Commuting ordinary differential operators of rank 2

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$$L_1 = \frac{d^n}{dx^n} + u_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + u_0(x),$$

$$L_2 = \frac{d^m}{dx^m} + v_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \cdots + v_0(x).$$

Lemma (Schur, 1905)

If $L_1L_2 = L_2L_1$ and $L_1L_3 = L_3L_1$, then $L_2L_3 = L_3L_2$.

Lemma (Burchnall, Chaundy, 1923)

If $L_1L_2 = L_2L_1$, then there exist a non-trivial polynomial $Q(\lambda, \mu)$ of two commuting variables such that $Q(L_1, L_2) = 0$.

Example

$$L_1 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \quad L_2 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3}$$

$$L_1^3 = L_2^2, \quad Q(\lambda, \mu) = \lambda^3 - \mu^2.$$

Spectral curve

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : Q(\lambda, \mu) = 0\}.$$

If $L_1\psi = \lambda\psi$ and $L_2\psi = \mu\psi$, then $(\lambda, \mu) \in \Gamma$.

rank of L_1 and L_2 is

$$l = \dim\{\psi : L_1\psi = \lambda\psi, \; L_2\psi = \mu\psi\}.$$

Baker–Akhiezer function $\psi(x,P)$

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_g\}$$

Γ is algebraic curve, $q \in \Gamma$, k^{-1} is a local parameter near q, $k^{-1}(q) = 0$, $\gamma_1, \ldots \gamma_g \in \Gamma$.

The Baker–Akhiezer function has the property:

1. $\psi = e^{kx}\left(1 + \frac{f(x)}{k} + \ldots\right)$

2. on $\Gamma \backslash q$ the BA-function ψ is meromorphic with the poles in $\gamma_1, \ldots, \gamma_g$

For $\gamma_1, \ldots, \gamma_g$ in general position the BA-function ψ there exists and unique.

Let f(P) be a meromorphic function on Γ with a unique pole in q of order n

$$f = k^n + c_{n-1}k^{n-1} + \cdots + c_0 + \frac{c_{-1}}{k} + \dots$$

$$\partial_x^n \psi = k^n e^{kx}(O(1)),$$

$$\partial_x^n \psi - f\psi = k^{n-1} e^{kx}\left(u_{n-1}(x) + O\left(\frac{1}{k}\right)\right),$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi - f\psi = k^{n-2} e^{kx}\left(u_{n-2}(x) + O\left(\frac{1}{k}\right)\right),$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi + \cdots + u_0(x)\psi = f\psi + e^{kx}\left(O\left(\frac{1}{k}\right)\right).$$

From the uniqueness of BA-function it follows that

$$L_1\psi(x,P) = f(p)\psi(x,P).$$

Let g(P) be a meromorphic function on Γ with unique pole in q of order m, then

$$L_2\psi(x,P) = g(P)\psi(x,P).$$

We have

$$(L_1L_2 - L_2L_1)\psi(x,P) = 0 \Rightarrow L_1L_2 = L_2L_1.$$

Example $\Gamma = \mathbb{C}P^2,\ q = \infty,\ k = z$

Baker–Akhiezer function $\psi = e^{xz}$

$$f = z^n + c_{n-1}z^{n-1} + \cdots + c_0,$$

$$\partial_x^n\psi + c_{n-1}\partial_x^{n-1}\psi + \cdots + c_0\psi = f\psi.$$

Example

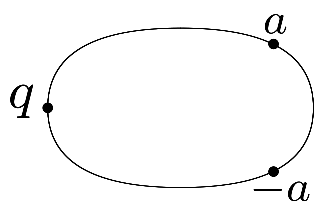
$$\Gamma = \mathbb{C}/\{2\omega\mathbb{Z} + 2\omega'\mathbb{Z}\}, \;\; q = 0,$$

$$\psi = e^{-x\zeta(z)}\frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$

$$\left(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x)\right)\psi(x,z) = \frac{1}{2}\wp'(z)\psi(x,z).$$

