# Algebro-geometric solutions of the Davey-Stewartson and the multi-component nonlinear Schrödinger equations

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# Introduction

# Integrable equations and algebraic geometry

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- 1968 - Lax, Novikov, Dubrovin, Krichever, Matveev, Its,...:
  $\rightarrow$ Lax pairs, Baker-Akhiezer functions
  $\rightarrow$ KP, Boussinesq, sine-Gordon, NLS equations...

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- 1973 - Fay's identity
- 1983 - Fay, Mumford: KdV, KP, sine-Gordon

# The nonlinear Schrödinger equation

- The nonlinear Schrödinger (NLS) equation, $\rho = \pm 1$

$$i\,\partial_t \psi + \partial_x^2 \psi + 2\,\rho\,|\psi|^2\,\psi = 0$$

- Integrability: Zakharov and Shabat (1971)
- Applications: hydrodynamics (deep water waves), plasma physics and nonlinear fiber optics

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## Reference

- E. Belokolos, A. Bobenko, V. Enolskii, A. Its, V. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, (1994)

# The Davey-Stewartson equations

- The (integrable) Davey-Stewartson equations, where $\alpha = \pm 1$ and $\rho = \pm 1$,

$$i\psi_t + \psi_{xx} - \alpha\psi_{yy} + 2(\Phi + \rho|\psi|^2)\psi = 0$$
$$\Phi_{xx} + \alpha\Phi_{yy} + 2\rho|\psi|^2_{xx} = 0$$

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## Reference

- T. Malanyuk, *Finite-gap solutions of Davey-Stewartson equations*, (1994)

# The multi-component nonlinear Schrödinger equation

- The multi-component nonlinear Schrödinger equation, $s_k = \pm 1$

$$i\,\partial_t \psi_j + \partial_x^2 \psi_j + 2\Big(\sum_{k=1}^{n} s_k |\psi_k|^2\Big)\psi_j = 0 \qquad j = 1, \dots, n$$

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## Reference


- J. Elgin, V. Enolski, A. Its, *Effective integration of the nonlinear vector Schrödinger equation*, (2007)

# Outline

# Theta functions

# Riemann surfaces

## Definition

A Riemann surface $\mathcal{R}$ is a connected one-dimensional complex analytic manifold

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Example: Nonsingular algebraic curves

**Theorem**

Any compact Riemann surface can be represented as a nonsingular compactified algebraic curve

# Covering of compact Riemann surfaces

$M$, $N$ compact Riemann surfaces

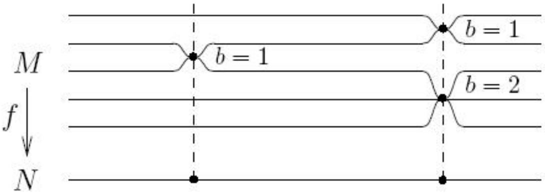


Figure: Covering of degree 5

# Topology of compact Riemann Surfaces

**Theorem (and Definition)**

Any compact Riemann surface $\mathcal{R}$ is homeomorphic to a sphere with handles. The number $g \in \mathbb{N}$ of handles is called the genus of $\mathcal{R}$.

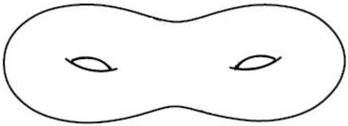

Figure: Sphere with 2 handles

# Homology basis

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- $\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{A}_g, \mathcal{B}_g$ canonical homology basis

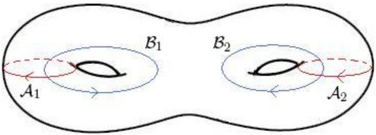


Figure: Canonical homology basis

# Holomorphic differentials

- $\omega_1, \ldots, \omega_g$ basis of holomorphic differentials normalized by

$$\int_{\mathcal{A}_k} \omega_j = 2i\pi\delta_{jk} \quad j,k = 1,\ldots,g$$

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$$\int_{\mathcal{A}_k} \omega_j = 2i\pi\delta_{jk} \quad j,k = 1,\dots,g$$

- $\mathbb{B} = (\int_{\mathcal{B}_k} \omega_j)$ matrix of $\mathcal{B}$-periods

# Theta functions with characteristics

Let $\mathbf{z} \in \mathbb{C}^g$ and $\delta', \delta'' \in \{0, \frac{1}{2}\}^g$

$$\Theta[\delta](\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\left(\mathbf{m}+\delta'\right), \mathbf{m}+\delta'\rangle + \langle \mathbf{m}+\delta', \mathbf{z}+2i\pi\delta''\rangle\right\}$$

