Hyperelliptic, Trigonal and Tetragonal Reductions of Benney's Equations

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Abstract

Reductions of Benney's equations are described by conformal mappings from the upper half p-plane to the slit upper half λ-plane. The slits are on prescribed non-intersecting curves, and their moving end points are the Riemann invariants. If the slits are straight lines, the mapping may be given in Schwartz-Christoffel form. In the case where the slits are all vertical, the mapping is given explicitly in terms of Kleinian σ functions of a hyperelliptic curve; a similar formula is found for a trigonal or tetragonal curve, when the slits make angles of $\pi/3$ or $\pi/4$ with the real λ-axis and one another.

Dispersionless PDE

The prototype of this class of system is the kinematic wave equation

$$u_t + uu_x = 0, \tag{1}$$

which is a dispersionless bi-Hamiltonian integrable system. It can be considered as the singular limit of the KdV as the dispersion term tends to zero. Until shocks form, its solution with initial data $u(x, 0) = f(x)$ is given implicitly by the hodograph form:

$$x - ut = f^{(-1)}(u). \tag{2}$$

Tsarev showed that this result can be generalised to a much wider class - if a system

$$\mathbf{u}_t + V\mathbf{u}_x = 0, \tag{3}$$

with an N component vector $\mathbf{u}$ and an $N \times N$ matrix V, can be transformed to diagonal form,

$$r^i_t + v^i(\mathbf{r})r^i_x = 0, \tag{4}$$

where the r^i are called the Riemann invariants, and the v^i the eigenvalues of V are the characteristic speeds, assumed real and distinct; and if it is Hamiltonian, its solutions may again be written in hodograph form

$$x - v^i t = w^i. \tag{5}$$

Here the w^i satisfy an over-determined (but consistent) set of linear equations.

I will be talking about the problem of a dispersionless Hamiltonian pde with infinitely many dependent variables, and the problem of how, for some classes of its solutions it may be reduced to an $N \times N$ system of this type. In particular I will look at the problem of constructing such classes of solutions - called reductions of the system.

Benney's Equations

Many of the most important examples of dispersionless integrable systems are included in the Benney hierarchy. These equations were first derived as a description of long waves on a shallow perfect fluid, but they can be written as Vlasov equations:

$$\frac{\partial f}{\partial t_2} + p\,\frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial f}{\partial p} = 0, \tag{6}$$

where the moments A_n are defined by

$$A_n = \int_{-\infty}^{\infty} p^n f \,\mathrm{d}p. \tag{7}$$

These moments satisfy the Benney moment equations:

$$(A_n)_t + (A_{n+1})_x + nA_{n-1}(A_0)_x = 0. \tag{8}$$

When can we find solutions $f(x,p,t_2)$, of this system which depend on (x,t_2) through only N Riemann invariants; that is, functions $\hat{\lambda}_i(x,t)$, satisfying

$$\frac{\partial\hat{\lambda}_i}{\partial t}+\hat{p}_i\,\frac{\partial\hat{\lambda}_i}{\partial x}=0, \tag{9}$$

for some characteristic speeds $\hat{p}_i(\hat{\lambda})$? Such a solution $f(p,\hat{\lambda}(x,t_2)$ must satisfy

$$(p-\hat{p}_i)\frac{\partial f}{\partial\hat{\lambda}_i}-\frac{\partial A_0}{\partial\hat{\lambda}_i}\frac{\partial f}{\partial p}=0 \tag{10}$$

and thus, on dividing by $\frac{\partial f}{\partial p}$, we get:

Löwner equations

$$\frac{\partial p}{\partial \widehat{\lambda}_i} = -\frac{\frac{\partial A_0}{\partial \widehat{\lambda}_i}}{p - \widehat{p}_i}. \tag{11}$$

Each of these Löwner equations, taken separately, can be used to describe the growth of a slit in the image of a conformal map from the p-plane to the λ-plane. The consistency conditions between these need to be satisfied; geometrically, the growth of one slit should not cause the other slits to move. Such solutions can be described by choosing N non-intersecting curves C_i in the upper half λ-plane, with end points $\widehat{\lambda}_i$, and bases $\widehat{\lambda}_i^0$ on the real axis. There is a unique conformal mapping from the upper half p-plane to the slit upper half λ-plane, with asymptotics $\lambda = p + \mathrm{O}(1/p)$ as $p \to \infty$.

If the curves are chosen *ab initio*, the consistency conditions are automatically satisfied. The solutions of those conditions are parameterised by N functions - and these slit mappings are parametrised by N curves. The question now arises, can we construct explicit non-trivial examples of these mappings? The obvious class to consider first is where the slits are all straight lines, and the mapping is then of Schwartz-Christoffel type.

