On a Deformation of Baker's Addition Formula

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The higher genus sigma function and applications

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<u>g=1</u>

$2\omega_1, 2\omega_2 \in \mathbb{C}$, lin. indep./$\mathbb{R}$ $\longrightarrow$ $\wp(u), \zeta(u), \sigma(u)$

AF: $$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u) = \begin{vmatrix} 1 & 1 \\ \wp(u) & \wp(v) \end{vmatrix}$$

DAF: $$\frac{\sigma(u+v+c)\sigma(u-v)}{\sigma(c)\sigma(u)^2\sigma(v)^2} = - \begin{vmatrix} \varphi(u) & \varphi(v) \\ \partial\varphi(u) & \partial\varphi(v) \end{vmatrix}$$

$$\varphi(u) = \frac{\sigma(u+c)}{\sigma(u)\sigma(c)} \qquad \partial = \frac{d}{du}$$

- DAF $\xrightarrow{c\to 0}$ AF.

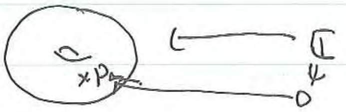
How to understand these formulas.

(1) expansion in a basis

(2) 3-term identity.

<u>(1)</u>

$X = \mathbb{C}/_{2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z}}$ =



$A(2)$ = the space of mero. functions on X which have poles only at P_∞ of order at most 2.

$= \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot\wp(u)$

w.r.t. u.

LHS of AF $\in A(2)$

AF = the expression of LHS as a lin. comb. of a basis of $A(2)$

Similarly for DAF:

- $L_c(2) = \mathbb{C}\cdot\varphi(u) \oplus \mathbb{C}\,\partial\varphi(u)$
- LHS of DAF $\in L_c(2)$

(2) 3-term identity.

$$\sigma(u+a_1)\sigma(u-a_1)\sigma(a_2+a_3)\sigma(a_2-a_3) + (a_1, a_2, a_3 \text{ - cyclic}) = 0$$

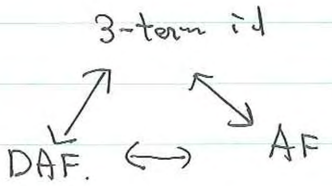
- AF $\xrightarrow{\text{substitution}}$ 3-term id.
  AF $\xleftarrow[\text{coeff. of } a_2^2 a_3^0]{\text{limit}}$ 3-term id.

Deformed case:

- ~~Similarly~~ Make a shift $u \to u+\frac{c}{2}$, $a_i \to a_i + \frac{c}{2}$ in 3-term id:

$$\sigma(u+a_1+c)\sigma(u-a_1)\sigma(a_2+a_3+c)\sigma(a_2-a_3) + (\text{cyclic}) = 0$$

- DAF $\longleftrightarrow$ 3-term id



A remarkable property of 3-term identity

3-term identity:

$$\sigma(z+a)\sigma(z-a)\sigma(b+c)\sigma(b-c)$$
$$+\sigma(z+b)\sigma(z-b)\sigma(c+a)\sigma(c-a)$$
$$+\sigma(z+c)\sigma(z-c)\sigma(a+b)\sigma(a-b)=0.$$

Make a shift $z \longrightarrow z-c$ and a change of variables

$$a=\frac{1}{2}(-x_1+x_2+y_1-y_2), \quad b=\frac{1}{2}(-x_1+x_2-y_1+y_2), \quad c=\frac{1}{2}(x_1+x_2-y_1-y_2).$$

We get

$$\sigma(z+y_1-x_1)\sigma(z+y_2-x_2)\sigma(x_2-y_1)\sigma(y_2-x_1)$$
$$+\sigma(z+y_2-x_1)\sigma(z+y_1-x_2)\sigma(x_2-y_2)\sigma(x_1-y_1)$$
$$+\sigma(z)\underline{\sigma(z-x_1-x_2+y_1+y_2)}\sigma(x_2-x_1)\sigma(y_1-y_2)=0.$$