Example $\Gamma = \mathbb{C}P^1/\{a \sim -a\},\ q = \infty,\ g_a = 1,\ k = z$



$$\psi = e^{xz}\left(1 + \frac{\xi(x)}{z-\gamma}\right),$$

$$\psi(a) = \psi(-a) \Rightarrow \xi(x) = \frac{(\gamma^2 - a^2)\,\mathrm{sinh}(ax)}{a\,\mathrm{cosh}(ax) + \gamma\,\mathrm{sinh}(ax)},$$

$$(\partial_x^2 - u(x))\psi = z^2\psi,\ u(x) = -\frac{2a^2(a^2-\gamma^2)}{(a\,\mathrm{cosh}(ax) + \gamma\,\mathrm{sinh}(ax))^2}.$$

Commuting differential operators correspond to the following data (D. Mumford):

a) Γ a complete curve over $\mathbb{C}$

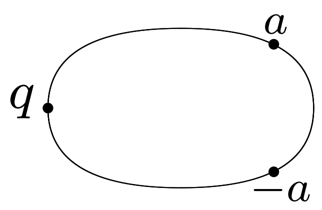
b) $q \in \Gamma$, smooth point, and isomorphism

$$T_q \cong \mathbb{C}$$

c) $\mathcal{F}$ torsion-free sheaf on Γ such that

$$h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0.$$

Example $\Gamma = \mathbb{C}P^1/\{a \sim -a\},\ q = \infty,\ g_a = 1,\ k = z$



$$\psi = e^{xz}\left(1 + \frac{\xi(x)}{z-\gamma}\right),$$

$$\psi(a) = 2\psi(-a) \Rightarrow \xi(x) = \frac{(\gamma^2 - a^2)(-2 + e^{2ax})}{2a - 2\gamma + e^{2ax}(a+\gamma)},$$

$$(\partial_x^2 - u(x))\psi = z^2\psi,\ u(x) = -\frac{16a^2e^{2ax}(a^2-\gamma^2)}{(2a - 2\gamma + e^{2ax}(a+\gamma))^2}.$$

Rank $l > 1$

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \dots, \gamma_{lg}, \alpha_1, \dots, \alpha_{lg}\}$$

$\alpha_i = (\alpha_{1i}, \dots, \alpha_{il-1})$ — vector

(γ, α) — Turin parameters define stable (in the sense of Mumford) vector bundle of rank l degree lg on Γ with holomorphic sections $\eta_1, \dots, \eta_l$

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{ij} \eta_j(\gamma_i).$$

Vector Baker–Akhiezer function $\psi(x,P) = (\psi_0(x,P),\dots,\psi_{l-1}(x,P))$:

1. $\psi(x,P) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right) \Psi_0(x,P)$, $\xi_0 = (1,0,\dots,0)$, $\frac{d}{dx}\Psi_0 = A\Psi_0$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k+u_0(x) & u_1(x) & u_2(x) & \dots & u_{l-1}(x) & 0 \end{pmatrix}$$

2. on $\Gamma - \{q\}$ ψ is meromorphic with the simple poles in $\gamma_1,\dots,\gamma_{lg}$

3. $\mathrm{Res}_{\gamma_i}\psi_j = \alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{l-1}$.

