Computing Bitangents of Quartics using Riemann Theta Functions

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Introduction

The connection between theta functions of odd characteristic and bitangents to quartic curves was known to Riemann. The work of Bobenko, Deconinck, Heil, van Hoeij, and Schmies [1, 2] allows one to compute the bitangents of a quartic. This poster describes this connection and presents an example computation using the `algcurves` package in Maple.

Geometry of Algebraic Curves

Let $C: f(x,y) = a_n(x)y^n + \cdots + a_1(x)y + a_0(x) = 0$ be a complex plane algebraic curve and $F(z_1,z_2,z_3) = z_3^d f(z_1/z_3, z_2/z_3)$ be its homogenization where d is the total degree of the curve. An algebraic curve defines an **n-sheeted covering** $y(x)$ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A **branch point** of this covering is a point $x = b \in \mathbb{C}$ such that $y(x)$ does not return to its original value when analytically continued around a small circle about $x = b$. That is, if $y(x)$ has any branch points then $y(x)$ is multi-valued.

The Riemann surface Γ of the curve C is a one-dimensional complex manifold where $y(x)$ is single-valued. Furthermore, it is a result from topology that the Riemann surface of an algebraic curve is homeomorphic to a genus g compact, connected surface.

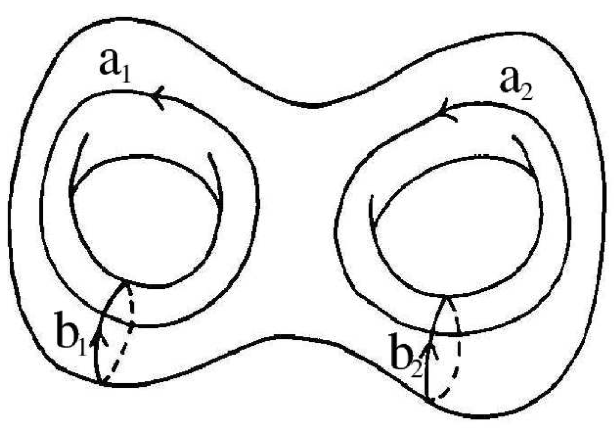

Figure 1: *A genus 2 Riemann surface with a canonical basis of cycles.*

A Riemann surface admits a homology basis $a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(\Gamma, \mathbb{Z})$ of cycles with intersection indices $a_i \circ a_j = b_i \circ b_j = 0, a_i \circ b_j = \delta_{ij}$ and a cohomology basis $\omega_1, \ldots, \omega_g \in H^0(\Gamma)$ of holomorphic differentials. These define a period matrix $\tilde{\Omega} = (A\, B) \in \mathbb{C}^{g \times 2g}$ where

$$A = (A_{ij})_{i,j=1}^g, \quad A_{ij} = \oint_{a_j} \omega_i,$$
$$B = (B_{ij})_{i,j=1}^g, \quad B_{ij} = \oint_{b_j} \omega_i.$$

With an appropriate choice of cohomology basis such that $\oint_{a_j} \omega_i = \delta_{ij}$, we have the normalized period matrix $\tilde{\Omega} = (I\, \Omega)$. Ω is called the **Riemann matrix** of the algebraic curve.

The Riemann Theta Function

Elliptic functions have application in computing elliptic integrals, solving differential equations, number theory, and more. Abelian functions are higher-genus generalizations of elliptic functions. We choose to study Riemann theta functions, however, since every abelian function can be written in terms of rational functions of theta functions and their derivatives. [1, 3]

First, given a Riemann surface Γ and its Riemann matrix Ω consider the lattice $\Lambda = \{m + \Omega n \mid m, n \in \mathbb{Z}^g\}$. Define the **Jacobian** of Γ

$$J(\Gamma) = \mathbb{C}^g / \Lambda.$$

Definition 1. *Let $\Omega \in \mathbb{C}^{g \times g}$ be a Riemann matrix; i.e. Ω is symmetric and $\operatorname{Im}(\Omega)$ is strictly positive definite. Then the Riemann theta function, $\theta : J(\Gamma) \to \mathbb{C}$ is defined as*

$$\theta(z|\Omega) = \sum_{n \in \mathbb{Z}} e^{2\pi i (\frac{1}{2} n \cdot \Omega n + z \cdot n)}.$$

