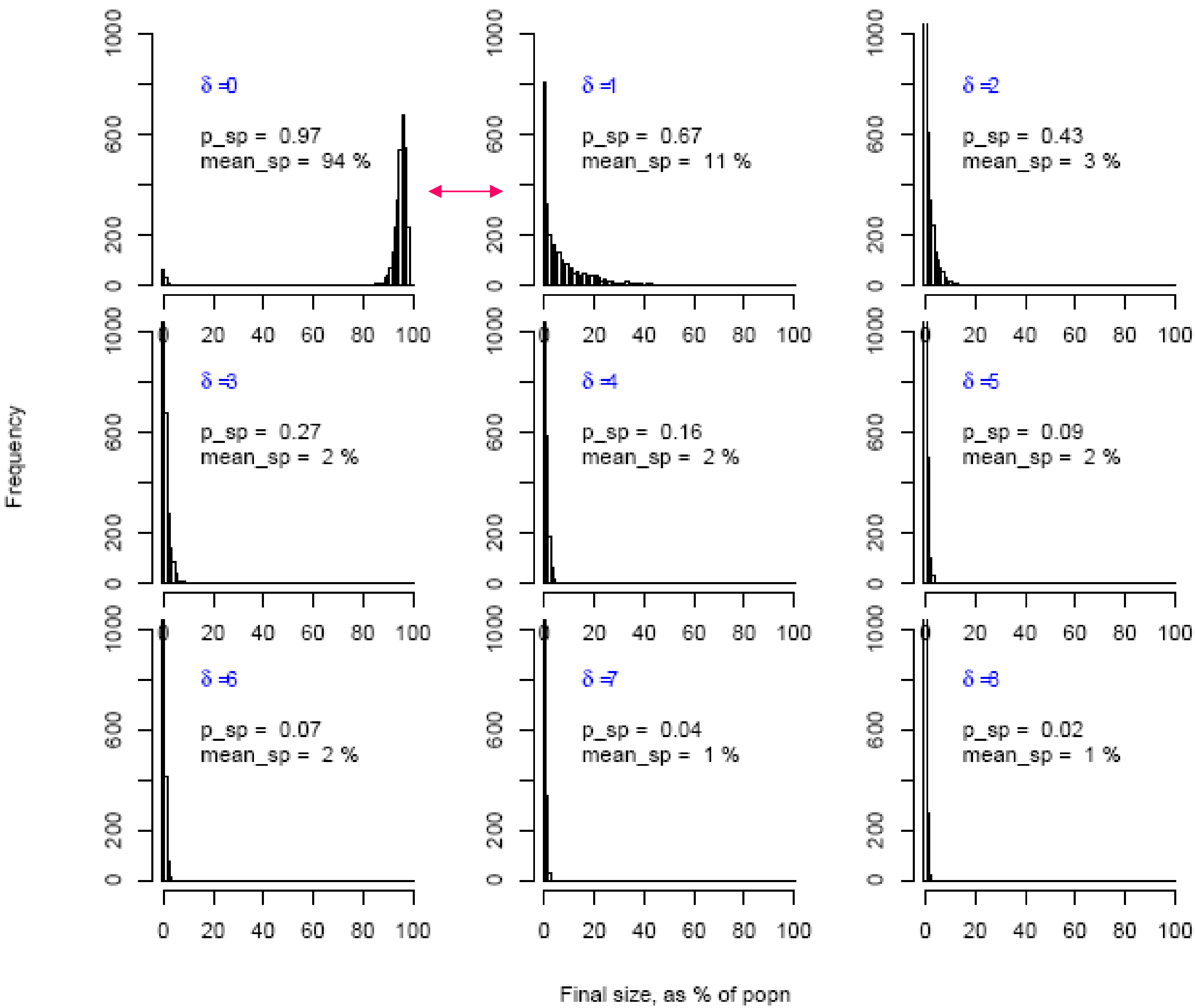


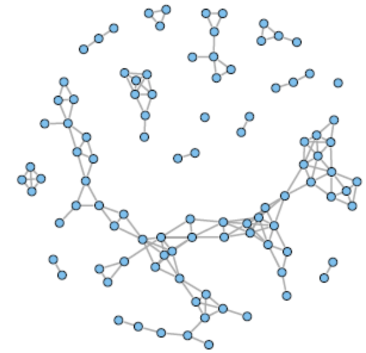


Simple Random Network – Full model



Geometric Random Network – Full model





Effect of network structure on thresholds

For n =1000 and total spreading rate λE(D)=6, threshold δ values are

Mean degree	S R G	B-A	R G G
2	2-3	not bimodal	not bimodal
4	3-4	7	not bimodal
6	~5	8	0-1
12	5-6	9	2-3

Thresholds in δ are higher for scale-free networks

effect of hubs - deterministic approx threshold $\delta = \lambda\left(\mu_K + \sigma_K^2/\mu_K\right)$

and much lower for random geometric graphs

effect of high clustering/correlation?