If f(P) is meromorphic function with the pole in q of order n, then there exist $L(f)$ such that

$$L(f)\psi(x,P) = f(P)\psi(x,P), \ \mathrm{ord}L(f) = ln.$$

Method of Turin parameters deformation (Krichever–Novikov method)

$$\frac{d^l}{dx^l}\psi_j = \chi_{l-1}\frac{d^{l-1}}{dx^{l-1}}\psi_j + \cdots + \chi_0\psi_j$$

χ_s — meromorphic on Γ, χ_s has lg simple poles $P_1(x), \dots, P_{lg}(x)$. In the neighbourhood of q the functions χ_s have the form

$$\chi_0(x, P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad j < l - 1,$$

$$\chi_{l-1}(x, P) = O(k^{-1}).$$

At the point $P_i(x)$ one has the expansions

$$\chi_j = \frac{c_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)).$$

Theorem *Parameters* $\gamma_i(x), \alpha_{ij}(x) = \frac{c_{ij}(x)}{c_{i,l-1}(x)}$, *and* $d_{ij}(x), 0 \le j \le l - 2, 1 \le i \le lg$ *satisfy the equation*

$$c_{i,l-1}(x) = -\gamma_i'(x),$$

$$d_{i0}(x) = \alpha_{i0}(x)\alpha_{i,l-2}(x) + \alpha_{i0}(x)d_{i,l-1}(x) - \alpha_{i0}'(x),$$

$$d_{ij}(x) = \alpha_{ij}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{ij}(x)d_{i,l-1}(x) - \alpha_{ij}'(x), j \ge 1.$$

Dixmier: $g = 1,\ l = 2$

$$L_1 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^2 - 2x,$$

$$L_2 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^3 - \frac{3}{2}\left(x\left(\frac{d^2}{dx^2} - x^3 - \alpha\right) + \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)x\right).$$

Krichever, Novikov: $g = 1,\ l = 2$

$$\Gamma : \mu^2 = P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3$$

$$L_1 = \left(\partial_x^2 + u\right)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1),$$

$$\gamma_1(x) = \gamma_0 + c(x),\ \gamma_2(x) = \gamma_0 - c(x),$$

$$u(x) = -\frac{1}{4c_x^2} + \frac{1}{2}\frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1, \gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0 + c, \gamma_0 - c) - \Phi^2(\gamma_1, \gamma_2)),$$

$$\Phi(\gamma_1, \gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

Operator L_2 can be find from the equation

$$L_2^2 = P_3(L_1).$$

Theorem (Grinevich)

Commuting operators L_1 and L_2 corresponding to an elliptic curve have rational coefficients if and only if

$$c(x) = \int_{q_(x)}^{\infty} \frac{dt}{\sqrt{P_3(t)}},$$

where $q(t)$ is a rational function.

If $\gamma_0 = 0$, and $q(x) = x$, we have the Dixmier operators.

Theorem (Grinevich, Novikov) *Operator L_1 is formally self-adjoint if and only if $\wp(\gamma_1) = \wp(\gamma_2)$.*

Rank $l = 2,\ g > 1$: self-adjoint case

Let L be an operator of the forth order of rank 2, then

$$\Gamma : w^2 = F(z) = z^{2g+1} + c_{2g} z^{2g} + \dots + c_0,$$

q is a branch point,

$$\sigma : \Gamma \to \Gamma,\ \sigma(z, w) = (z, -w).$$

We have

$$\psi'' = \chi_0 \psi + \chi_1 \psi',$$

where $\psi = (\psi_1, \psi_2)$ is a Baker–Akhiezer function.

Theorem (M.) *If*

$$\chi_1(x, P) = \chi_1(x, \sigma(P)),$$

then operator L is self-adjoint

$$L = L^* = (\partial_x^2 + V(x))^2 + W(x).$$

If $g = 2$, and L is self-adjoint then $\chi_1(x, P) = \chi_1(x, \sigma(P))$.