This series converges absolutely in z and Ω and uniformly on compact sets. Note that the function is periodic with period 1 in each z component and is quasi-periodic in the columns of Ω; a shift $z \mapsto z + \Omega e_j$ produces a scaling factor $\exp(2\pi i(-\frac{1}{2}\Omega_{jj} - z_j)$.

Definition 2. *Given $\alpha, \beta \in [0,1)^g$ we define the Riemann theta function with characteristic $[\alpha, \beta]$:*

$$\theta[\alpha,\beta](z|\Omega) = \sum_{n \in \mathbb{Z}} e^{2\pi i (\frac{1}{2}(n+\alpha)\cdot\Omega(n+\alpha) + (z+\beta)\cdot(n+\alpha))}.$$

Note

$$\theta[\alpha,\beta](z|\Omega) = e^{2\pi i (\alpha\cdot\Omega\alpha + z\cdot\alpha + \frac{1}{2}\alpha\cdot\beta)} \theta(z + \Omega\alpha + \beta|\Omega).$$

Theorem 3. *Up to sign there are 2^{2g} different theta functions with characteristics $\alpha, \beta \in \{0, \frac{1}{2}\}^g$. Of these, $2^{g-1}(2^g+1)$ are even functions and $2^{g-1}(2^g-1)$ are odd. They are precisely those for which $2\alpha \cdot \beta \equiv 0 \pmod 2$ and $2\alpha \cdot \beta \equiv 1 \pmod 2$, respectively. (Such characteristics are called even and odd characteristics, respectively.)*

Bitangents of Quartics

A **bitangent line** to a plane curve C is a line tangent to C at two distinct points. Bézout's theorem implies that a real plane curve with a bitangent must have degree at least 4. In 1839, Plücker showed that the number of real bitangents of any quartic must by 28, 16, or a number less than 9.

It is well known that there is a bijection between odd theta functions and bitangents to quartics. In particular, the correspondence is given by the following theorem.

Theorem 4. *Given a plane quartic curve C (with homogenous coordinates z_1, z_2, z_3) with its Riemann matrix Ω and corresponding Riemann theta function, θ; its bitangent lines are given by the linear term of the Taylor expansions*

$$\sum_{i=1}^{3} \frac{\partial \theta[\alpha,\beta](0|\Omega)}{\partial z_i} z_i = 0$$

for all odd characteristics $[\alpha, \beta]$.

Example: The Edge Quartic

An example quartic, suggested by Plaumann, Sturmfels, and Vinzant [4] is the Edge quartic from the family of quartics studied by William L. Edge. (1938) Its equation in homogenous coordinates is

$$25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2) = 0.$$

The curve corresponds to a genus 3 Riemann surface and, hence, there are 28 odd theta characteristics. Using Deconinck and van Hoeij's `algcurves` package in Maple, which can compute Riemann matrices of algebraic curves and the corresponding Riemann theta function to desired accuracy, we can use Theorem 4 to compute all of its bitangent lines.