Solve the "four point function" $\sigma(z - x_1 - x_2 + y_1 + y_2)$ in terms of "two point functions" we get

$$\sigma(z - x_1 - x_2 + y_1 + y_2) = \frac{\sigma(z)^{-1} \prod_{i.j} \sigma(y_j - x_i)}{\sigma(x_2 - x_1)\sigma(y_1 - y_2)} \begin{vmatrix} \frac{\sigma(z+y_1-x_1)}{\sigma(y_1-x_1)} & \frac{\sigma(z+y_2-x_1)}{\sigma(y_2-x_1)} \\ \frac{\sigma(z+y_1-x_2)}{\sigma(y_1-x_2)} & \frac{\sigma(z+y_2-x_2)}{\sigma(y_2-x_2)} \end{vmatrix} .$$

If we make a shift $z \to z - x_3 + y_3$ in this formula, then "6 point function" is expressed by "4-point function". Applying this formula again we get the formula expressing the 6-point function in terms of 2-point functions. In this way 2n-point function is expressed in terms 2-point functions.

Frobenius' Formula

$$\frac{\sigma(z-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i)}{\sigma(z)}\frac{\prod_{i<j}\sigma(x_j-x_i)\sigma(y_i-y_j)}{\prod_{i,j}\sigma(y_j-x_i)}=\det\left(\frac{\sigma(z+y_j-x_i)}{\sigma(z)\sigma(y_j-x_i)}\right)_{1\le i,j\le n}$$

Limits of Frobenius' Formula

Take a limit $x_n \to 0,\ldots,x_1 \to 0$ and set $z = c$ in Frobenius' formula we get a deformation of higher degree addition formula:

$$\frac{\sigma(y_1 + \cdots + y_n + c)\prod_{i<j}\sigma(y_i - y_j)}{\sigma(c)\prod_{i=1}^{n}\sigma(y_i)^n} = \frac{(-1)^{\frac{1}{2}(n-1)(n-2)}}{\prod_{k=1}^{n-1}k!}\begin{vmatrix} 1 & \cdots & 1 \\ \varphi(y_1) & \cdots & \varphi(y_n) \\ \vdots & & \vdots \\ \partial^{n-1}\varphi(y_1) & \cdots & \partial^{n-1}\varphi(y_n) \end{vmatrix},$$

where

$$\varphi(y) = \frac{\sigma(y+c)}{\sigma(y)\sigma(c)}$$

Frobenius-Stickelberger's Formula

Take further the limit $c \to 0$ we get:

$$\frac{\sigma(y_1 + \cdots + y_n) \prod_{i<j} \sigma(y_i - y_j)}{\prod_{i=1}^{n} \sigma(y_i)^n} = \frac{(-1)^{\frac{1}{2}(n-1)(n-2)}}{\prod_{k=1}^{n-1} k!} \begin{vmatrix} 1 & \cdots & 1 \\ \wp(y_1) & \cdots & \wp(y_n) \\ \vdots & & \vdots \\ \wp^{(n-2)}(y_1) & \cdots & \wp^{(n-2)}(y_n) \end{vmatrix} .$$

To summarize we get a diagram

$$
\begin{array}{ccccc}
\text{DAF} & \Longleftrightarrow & \boxed{\text{3-term identity}} & \Longleftrightarrow & \text{addition formula (AF)} \\
\Uparrow & & \Downarrow & & \Uparrow \\
\text{DFSF} & \Longleftarrow & \text{Frobenius' formula} & \Longrightarrow & \text{Frobenius-Stickelberger's formula (FSF).}
\end{array}
$$

- At a glance addition formula does not have a power to produce Frobenius-Stickelberger's formula.

Two Possible Generalizations of This Diagram

(1) higher genus $\cdots$ Fay's formula, addition formula of the tau function of the KP hierarchy,

(2) higher dimension ???