$$w^2 = F(z) = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0,$$

$$\chi_0 = -\frac{1}{2}\frac{H_1(x)\gamma_1'(x)}{z-\gamma_1(x)} - \cdots - \frac{1}{2}\frac{H_g(x)\gamma_g'(x)}{z-\gamma_g(x)} + \frac{w}{2(z-\gamma_1)\dots(z-\gamma_g)} + \frac{\kappa(x)}{2},$$

$$\chi_1(x,P) = -\frac{\gamma_1'(x)}{z-\gamma_1(x)} - \cdots - \frac{\gamma_g'(x)}{z-\gamma_g(x)},$$

$$P_1(x) = (\gamma_1, \sqrt{F(\gamma_1)}), \dots, P_g(x) = (\gamma_g, \sqrt{F(\gamma_g)}),$$

$$P_{g+1}(x) = (\gamma_1, -\sqrt{F(\gamma_1)}), \dots, P_{2g}(x) = (\gamma_g, -\sqrt{F(\gamma_g)}),$$

$$\sigma(P_i) = P_{i+g},\ 1 \le i \le g.$$

$$d_{1,1} = d_{g+1,1} = -\frac{\gamma_2'(x)}{\gamma_1(x) - \gamma_2(x)} - \cdots - \frac{\gamma_g'(x)}{\gamma_1(x) - \gamma_g(x)},$$

$$\ldots\ldots$$

$$d_{g,1} = d_{2g,1} = -\frac{\gamma_1'(x)}{\gamma_g(x) - \gamma_1(x)} - \cdots - \frac{\gamma_{g-1}'(x)}{\gamma_g(x) - \gamma_{g-1}(x)}.$$

$$\alpha_i = \frac{H_i}{2} - \frac{w(\gamma_i)}{2\gamma_i'(\gamma_i - \gamma_1)\ldots(\gamma_i - \gamma_{i-1})(\gamma_i - \gamma_{i+1})\ldots(\gamma_i - \gamma_g)},$$

$$\ldots\ldots$$

$$\alpha_{i+g} = \frac{H_i}{2} + \frac{w(\gamma_i)}{2\gamma_i'(\gamma_i - \gamma_1)\ldots(\gamma_i - \gamma_{i-1})(\gamma_i - \gamma_{i+1})\ldots(\gamma_i - \gamma_g)}, \ \ 1 \le i \le g,$$

$$d_{i0} = -\frac{1}{2}\frac{H_1\gamma_1'}{\gamma_i - \gamma_1} - \dots - \frac{1}{2}\frac{H_{i-1}\gamma_{i-1}'}{\gamma_i - \gamma_{i-1}} - \frac{1}{2}\frac{H_{i+1}\gamma_{i+1}'}{\gamma_i - \gamma_{i+1}} - \dots - \frac{1}{2}\frac{H_g\gamma_g'}{\gamma_i - \gamma_g} +$$

$$+\partial_z\left(\frac{w(z)}{2(z-\gamma_1)\dots(z-\gamma_{i-1})(z-\gamma_{i+1})\dots(z-\gamma_g)}\right)|_{z=\gamma_i} + \frac{\kappa}{2},$$

$$\dots\dots$$

$$d_{i+g,0} = -\frac{1}{2}\frac{H_1\gamma_1'}{\gamma_i - \gamma_1} - \dots - \frac{1}{2}\frac{H_{i-1}\gamma_{i-1}'}{\gamma_i - \gamma_{i-1}} - \frac{1}{2}\frac{H_{i+1}\gamma_{i+1}'}{\gamma_i - \gamma_{i+1}} - \dots - \frac{1}{2}\frac{H_g\gamma_g'}{\gamma_i - \gamma_g} -$$

$$-\partial_z\left(\frac{w(z)}{2(z-\gamma_1)\dots(z-\gamma_{i-1})(z-\gamma_{i+1})\dots(z-\gamma_g)}\right)|_{z=\gamma_i} + \frac{\kappa}{2}.$$

We have $2g$ equations on $\gamma_1, \dots, \gamma_g, H_1, \dots, H_g, \kappa$

$$L_i = d_{i0}(x) - (\alpha_i^2(x) + \alpha_i(x) d_{i1}(x) - \alpha_i'(x)) = 0, \ 1 \le i \le 2g.$$