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$$\Theta(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\,\mathbf{m}, \mathbf{m} \rangle + \langle \mathbf{m}, \mathbf{z} \rangle\right\}$$

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$$\Theta[\delta](\mathbf{z}) = \Theta(\mathbf{z}+2i\pi\delta''+\mathbb{B}\,\delta')\,\exp\left\{\tfrac{1}{2}\langle \mathbb{B}\,\delta', \delta'\rangle + \langle \mathbf{z}+2i\pi\delta'', \delta'\rangle\right\}$$

# Fay's identities

# Cross-ratio functions

## On $\hat{\mathbb{C}}$

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- $\lambda_0(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$
- $\lambda_0(a,b,c,d) + \lambda_0(a,c,b,d) = 1$

# Cross-ratio functions

## On a compact Riemann surface of genus $g > 0$

- $\delta$ nonsingular odd characteristic

$$\lambda_g(a,b,c,d) = \frac{\Theta[\delta](\int_c^a \omega)\,\Theta[\delta](\int_d^b \omega)}{\Theta[\delta](\int_d^a \omega)\,\Theta[\delta](\int_c^b \omega)}$$

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- **Fay's identity.** Let $\mathbf{z} \in \mathbb{C}^g$ and $a,b,c,d \in \mathcal{R}_g$

$$\Theta(\mathbf{z} + \int_c^a \omega)\,\Theta(\mathbf{z} + \int_b^d \omega)\,\lambda_g(a,c,b,d) + \Theta(\mathbf{z} + \int_b^a \omega)\,\Theta(\mathbf{z} + \int_c^d \omega)\,\lambda_g(a,b,c,d) = \Theta(\mathbf{z})\,\Theta(\mathbf{z} + \int_c^a \omega + \int_b^d \omega)$$

# Notations

- Let $a \in \mathcal{R}_g$ and $k_a$ a local parameter

$$\omega_j(a) = (V_{a,j} + W_{a,j}\, k_a + U_{a,j}\, \frac{k_a^2}{2!} + \ldots)\, dk_a$$

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- $F : \mathbb{C}^g \longmapsto \mathbb{C}$

$$D_a F(\mathbf{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\mathbf{z}) V_{a,j}$$

$$D'_a F(\mathbf{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\mathbf{z}) W_{a,j}$$

# Corollaries

## Corollary 1

$$D_b \ln \frac{\Theta(\mathrm{z} + \int_c^a \omega)}{\Theta(\mathrm{z})} = p_1 + p_2 \frac{\Theta(\mathrm{z} + \int_b^a \omega)\, \Theta(\mathrm{z} + \int_c^b \omega)}{\Theta(\mathrm{z} + \int_c^a \omega)\, \Theta(\mathrm{z})}$$

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## Corollary 2

$$D_a D_b \ln \Theta(\mathbf{z}) = q_1 + q_2 \frac{\Theta(\mathbf{z} + \int_a^b \omega)\,\Theta(\mathbf{z} + \int_b^a \omega)}{\Theta(\mathbf{z})^2}$$

# Corollaries

## New corollary

$$D_a' \ln \frac{\Theta(\mathbf{z} + \int_a^b \omega)}{\Theta(\mathbf{z})} + D_a^2 \ln \frac{\Theta(\mathbf{z} + \int_a^b \omega)}{\Theta(\mathbf{z})} + \left( D_a \ln \frac{\Theta(\mathbf{z} + \int_a^b \omega)}{\Theta(\mathbf{z})} - K_1 \right)^2$$

$$+ K_2 + 2\, D_a^2 \ln \Theta(\mathbf{z}) = 0$$

# Associated system of n-NLS$^s$ and reduction condition

$$\text{n-NLS}^s: \quad i\,\partial_t \psi_j + \partial_x^2 \psi_j + 2\,\Big(\sum_{k=1}^{n} s_k |\psi_k|^2\Big)\,\psi_j = 0 \qquad j = 1, \ldots, n$$

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- Associated system of n-NLS$^s$

$$i\,\partial_t\psi_j + \partial_x^2\psi_j + 2\,\Big(\sum_{k=1}^{n} \psi_k\,\psi_k^*\Big)\,\psi_j = 0$$

$$-i\,\partial_t\psi_j^* + \partial_x^2\psi_j^* + 2\,\Big(\sum_{k=1}^{n} \psi_k\,\psi_k^*\Big)\,\psi_j^* = 0$$