$g = 2$: Baker's Addition Formula (BAF)

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \begin{vmatrix} 1 & 1 \\ \wp_{33}(u) & \wp_{33}(v) \end{vmatrix} + \begin{vmatrix} \wp_{11}(u) & \wp_{11}(v) \\ \wp_{13}(u) & \wp_{13}(v) \end{vmatrix}.$$

- ○ $\sigma(u) = \sigma(u_1, u_3)$
- ○ $\sigma(-u) = -\sigma(u), \quad \sigma(u) = -u_3 + \frac{1}{3}u_1^3 + \cdots$
- ○ $\wp_{i_1 \ldots i_n}(u) = -\partial_{i_1} \cdots \partial_{i_n} \log \sigma(u)$,
- ○ $A(0) = A(1) = \mathbb{C}$, $A(2) = \mathbb{C} \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{13} \oplus \mathbb{C}\wp_{33}$

$g = 2$: A Deformation of BAF

By the expansion in a "basis" we can find the following formula.

$$\frac{\sigma(u+v+c)\sigma(u-v)}{\sigma(c)\sigma(u)^2\sigma(v)^2} = -\frac{\sigma_1(c)}{2\sigma(c)}\begin{vmatrix} \varphi_1(u) & \varphi_1(v) \\ \varphi_2(u) & \varphi_2(v) \end{vmatrix} + \frac{\sigma_{11}(c)}{2\sigma(c)}\begin{vmatrix} \varphi_1(u) & \varphi_1(v) \\ \partial_1\varphi_1(u) & \partial_1\varphi_1(v) \end{vmatrix}$$

$$+\frac{1}{2}\begin{vmatrix} \partial_1\varphi_1(u) & \partial_1\varphi_1(v) \\ \varphi_2(u) & \varphi_2(v) \end{vmatrix} + \begin{vmatrix} \varphi_1(u) & \varphi_1(v) \\ \partial_3\varphi_1(u) & \partial_3\varphi_1(v) \end{vmatrix}$$

- $\varphi_1(u) = \dfrac{\sigma(u+c)}{\sigma(c)\sigma(u)}, \qquad \partial_i = \dfrac{\partial}{\partial u_i}.$
- $\varphi_2(u) = \dfrac{D_1^2\sigma(u+c)\cdot\sigma(u)}{\sigma(c)\sigma(u)^2} = \dfrac{\sigma_{11}(u+c)\sigma(u) - 2\sigma_1(u+c)\sigma_1(u) + \sigma(u+c)\sigma_{11}(u)}{\sigma(c)\sigma(u)^2},$
- $L_c(2) = \mathbb{C}\varphi_1 \oplus \mathbb{C}\partial_1\varphi_1 \oplus \mathbb{C}\partial_3\varphi_1 \oplus \mathbb{C}\varphi_2$
- Taking $c_1 \to 0$ firstly and $c_3 \to 0$ secondly we get BAF.

Problem

- What is the two dimensional analogue of the 3-term identity ?

A Derivation of 3-term identity of $g = 1$

Since

$$\dim(\text{the space of degree two theta functions}) = 2$$

three functions

$$\sigma(u + a_i)\sigma(u - a_i),\ i = 1, 2, 3$$

are linearly dependent. Thus we have a relation

$$\sum_{i=1}^{3} A_i \sigma(u + a_i)\sigma(u - a_i) = 0.$$

Put $u = a_i$, $i = 1, 2, 3$ and get linear equations for A_i. By solving this equation we get

$$A_1 = f_{23}, \quad A_2 = f_{31}, \quad A_3 = f_{12}, \quad f_{ij} = \sigma(a_i + a_j)\sigma(a_i - a_j).$$

It gives the 3-term identity.