These equations can be reduced the $g-1$ equations on $\gamma_1, \dots, \gamma_g$. From

$$L_i - L_{i+g} = 0$$

one can express H_i in terms of $\gamma_1, \dots \gamma_g$. From

$$L_i + L_{i+g} = 0$$

one can express κ in terms of $\gamma_1, \dots \gamma_g$, and H_i. We have $g-1$ equations on $\gamma_1, \dots, \gamma_g$. Operator L has the form

$$L = L^* = (\partial_x^2 + V(x))^2 + W(x),$$

where

$$V = -\frac{\kappa}{2}, \ W = -\frac{1}{2}(\gamma_1 + \cdots + \gamma_g).$$

Example: $g = 2$ The equations on γ_1, γ_2 have the form

$$4\gamma_1'^4\gamma_2'^2+\gamma_2'^2(c_0+c_3\gamma_1^3+c_4\gamma_1^4+\gamma_1^5-\gamma_2^2\gamma_1''^2+\gamma_1^2(c_2-\gamma_1''^2)+\gamma_1(c_1+2\gamma_2\gamma_1''^2))$$

$$+2(\gamma_2-\gamma_1)\gamma_1'^3\gamma_2'\gamma_2''+2(\gamma_1-\gamma_2)\gamma_1'\gamma_2'^2((\gamma_1-\gamma_2)\gamma_1'''-\gamma_2'\gamma_1'')$$

$$-\gamma_1'^2(c_0+c_3\gamma_2^3+c_4\gamma_2^4+\gamma_2^5$$

$$+4\gamma_2'^4-\gamma_1^2\gamma_2''^2+6\gamma_1\gamma_2'^2(\gamma_1''+\gamma_2'')+2\gamma_1^2\gamma_2'\gamma_2'''+\gamma_2^2(c_2-\gamma_2''^2+2\gamma_2'\gamma_2''')$$

$$+\gamma_2(c_1+2\gamma_1\gamma_2''^2-6\gamma_2'^2(\gamma_1''+\gamma_2'')-4\gamma_1\gamma_2'\gamma_2'''))=0.$$

If

$$\gamma_1 = \gamma, \quad \gamma_2 = \gamma + c,$$

we have

$$(c^4 + c_1 + cc_2 + c^2 c_3) + (5c_3 + 2c_2 + 3cc_3)\gamma + (10c^2 + 3c_3)\gamma^2$$

$$+10c\gamma^3 + 5\gamma^4 - 16\gamma'^2\gamma'' = 0,$$

$$\gamma^{-1}(y) = \int \left(\frac{4}{3P(y)}\right)^{\frac{1}{4}} dy,$$

$P = y^5 + \frac{5}{2}cy^4 + \frac{1}{3}(10c^2 + 3c_3)y^3 + \frac{1}{2}(5c_3 + 2c_2 + 3cc_3)y^2 + (c^4 + c_1 + cc_2 + c^2c_3)y + \delta,$
$y = \gamma(x), \delta, c \in \mathbb{C}.$

Example Let Γ is given by the equation

$$w^2 = z^5 - \frac{10}{3}z^3 + \frac{7}{3}z$$

Let

$$c = 2, \quad \delta = 1, \quad \gamma = \frac{1024}{x^4} - 1.$$

Then L has the form

$$\partial_x^4 + \left(\frac{x^6}{49152} - \frac{425}{6x^2}\right)\partial_x^2 + \left(\frac{x^5}{8192} + \frac{425}{3x^3}\right)\partial_x + \frac{x^{12}}{9663676416} - \frac{245x^4}{589824} + \frac{2569}{144x^4}.$$