Schwartz-Christoffel mappings

If the upper half p-plane is mapped into an N-gonal domain of the λ-plane, with $\lambda \simeq p + \mathrm{O}(1/p)$, as $p \to \infty$, and the points p_i are mapped to the vertices λ_i with internal angles $\pi\alpha_i$, satisfying $\sum_{i=1}^{N} \pi(1-\alpha_i) = 0$, the mapping is given by

$$\lambda = \int^p \left(\prod_{i=1}^{N} \frac{1}{(p'-p_i)^{(1-\alpha_i)}} \right) \mathrm{d}p'. \qquad (12)$$

for then near $p = p_i$, $\lambda \simeq (p-p_i)^{\alpha_i}$.

The integrand is algebraic if the α_i are all rational. For instance, with $\alpha_i = 1/2$, if we take the real p-axis, and mark N intervals $I_i = [p_i^-, p_i^+]$ on it, and a point $\widehat{p}_i \in I_i$, then the mapping

$$\lambda = p + \int_{-\infty}^{p} \left(\frac{\prod_{i=1}^{N}(p'-\widehat{p}_i)}{\sqrt{\prod_{i=1}^{N}(p'-p_i^-)(p'-p_i^+)}} - 1 \right) \mathrm{d}p' \qquad (13)$$

takes the upper half p-plane to the upper half λ-plane with N vertical slits in it, stretching from λ_i^0, the image of $p_i^{\pm}$, to $\widehat{\lambda}_i$, the image of $\widehat{p}_i$. The conditions

$$\int_{p_k^-}^{p_k^+} \left(\frac{\prod_{i=1}^{N}(p' - \widehat{p}_i)}{\sqrt{\prod_{i=1}^{N}(p' - p_i^-)(p' - p_i^+)}} - 1 \right) \mathrm{d}p' = 0 \tag{14}$$

ensure that the images of $p_k^{\pm}$ are the same point λ_k^0 of the real λ axis. These image points λ_k^0 should be independent of (x,t), so the resulting mapping is found to depend on N free parameters, say the end points λ_k.

A Hyperelliptic Curve

The integral

$$\begin{aligned}\lambda &= p + \int_{-\infty}^{p} \Big(\frac{\prod_{i=1}^{N}(p' - \widehat{p}_i)}{\sqrt{\prod_{i=1}^{N}(p' - p_i^-)(p' - p_i^+)}} - 1 \Big) \mathrm{d}p' \\ &= p + \int_{-\infty}^{p} \varphi(p')\mathrm{d}p' \qquad (15)\end{aligned}$$

is an integral of a second kind differential on a hyperelliptic Riemann surface (elliptic for N=2). It helps to move one branch point to infinity; we take:

$$\begin{aligned} p &= p_N^+ - 1/t, \\ p_i^{\pm} &= p_N^+ - 1/t_i^{\pm}, \qquad (16)\end{aligned}$$

and the integrand becomes:

$$\begin{aligned}\varphi(p)\,\mathrm{d}p &= k\left(\frac{\sum_{i=0}^{g+1} a_i\, t^i}{s}\right)\frac{\mathrm{d}t}{t^2} \\ &= k\left(\sum_{i=2}^{g+1} a_i\, t^{i-2}\right)\frac{\mathrm{d}t}{s} \\ &\quad + k\left(\frac{a_1}{t} + \frac{a_0}{t^2}\right)\frac{\mathrm{d}t}{s}, \qquad (17)\end{aligned}$$

for some coefficients a_i, k.

The terms in positive powers of t are holomorphic differentials, and those in negative powers, meromorphic, on the genus $g = N - 1$ hyperelliptic Riemann surface

$$\begin{aligned} s^2 &= \mu_0 + \mu_1 t + \ldots + \mu_{2g} t^{2g} + 4 t^{2g+1} \\ &= 4(t - t_N^-) \prod_{i=1}^{N-1} (t - t_i^+)(t - t_i^-). \end{aligned} \quad (18)$$

Since

$$\sum_{i=1}^{n} (p_i^- + p_i^+ - 2\widehat{p}_i) = 0,$$

so that $\lambda(p)$ will have the required asymptotics:

$$\lambda(p) \simeq p + \mathrm{O}(1/p) \quad \text{as} \quad p \to \infty,$$

the residue at $t = 0$ of the meromorphic part is identically zero. To construct the mapping, it is enough to find a function which has the correct principal part at this singularity; this function and the required Schwartz-Christoffel mapping then differ by at most a holomorphic term.

Kleinian σ-functions.

We define the Abel map in the usual way in terms of the holomorphic differentials, fixing the base point at $t = \infty$.

$$u_i = \int_\infty^t t'^{i-1} \frac{\mathrm{d}t'}{s'}, \quad i = 1, \ldots, g. \qquad (19)$$

If we define a basis of cycles
$\{\mathfrak{a}_i, \mathfrak{b}_i\}, i = 1, .., g$, as in the diagram,

then we define the period matrices

$$\begin{aligned} 2\omega &= \left(\int_{\mathfrak{a}_i} \mathrm{d}u_j \right) \\ 2\omega' &= \left(\int_{\mathfrak{b}_i} \mathrm{d}u_j \right) \end{aligned}$$

whose $2g$ columns define a lattice Λ in $\mathbb{C}^g$, then the Abel map is defined uniquely up to translations by Λ. We write $\mathrm{Jac} = \mathbb{C}^g/\Lambda$.