Two Dimensional Case

In this case

$$\dim(\text{the space of degree two theta functions}) = 2^2 = 4.$$

Thus five functions

$$\sigma(u + a_i)\sigma(u - a_i),\ i = 1, 2, 3, 4, 5$$

are linearly dependent and we have a relation of the form

$$\sum_{i=1}^{5} A_i \sigma(u + a_i)\sigma(u - a_i) = 0.$$

Put $u = a_i$, $i = 1, 2, 3, 4, 5$ and get linear equations for A_i. By solving this equation we get some equation.

$g = 2$: 5-Term Identity

$$
\begin{aligned}
&\underline{\sigma(u+a_1)\sigma(u-a_1)}\Big\{\ \sigma(a_2+a_3)\sigma(a_2-a_3)\sigma(a_4+a_5)\sigma(a_4-a_5) \\
&\qquad +\sigma(a_2+a_4)\sigma(a_2-a_4)\sigma(a_5+a_3)\sigma(a_5-a_3)+\sigma(a_2+a_5)\sigma(a_2-a_5)\sigma(a_3+a_4)\sigma(a_3-a_4)\Big\} \\
&+\underline{\sigma(u+a_2)\sigma(u-a_2)}\Big\{\ \sigma(a_3+a_4)\sigma(a_3-a_4)\sigma(a_5+a_1)\sigma(a_5-a_1) \\
&\qquad \sigma(a_3+a_5)\sigma(a_3-a_5)\sigma(a_1+a_4)\sigma(a_1-a_4)+\sigma(a_3+a_1)\sigma(a_3-a_1)\sigma(a_4+a_5)\sigma(a_4-a_5)\Big\} \\
&+\underline{\sigma(u+a_3)\sigma(u-a_3)}\Big\{\ \sigma(a_4+a_5)\sigma(a_4-a_5)\sigma(a_1+a_2)\sigma(a_1-a_2) \\
&\qquad +\sigma(a_4+a_1)\sigma(a_4-a_1)\sigma(a_2+a_5)\sigma(a_2-a_5)+\sigma(a_4+a_2)\sigma(a_4-a_2)\sigma(a_5+a_1)\sigma(a_5-a_1)\Big\} \\
&+\underline{\sigma(u+a_4)\sigma(u-a_4)}\Big\{\ \sigma(a_5+a_1)\sigma(a_5-a_1)\sigma(a_2+a_3)\sigma(a_2-a_3) \\
&\qquad +\sigma(a_5+a_2)\sigma(a_5-a_2)\sigma(a_3+a_1)\sigma(a_3-a_1)+\sigma(a_5+a_3)\sigma(a_5-a_3)\sigma(a_1+a_2)\sigma(a_1-a_2)\Big\} \\
&+\underline{\sigma(u+a_5)\sigma(u-a_5)}\Big\{\ \sigma(a_1+a_2)\sigma(a_1-a_2)\sigma(a_3+a_4)\sigma(a_3-a_4) \\
&\qquad +\sigma(a_1+a_3)\sigma(a_1-a_3)\sigma(a_4+a_2)\sigma(a_4-a_2)+\sigma(a_1+a_4)\sigma(a_1-a_4)\sigma(a_2+a_3)\sigma(a_2-a_3)\Big\} \\
&=0.
\end{aligned}
$$

Properties of 5-Term Identity

(1) BAF $\Longleftrightarrow$ 5-term identity

(2) DBAF $\Longleftrightarrow$ 5-term identity

(3) 5-term identity $\Longrightarrow$ Fay's trisecant formula (3-term identity)