Rank $l = 2,\ g > 1$: non self-adjoint case

$$\Gamma : w^2 = F(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0,$$

$q = (0, \sqrt{c_0}), c_0 \neq 0$ is not a branch point. The functions $\chi_0(x, P)$ and $\chi_1(x, P)$ have the form

$$\chi_0(x, P) = -\frac{1}{2}\frac{H_1(x)\gamma_1'(x)}{z - \gamma_1(x)} - \cdots - \frac{1}{2}\frac{H_g(x)\gamma_g'(x)}{z - \gamma_g(x)} + \frac{1}{2z} + \frac{\kappa(x)}{2} +$$

$$(-1)^g \frac{1}{2}\frac{w\gamma_1(x)\ldots\gamma_g(x)}{z(z - \gamma_1(x))\ldots(z - \gamma_g(x))},$$

$$\chi_1(x, P) = -\frac{\gamma_1'(x)}{z - \gamma_1(x)} - \cdots - \frac{\gamma_g'(x)}{z - \gamma_g(x)} - \frac{\gamma_1'(x)}{\gamma_1(x)} - \cdots - \frac{\gamma_g'(x)}{\gamma_g(x)}.$$

Main observation:

$$\Gamma : w^2 = z^{2g+2} + c_{g+1} z^{g+1} + c_0,$$

$$\gamma_2 = a_2 \gamma_1, \dots, \gamma_g = a_g \gamma_1, \;\; a_j \in \mathbb{C},$$

where γ_1 satisfies the equation

$$(\gamma_1')^2 = \gamma_1^{g+2} + c\gamma_1, \;\; c \in \mathbb{C}.$$

Then equations on γ_k are reduced to algebraic equations on a_k, c, c_{g+1}, c_0.

Example: $l=2,\ g=2\ w^2=z^6+2z^3+\frac{1}{4},\ \gamma_1(x)=-\gamma_2(x)=\frac{1}{x},$

$$L_1=\partial_x^6-\frac{3x^4}{16}\partial_x^4-\frac{3x^3}{2}\partial_x^3+\frac{3}{256}(x^8-576x^2)\partial_x^2+\left(\frac{3x^7}{32}-6x\right)\partial_x-\frac{x^{12}}{4096}+\frac{23x^6}{64}-\frac{3}{2},$$

$$L_2=\partial_x^8-\frac{x^4}{4}\partial_x^6-3x^3\partial_x^5+\frac{1}{128}(3x^8-2368x^2)\partial_x^4+\frac{1}{8}(3x^7-320x)\partial_x^3$$
$$-\left(\frac{x^{12}}{1024}-\frac{45x^6}{16}+30\right)\partial_x^2-\left(\frac{3x^{11}}{256}-\frac{35x^5}{4}\right)\partial_x+\frac{x^{16}-3712x^{10}+716800x^4}{65536},$$

$$L_3=\partial_x^{10}-\frac{5x^4}{16}\partial_x^8-5x^3\partial_x^7+\frac{5}{128}(x^8-1024x^2)\partial_x^6+\frac{5}{16}(3x^7-416x)\partial_x^5$$
$$-\left(\frac{5x^{12}}{2048}-\frac{85x^6}{8}+150\right)\partial_x^4-\frac{5}{256}(3x^{11}-3136x^5)\partial_x^3$$
$$+\frac{5(x^{16}-8192x^{10}+2437120x^4)}{65536}\partial_x^2+\frac{5}{4096}(x^{15}-2464x^9+215040x^3)\partial_x$$
$$-\frac{x^{20}}{1048576}+\frac{15x^{14}}{2048}-\frac{1505x^8}{256}+\frac{525x^2}{4}.$$