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- Reduction condition

$$\psi_j^* = s_j\,\overline{\psi_j}$$

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*The following functions for $j = 1, \ldots, n$ are solutions of the associated system of n-NLS$^s$*

$$\psi_j(x,t) = A_j \, \frac{\Theta(\mathbf{Z} - \mathbf{D} + \int_{a_{n+1}}^{a_j} \omega)}{\Theta(\mathbf{Z} - \mathbf{D})} \, \exp i(-E_j x + N_j t)$$

$$\psi_j^*(x,t) = \frac{q_2(a_{n+1}, a_j)}{A_j} \, \frac{\Theta(\mathbf{Z} - \mathbf{D} - \int_{a_{n+1}}^{a_j} \omega)}{\Theta(\mathbf{Z} - \mathbf{D})} \, \exp i(E_j x - N_j t)$$

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*where*

$$\mathbf{Z} = i\mathbf{V}_{a_{n+1}} x + i\mathbf{W}_{a_{n+1}} t$$

*$E_j$, $N_j$ depend on $a_j$ and $a_{n+1}$, and $\mathbf{D} \in \mathbb{C}^g$, $A_j \in \mathbb{C}^*$ are arbitrary constants.*

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$$i\,\partial_t\psi_1 + \partial_x^2\psi_1 + 2\left(\sum_{k=1}^{n}\psi_k\psi_k^*\right)\psi_1 = 0$$

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$$i\,\partial_t\psi_1 + \partial_x^2\psi_1 + 2\left(\sum_{k=1}^n \psi_k\psi_k^*\right)\psi_1 = 0$$

1. $a = a_{n+1}$ and $b = a_1$

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$$\left(D_{a_{n+1}} \ln \frac{\Theta(\mathbf{z}+\int_{a_{n+1}}^{a_1}\omega)}{\Theta(\mathbf{z})} - K_1\right)^2 + K_2 + 2\,D^2_{a_{n+1}} \ln\Theta(\mathbf{z}) = 0$$

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3. $D_{a_{n+1}}D_{a_k}\ln\Theta(\mathbf{z}) = q_1 + q_2\,\dfrac{\Theta(\mathbf{z}+\int_{a_{n+1}}^{a_k}\omega)\,\Theta(\mathbf{z}+\int_{a_k}^{a_{n+1}}\omega)}{\Theta(\mathbf{z})^2}$

# Theta-functional solutions of n-NLS$^s$

# Real Riemann surfaces

## Definitions

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- The set of fixed points forms connected components which are called the real ovals of the involution $\tau$
- Denote by $\mathcal{R}_g(\mathbb{R})$ the set of fixed points. It consists of $k$ disjoint topological circles, $0 \leq k \leq g+1$

# Real Riemann surfaces

## Examples

- Curve with $g+1$ real ovals: $\tau(z,w) = (\bar{z}, \bar{w})$

$$w^2 = \prod_{k=1}^{2g+1} (z - \lambda_k), \quad \lambda_k \in \mathbb{R}, \quad \lambda_1 < \ldots < \lambda_{2g+1}$$

Real ovals are over the intervals $[\lambda_1, \lambda_2], \ldots, [\lambda_{2g+1}, +\infty]$

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- Curve without real oval: $\tau(z,w)=(\bar{z},\bar{w})$

$$w^2=-\prod_{k=1}^{g+1}(z-\lambda_k)(z-\overline{\lambda_k})$$

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- $\mathcal{R}_g$ is a *dividing* surface if $\mathcal{R}_g(\mathbb{R})$ separates $\mathcal{R}_g$
- $\mathcal{R}_g$ is *non-dividing* if $\mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R})$ remains connected

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## Symmetric homology basis (V. Vinnikov's paper 1993)

# Real Riemann surfaces

## Symmetric homology basis (V. Vinnikov's paper 1993)

There exists a canonical homology basis $(\mathcal{A},\mathcal{B}) = (\mathcal{A}_1 \ldots \mathcal{A}_g, \mathcal{B}_1 \ldots \mathcal{B}_g)$ such that

$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0_g \\ \mathbb{H} & -\mathbb{I}_g \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}$$

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5. $\mathbf{D} \in \mathbb{R}^g$

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- What is the general form of a dividing algebraic curve?