When E(D)=6, mean geodesic distances of these networks were

4.1 (SRG), 4.3 (B-A), 16 (RGG)

For a homogeneously mixing population, threshold is δ = 6.

Effect of network correlation on thresholds

Rewiring algorithm (*cf* Xulvi-Brunet and Sokolov (2004))

- Take an uncorrelated network with a fixed degree distribution
- Choose two edges at random
- With probability α, rewire the 4 nodes joining the two nodes with the highest degrees and the two with the lowest
- Otherwise the nodes are rewired at random
- If one or both new edges already exists the step is discarded.

The larger α, the larger the limiting (positive) correlation.

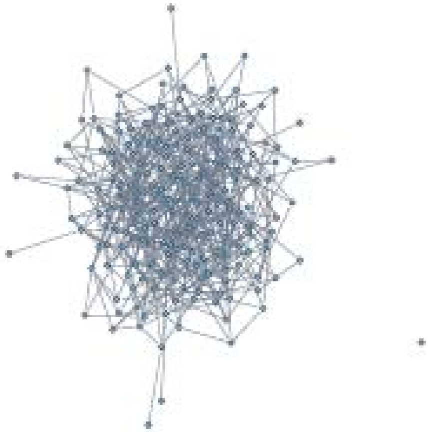
For negative correlations, join nodes with largest and smallest degrees.

Effect of network clustering on thresholds

Rewiring algorithm (*cf* Bansal et al (2009))

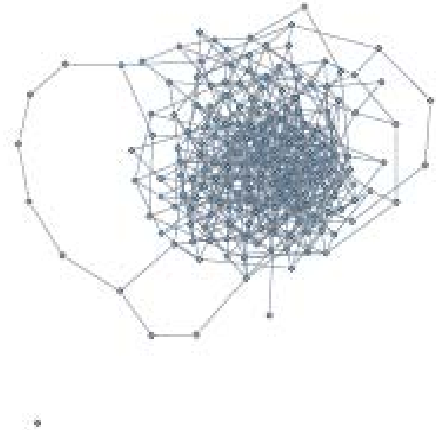
- Take an uncorrelated network with a fixed degree distribution
- Choose a random node x having at least 2 neighbours
- Randomly select two of its neighbours y_1,y_2 (each with degree at least 2)
- Randomly select neighbours z_1 of y_1, z_2 of y_2
- Delete edges (y_1,z_1) and (y_2,z_2) and add edge (y_1,y_2) to create triangle (x,y_1,y_2), and edge (z_1,z_2)
- Step retained if cluster coefficient increases, otherwise discarded

n=200,
E(D)=6



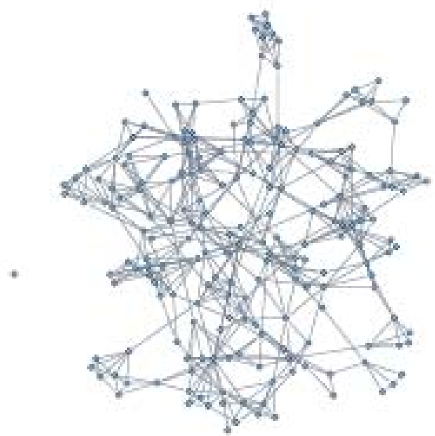
SRG

(a) Simple random graph



SRG + correlation

(b) Re-wired to give 0.59 correlation.