Example: $l = 2, \ g = 4$

$$w^2 = F(z) = z^{10} - 3z^5 + 1,$$

$$\gamma_1 = \frac{-a}{\sqrt{x}}, \ \gamma_2 = \frac{a}{\sqrt{x}}, \ \gamma_3 = \frac{b}{\sqrt{x}}, \ \gamma_4 = \frac{-b}{\sqrt{x}},$$

$$a = (-1)^{\frac{1}{4}}(1 + i\sqrt{3})^{\frac{1}{4}}, \ b = \frac{1}{2}(-1)^{\frac{1}{4}}(1 + i\sqrt{3})^{\frac{5}{4}}.$$

$$L_1 = \partial_x^{10} + \frac{5x^3}{12\sqrt{2}}\partial_x^8 + \frac{5x^2}{\sqrt{2}}\partial_x^7 + \frac{5}{144}(396\sqrt{2}x + x^6)\partial_x^6 + \frac{5}{8}(36\sqrt{2} + x^5)\partial_x^5 +$$

$$\frac{5x^4(3528 + \sqrt{2}x^5)}{3456}\partial_x^4 + \frac{5x^3(760 + \sqrt{2}x^5)}{192}\partial_x^3 +$$

$$+\frac{5x^2(622080 + 3384\sqrt{2}x^5 + x^{10})}{82944}\partial_x^2 +$$

$$\frac{5x(36288 + 960\sqrt{2}x^5 + x^{10})}{6912}\partial_x + \frac{23}{4} + \frac{61x^5}{32\sqrt{2}} + \frac{5x^{10}}{1536} + \frac{x^{15}}{995328\sqrt{2}},$$

$$L_2 = \partial_x^{12} + \frac{x^3}{2\sqrt{2}}\partial_x^{10} + \frac{15x^2}{2\sqrt{2}}\partial_x^9 + \frac{2424\sqrt{2}x + 5x^6}{96}\partial_x^8 + \left(\frac{109}{\sqrt{2}} + \frac{5x^5}{4}\right)\partial_x^7 +$$

$$\left(\frac{109x^4}{8} + \frac{5x^9}{864\sqrt{2}}\right)\partial_x^6 + \frac{9680x^3 + 10\sqrt{2}x^8}{128}\partial_x^5$$

$$+\frac{x^2(6072192 + 26064\sqrt{2}x^5 + 5x^{10}}{27648}\partial_x^4 + \left(294x + \frac{377x^6}{32\sqrt{2}} + \frac{5x^{11}}{1152}\right)\partial_x^3$$

$$+\frac{42964992 + 6535296\sqrt{2}x^5 + 14736x^{10} + \sqrt{2}x^{15}}{331776}\partial_x^2 +$$

$$\frac{28237824\sqrt{2}x^4 + 181376x^9 + 40\sqrt{2}x^{14}}{884736}\partial_x +$$

$$\frac{463822848\sqrt{2}x^3 + 9092736x^8 + 5832\sqrt{2}x^{13} + x^{18}}{23887872}.$$

Open problems:

1. To find examples of operators rank 2

a) In the self-adjoint case $g > 2$

b) To find operators with polynomial coefficients in the case $g = 3$ (maybe there are exist only in the case $g = 2^n$)

c) In the non self-adjoint case $g > 4$

2. To proof in the case $g > 2$ that if an operator of order 4 of rank 2 is self-adjoint then

$$\chi_1(x, P) = \chi_1(x, \sigma(P)),$$

i.e.

$$L = L^* = (\partial_x^2 + V(x))^2 + W(x).$$

3. To find rank 2 solutions of Boussinesq equation in the case $g > 1$.

4.

$$\psi'' = \chi_0 \psi + \chi_1 \psi',$$

$$\chi_0 = -\frac{2\sqrt{2}x^3 z - 12\sqrt{2}xz^2 + 2x^4 z^3 - 12x^2 z^4 + (24 + \sqrt{2}x^5)z^5 - 24(1+w)}{24z(2+\sqrt{2}xz^2 + x^2 z^4)},$$

$$\chi_1 = \frac{z^2(\sqrt{2} + 2xz^2)}{2 + \sqrt{2}xz^2 + x^2 z^4}$$

$$w^2 = z^{10} - 3z^5 + 1.$$

Why ψ is BA?

Publications:

A ring of commuting differential operators of rank 2 corresponding to a curve of genus 2. Sb. Math. 2004. V. 195. N. 5. P. 711-722.

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