$\rightarrow$ Construction of M-curves by Harnack's method

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- Putting $D_k = \frac{1}{2}\mathbb{B}_{kk} + d_k$

$$\Theta(\mathbf{Z}-\mathbf{D}) \longrightarrow$$

$$\sum_{\mathbf{m}\in\{0,1\}^g} \exp\left\{ \sum_{1\le i<k\le g} \mathbb{B}_{ik}\, m_i m_k + \sum_{k=1}^{g} m_k\,(Z_k - d_k) \right\}$$

# Solutions in genus zero of 4-NLS$^s$

$$i\,\partial_t\psi_1 + \partial_x^2\psi_1 + 2\,\Big(\sum_{k=1}^{4} s_k|\psi_k|^2\Big)\,\psi_1 = 0$$

$$i\,\partial_t\psi_2 + \partial_x^2\psi_2 + 2\,\Big(\sum_{k=1}^{4} s_k|\psi_k|^2\Big)\,\psi_2 = 0$$

$$i\,\partial_t\psi_3 + \partial_x^2\psi_3 + 2\,\Big(\sum_{k=1}^{4} s_k|\psi_k|^2\Big)\,\psi_3 = 0$$

$$i\,\partial_t\psi_4 + \partial_x^2\psi_4 + 2\,\Big(\sum_{k=1}^{4} s_k|\psi_k|^2\Big)\,\psi_4 = 0$$

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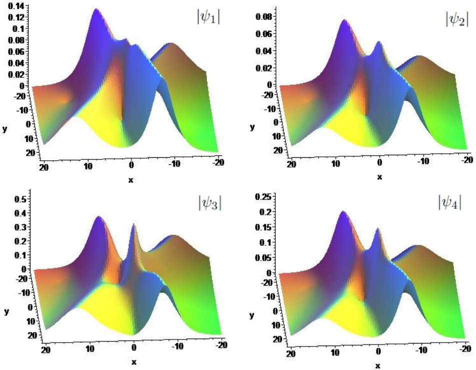


Figure: Bright 2-solitons of 4-NLS$^s$ with $s = (1, 1, 1, 1)$

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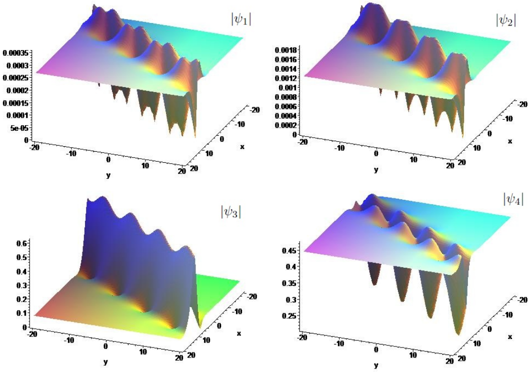


Figure: Breather of 4-NLS$^s$ with $s = (-1, -1, 1, -1)$

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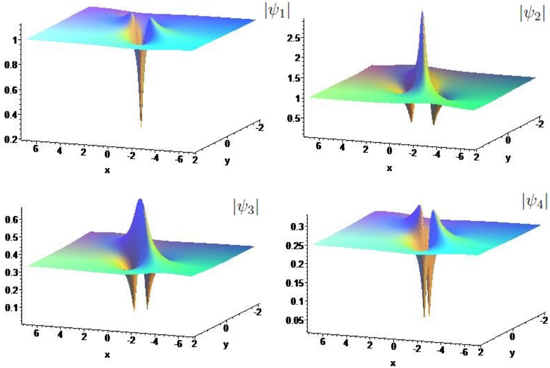


Figure: Rational breather of 4-NLS$^s$ with $s = (1, 1, 1, 1)$

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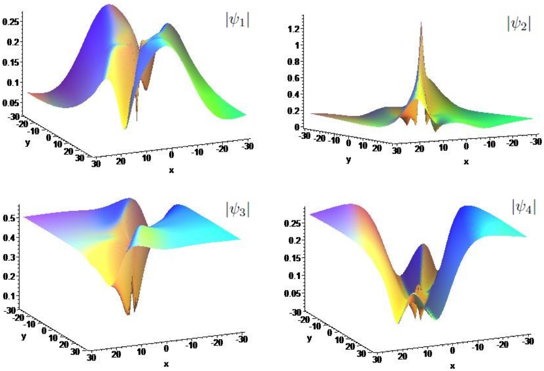


Figure: 4-rational breather of 4-NLS$^s$ with $s = (1, 1, 1, 1)$

# Thank you