Isomonodromic τ-function as higher genus analog of Dedekind η-function

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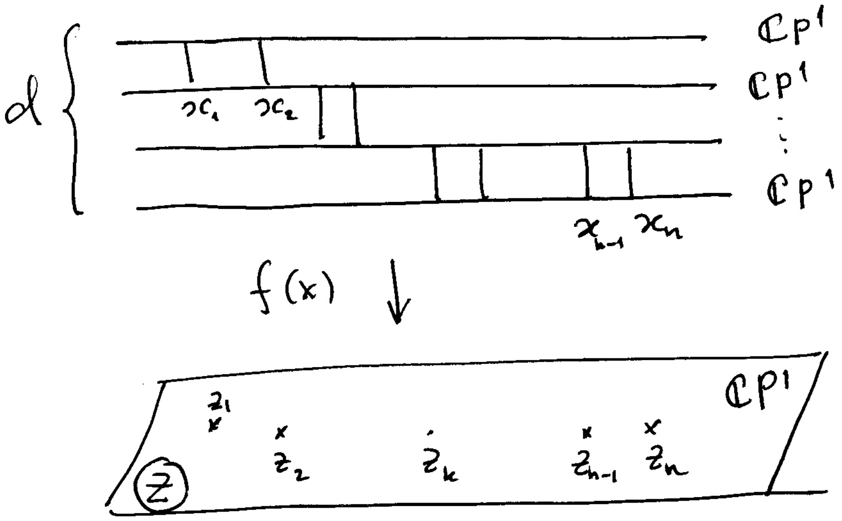
[based on joint works with A. Kokotov and P. Zograf]

$\mathcal{H}_{g,d}$ – space of meromorphic functions with simple critical points on Riemann surfaces of genus g (Hurwitz space); multiplicities of poles are fixed



$df(x_k) = 0$

$z_k := f(x_k)$

f – merom. funct of degree d on R.S $\mathcal{L}$ of genus g.

$W(x,y) := d_x d_y \log E(x,y)$ ~ canonical bimeromorphic differential

(E: prime-form)

$$W(x,y) \underset{x\sim y}{\sim} \left\{ \frac{1}{(\xi(x)-\xi(y))^2} + \frac{1}{6} S_B(\xi(x)) + \dots \right\} d\xi(x)\, d\xi(y)$$

(ξ: local parameter; S_B: Bergman proj. connection)

Definition of τ-function on Hurwitz space:

$$\boxed{\frac{\partial}{\partial z_k}\log\tau(\mathcal{L},f) = -\frac{1}{6}\,\mathrm{Res}\big|_{x_k}\frac{S_B - S_{df}}{df}}$$

$S_{df} := \{f(x), \xi(x)\}$; $\xi(x)$ - local parameter

Schwarzian derivative $\qquad S_B - S_{df}$ - quadratic differential

origin of τ:
- isomonodromic τ-function of RH problem of Hurwitz Frobenius manifolds
- isomonodromic τ-function of a RH problem with quasi-permutation monodromies equals $\tau^{-1/2}\,\Theta_{pq}(0)$
- genus 1 partition function of Hermitian matrix models
- holomorphic factorization of $\det\Delta$

explicit formula for τ: $\quad \not\!\!4\ (df) := \sum_i d_i \mathcal{D}_i$ divisor

system of distinguished local parameters on $\mathcal{L}$:

at x_j : $\xi(x) = (f(x) - f(x_j))^{1/(d_j+1)}$

at pole ∞^k of f: $\xi(x) = f(x)^{1/(d_j+1)}$

Example $f(x) : \mathbb{C}P^1 \to \mathbb{C}P^1$ ~ polynomial ; then

$$\tau = \mathrm{Resultant}(f', f'')$$

dependence on symplectic basis: $\quad \gamma = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g)$

$$\frac{\tau(\mathcal{L}^{(\tilde{a},\tilde{b})}, f)}{\tau(\mathcal{L}^{(a,b)}, f)} = \{\text{root of } 1\}\cdot\det(C\mathbb{B} + D)$$

matrix of b-periods

In general:

$$\tau = \frac{\left\{\left(\sum \omega_\alpha(x) \frac{\partial}{\partial u_\alpha}\right)^g \Theta(u\|\mathbb{B})\Big|_{u=K^x}\right\}^{2/3}}{W[\omega_1,\omega_2,\dots,\omega_g](x)} \frac{\prod_{j<l} E(\mathcal{D}_j,\mathcal{D}_l)^{\frac{d_j d_l}{6}}}{\prod_j E(x,\mathcal{D}_j)^{\frac{(g-1)d_j}{3}}} [df(x)]^{\frac{g-1}{3}}$$

Wronskian of normalized holomorphic 1-forms on $\mathcal{L}$.

this expression is in fact independent of $x \in \mathcal{L}$

hyperelliptic curves ($d=2$):

$$\tau = (\det \mathcal{A})^{1/2} \prod_{j<k}(z_j - z_k) \quad ; \quad \mathcal{A}_{\alpha\beta} = \oint_{a_\alpha} \frac{z^{\beta-1}\,dz}{\sqrt{P_{2g+2}(z)}}$$

$\approx \det \bar\partial_0$ in perturbative string theory; partition function of Ashkin-Teller model

analog of τ on moduli space of tori with periods (A,B):

$$\tau = \eta(B/A) \quad ; \quad \eta \text{ - Dedekind } \eta\text{-function}$$

in any genus: $$\hat\eta = \frac{\tau^{24(n-1)}}{\prod_{j<k}(z_j-z_k)^6}$$ ~ invariant under Möbius transformation in z-plane

section of line bundle $\lambda^{24(n-1)}$;

λ – Hodge line bundle on Hurwitz space; $\hat\eta$ – no poles no zeros inside of $\mathcal{H}_{g,n}$

$$\boxed{\hat\eta \longleftrightarrow \text{moduli-dependent constant in expression of } \sigma\text{-function via } \Theta\text{-function.}}$$