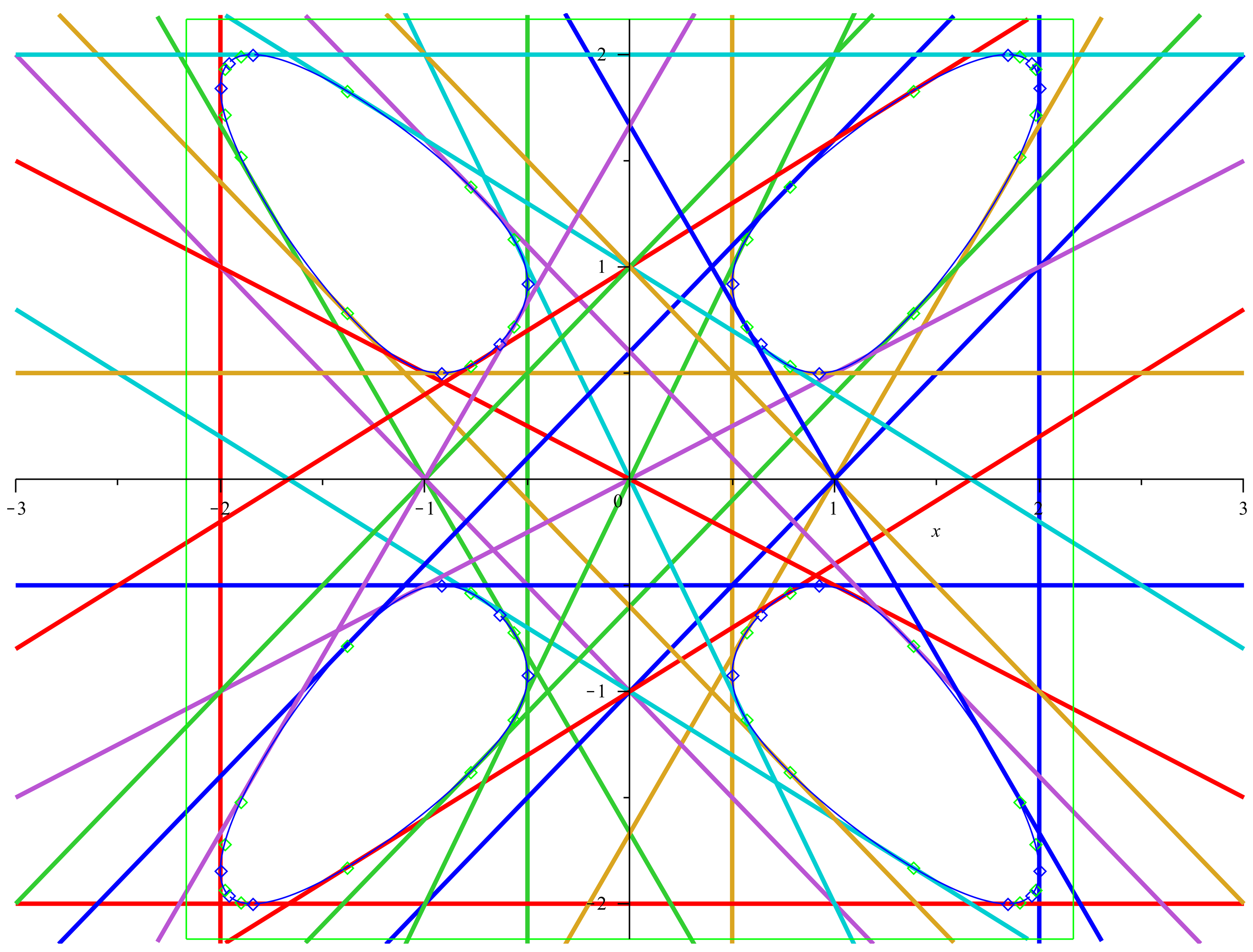

Figure 2: *The Edge quartic with 28 real bitangents. (Four at infinity.)*

References

[1] Bernard Deconinck, Matthias Heil, Alexander Bobenko, Mark van Hoeij, and Marcus Schmies, *Computing Riemann theta functions*, Mathematics of Computation **73** (2003), no. 247, 1417–1442.

[2] Bernard Deconinck and Mark van Hoeij, *Computing Riemann matrices of algebraic curves*, Physica D: Nonlinear Phenomena (2001).

[3] Jun ichi Igusa, *Theta functions*, Springer-Verlag, New York, 1972.

[4] Daniel Plaumann, Bernd Sturmfels, and Cynthia Vinzant, *Quartic curves and their bitangents*, (preprint).

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