A restriction of 5-Term Identity

Make the following substitution in 5-term identity:

$$a_1 = \frac{1}{2}\left(x_1 + x_2 - y_1 - y_2\right),$$

$$a_2 = \frac{1}{2}\left(-x_1 + x_2 + y_1 - y_2\right),$$

$$a_3 = \frac{1}{2}\left(-x_1 + x_2 - y_1 + y_2\right),$$

$$a_4 = \frac{1}{2}\left(x_1 - x_2 + y_1 + y_2\right),$$

$$a_5 = \frac{1}{2}\left(x_1 - x_2 - y_1 - y_2\right).$$

Then we get

$$
\begin{aligned}
&\sigma(u+x_1+x_2-y_1-y_2)\sigma(u)\Big\{\sigma(-x_1+x_2)\sigma(y_1-y_2)\sigma(x_1-x_2)\sigma(y_1+y_2)\\
&\quad+\underline{\sigma(y_1)}\sigma(-x_1+x_2-y_2)\underline{\sigma(-y_1)}\sigma(x_1-x_2-y_2)+\underline{\sigma(-y_2)}\sigma(-x_1+x_2+y_1)\underline{\sigma(y_2)}\sigma(-x_1+x_2-y_1)\Big\}\\
&+\sigma(u+x_2-y_2)\sigma(u+x_1-y_1)\Big\{\underline{\sigma(y_2)}\sigma(-x_1+x_2-y_1)\sigma(x_1-y_1-y_2)\underline{\sigma(-x_2)}\\
&\quad\underline{\sigma(-y_1)}\sigma(-x_1+x_2+y_2)\underline{\sigma(x_1)}\sigma(x_2--y_1-y_2)+\sigma(x_2-y_1)\sigma(-x_1+y_2)\sigma(x_1-x_2)\sigma(y_1+y_2)\Big\}\\
&+\sigma(u+x_2-y_1)\sigma(u+x_1-y_2)\Big\{\sigma(x_1-x_2)\sigma(y_1+y_2)\sigma(x_2-y_2)\sigma(x_1-y_1)\\
&\quad+\underline{\sigma(x_1)}\sigma(-x_2+y_1+y_2)\underline{\sigma(-y_2)}\sigma(-x_1+x_2+y_1)+\underline{\sigma(y_1)}\sigma(x_1-x_2+y_2)\sigma(x_1-y_1-y_2)\underline{\sigma(-x_2)}\Big\}\\
&+\sigma(u+x_1)\sigma(u+x_2-y_1-y_2)\Big\{\sigma(x_1-y_1-y_2)\underline{\sigma(-x_2)}\sigma(-x_1+x_2)\sigma(y_1-y_2)\\
&\quad+\underline{\sigma(-y_2)}\sigma(x_1-x_2-y_1)\sigma(x_2-y_1)\sigma(-x_1+y_2)+\underline{\sigma(-y_1)}\sigma(x_1-x_2-y_2)\sigma(x_2-y_2)\sigma(x_1-y_1)\Big\}\\
&+\sigma(u+x_1-y_1-y_2)\sigma(u+x_2)\Big\{\sigma(x_2-y_2)\sigma(x_1-y_1)\underline{\sigma(y_2)}\sigma(-x_1+x_2-y_1)\\
&\quad+\sigma(x_2-y_1)\sigma(x_1-y_2)\underline{\sigma(y_1)}\sigma(x_1-x_2+y_2)+\underline{\sigma(x_1)}\sigma(x_2-y_1-y_2)\sigma(-x_1+x_2)\sigma(y_1-y_2)\Big\}\\
&=0.
\end{aligned}
$$

We further set x_i, y_j to be abel images of points on a curve. Then

$$\sigma(x_i) = \sigma(y_i) = 0 \qquad \text{for any } i$$

and

$$\sigma(u + x_1 + x_2 - y_1 - y_2)\sigma(u)\sigma(-x_1 + x_2)\sigma(y_1 - y_2)\underline{\sigma(x_1 - x_2)\sigma(y_1 + y_2)}$$

$$+\sigma(u + x_2 - y_2)\sigma(u + x_1 - y_1)\sigma(x_2 - y_1)\sigma(-x_1 + y_2)\underline{\sigma(x_1 - x_2)\sigma(y_1 + y_2)}$$

$$+\sigma(u + x_2 - y_1)\sigma(u + x_1 - y_2)\underline{\sigma(x_1 - x_2)\sigma(y_1 + y_2)}\sigma(x_2 - y_2)\sigma(x_1 - y_1) = 0.$$