SRG + clustering

(c) Re-wired to give 0.59 clustering coefficient

RGG

(d) Random geometric graph

Comparison of correlated SRG, clustered SRG and RGG

Poisson degrees, n=1000, λ =1, E(D)=6, p=0.1

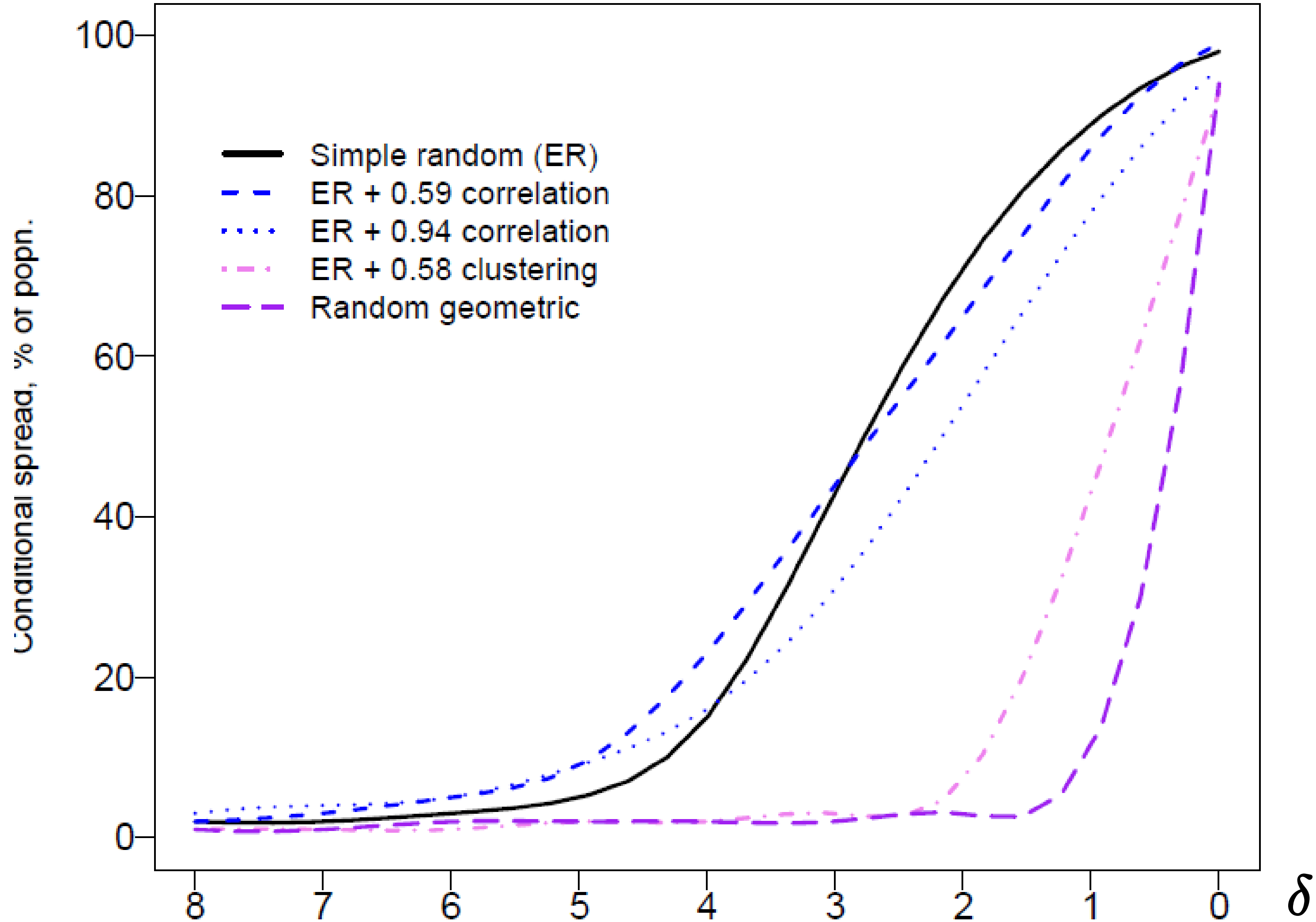
Network	correlation	clustering	geodesic dist	threshold δ
SRG	0.02	0.00	4.1	~5
SRG + corr	0.59	0.01	4.3	~5
SRG + CORR	0.94	0.04	6.1	5-6
SRG + clust	0.04	0.58	7.7	1-2
RGG	0.56	0.58	16	not bimodal

Little effect of increasing SRG correlation to level of RGG or more (by this algorithm)

Increasing clustering in SRG has marked effect in reducing threshold

Note Increasing correlation in B-A network (to 0.3-0.4, with clustering also increasing to about 0.4) also reduces threshold, but not to level of RGG

Conditional distribution of spread as a function of δ



Link with spectral graph theory

Multitype population – reproduction ratio is the largest eigenvalue of the first generation matrix (branching process approx).

Finite network, with each node a separate "type" and constant contact rates between nodes – for SIR, first generation matrix is $\lambda/\delta \times$ adjacency matrix A

How does the largest eigenvalue, and the spectrum more generally, of the adjacency matrix for SRG, BA and RGG graphs relate to properties such as clustering and correlation?

Spectrum of adjacency matrix A

kth spectral moment: $S_k = \sum_i \lambda_i^k$ = no of cycles of length k

$S_0 = n$, S_1=no of self loops (we exclude these),
S_2 = 2xno of edges = total degree,
S_3= 6xno of triangles *etc*

Cluster coeffft = S_3/[no of paths of length 2]

SRG: limiting spectral density – semicircular (Wigner's) law
BA: eigenvalues have a power-law distribution
RGG: spectrum is not symmetric about 0
notable singularity at -1, due to presence of cliques

Summary

- Rumours on networks
  - thresholds
  - final size distribution
  - effects of network structure
    - * variability of degree distribution
    - * for fixed degree distribution, clustering appears more influential than correlation in controlling spread?

- How do thresholds depend on network structure?
- What network properties are most influential in determining spread?

Refs:
Isham, Harden and Nekovee (2009) Physica A, 389, 561-576
Isham, Kaczmarska and Nekovee (2011) Phys. Review E 83, 046128