Dividing the equation by $\sigma(x_1 - x_2)\sigma(y_1 + y_2)$ we finally get

Fay's Trisecant Formula for $g = 2$

$$\sigma(u + x_1 + x_2 - y_1 - y_2)\sigma(u)\sigma(-x_1 + x_2)\sigma(y_1 - y_2)$$

$$+\sigma(u + x_2 - y_2)\sigma(u + x_1 - y_1)\sigma(x_2 - y_1)\sigma(-x_1 + y_2)$$

$$+\sigma(u + x_2 - y_1)\sigma(u + x_1 - y_2)\sigma(x_2 - y_2)\sigma(x_1 - y_1) = 0.$$

Remaining Important Problem

- Can 5-term identity generate higher degree addition formula ?

g-dimensional case: (2^g+1)-Term Identity

Let $\sigma(u)$, $u = (u_1, ..., u_g)$ be any odd theta function of degree one. Set

- $f_{ij} = \sigma(a_i + a_j)\sigma(a_i - a_j) = -f_{ji} \qquad i, j = 0, ..., n, \quad n = 2^g + 1,$
- $M = (f_{ij})$: $(2^g + 2) \times (2^g + 2)$ skew symmetric matrix

We have

$$\sum_{i=1}^{n} (-1)^{i-1} \sigma(u + a_i)\sigma(u - a_i) \ \mathrm{pf}(M_{0,i}^{0,i}) = 0.$$

Here $M_{0,i}^{0,i}$ is the matrix obtained from M by removing 0-th and i-th rows and columns. For example

$$M_{0,1}^{0,1} = \begin{pmatrix} 0 & f_{23} & \cdots & f_{2n} \\ f_{32} & 0 & \cdots & f_{3n} \\ \vdots & \ddots & & \vdots \\ f_{n2} & f_{n3} & \cdots & 0 \end{pmatrix} \qquad 2^g \times 2^g \text{ matrix.}$$

Generating Function of Addition Formula

By expanding the Pfaffian further we have

$$\sigma(u+v)\sigma(u-v)\ \mathrm{pf}(M_{0,1}^{0,1})$$

$$= \sum_{2\le i<j\le n} (-1)^{i+j-1} \begin{vmatrix} \sigma(u+a_i)\sigma(u-a_i), & \sigma(v+a_i)\sigma(v-a_i) \\ \sigma(u+a_j)\sigma(u-a_j), & \sigma(v+a_j)\sigma(v-a_j) \end{vmatrix} \mathrm{pf}(M_{0,1,i,j}^{0,1,i,j}).$$

- The formula is symmetric with respect to any permutation of $a_2, \ldots a_n$. up to sign.
- (2^g+1)-term identity $\Longleftrightarrow$ AF
- (2^g+1)-term identity $\Longleftrightarrow$ DAF

Conjecture

Consider (n,s)-curves $y^n + x^s + \sum \lambda_{ij} x^i y^j = 0$ such that $1/24(n^2-1)(s^2-1)$ is odd and the corresponding odd sigma function $\sigma(u)$. Then the Schur limit $(\forall \lambda_{ij} \to 0)$ of $\mathrm{pf}(\mathrm{M}_{0,1}^{0,1})$ is not zero.

The conjecture implies that

- the addition formula has polynomial coefficients in $\lambda_{ij}'s$,
- if the coeficient of $a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ in the Schur limit of $\mathrm{pf}(\mathrm{M}_{0,1}^{0,1})$ is not zero $D^{\alpha_i}\sigma(u) \cdot \sigma(u), \ i = 2, ..., 2^g + 1$ is a basis of degree two theta functions.

Remaining Important Problems

(1) Can $(2^g + 1)$-term identity generate higher degeree addition formula?

(2) How $(2^g + 1)$-term identity tends to Fay's formula in the case of Jacobians?