We also need the associated basis of second kind meromorphic differentials:

$$\mathrm{d}r_i = \sum_{k=i}^{2g+1-i} (1 + i - k)\, \mu_{1+i+k} \frac{t^k \mathrm{d}t}{4s}, \quad (i = 1, 2, \ldots, g) \qquad (20)$$

with period matrices η and η'. These period matrices are used to form the matrix

$$\mathfrak{M} = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}.$$

Then the σ function is, up to a factor which is irrelevant to our purposes:

$$\sigma(\mathbf{u};\mathfrak{M}) = \exp\left(\frac{1}{2}\mathbf{u}^{\mathsf{T}}\eta\,\omega^{-1}\mathbf{u}\right)$$

$$\times \sum_{\mathbf{m}\in\mathbb{Z}^4} \exp(i\pi(\mathbf{m}^T\tau\mathbf{m} + 2\mathbf{m}^T((2\omega)^{-1}\mathbf{u} - \Delta))$$

where Δ is the Riemann constant for the base point $t=\infty$, and $\tau = \omega'\omega^{-1}$. Then σ will vanish on the strata Θ_k of the Θ-divisor given by:

$$\Theta_k = \{\mathbf{u} \in \mathrm{Jac} : \mathbf{u} = \sum_{j=1}^{k} \int_{\infty}^{t_j} \mathrm{d}\mathbf{u}\}$$

and evidently

$$\{\mathbf{0}\} \subset \Theta_1 \subset \Theta_2 \subset \ldots \subset \mathrm{Jac}.$$

Here, as we are interested in a 1-dimensional integral, it makes sense to write it as a path integral on Θ_1, the Abel image of the curve.

Jacobi inversion formula on Θ_1

Using a result of Jorgenson, we may show that on

$$\Theta_{g-1}, \quad \sigma = 0,$$

and on

$$\Theta_{g-i}, \quad \sigma = \sigma_g = \sigma_{g-1} = ... = \sigma_{g-i+2} = 0.$$

In particular for a hyperelliptic curve, on Θ_1 only the two derivatives σ_1 and σ_2 are non-vanishing, and it follows that

$$t = \frac{\mathrm{d}u_2}{\mathrm{d}u_1} = -\frac{\sigma_1}{\sigma_2}.$$

This result was used by Enolski, Pronine and Richter to find the motion of a double pendulum in the absence of gravity - that problem reduces to the inversion of a 2nd kind meromorphic differential on Θ_1 for a genus 2 hyperelliptic curve. Similar problems have been studied elsewhere, e.g. by Abenda and Fedorov.

Our integral, or rather, its meromorphic part, then becomes

$$\varphi_2(\mathbf{u})\,\mathrm{d}u_1 = \left(\frac{{\sigma_2}^2}{{\sigma_1}^2}(\mathbf{u}) - \frac{1}{2}\frac{\mu_1}{\mu_0}\frac{\sigma_2}{\sigma_1}(\mathbf{u})\right)\mathrm{d}u_1$$

This is singular only where $t = 0$, ($p = \infty$) that is at the 2 points $\pm\mathbf{u}_0$ where $\sigma_1 = 0$.

We expand in a Laurent series in powers of $w_1 = u_1 - (\mathbf{u}_0)_1$, getting

$$\frac{\sigma_2}{\sigma_1}(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) =$$

$$\frac{(\sigma_2) + (\sigma_{12})\,w_1 + \cdots}{(\sigma_{11})\,w_1 + \ldots} =$$

$$\left(\frac{\sigma_2}{\sigma_{11}}\right) w_1^{-1} + \mathrm{O}(1),$$

and

$$\frac{{\sigma_2}^2}{{\sigma_1}^2}(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) =$$

$$\frac{{\sigma_2}^2 + (2\,\sigma_2\,\sigma_{12})\,w_1 + \ldots}{{\sigma_{11}}^2\,w_1^2 + (\sigma_{11}\,\sigma_{111})\,w_1^3 + (2\,\sigma_{11}\,\sigma_{12})\,w_1 w_2 + \ldots} =$$

$$\left(\frac{{\sigma_2}^2}{{\sigma_{11}}^2}\right) w_1^{-2} +$$

$$\left(2\frac{\sigma_2\,\sigma_{12}}{{\sigma_{11}}^2} - \frac{{\sigma_2}^2\,\sigma_{111}}{{\sigma_{11}}^3} - \sqrt{\mu_0}\frac{{\sigma_2}^2\,\sigma_{12}}{{\sigma_{11}}^3}\right) w_1^{-1} + O(1).$$

Here all the derivatives of σ are evaluated at $\mathbf{u}_0$. At this point one may show (consider the Taylor series of σ near $\mathbf{u}_0$), that

$$\sigma_{11}(\mathbf{u}_0) = -\sqrt{\mu_0}\,\sigma_2(\mathbf{u}_0)$$

and

$$\sigma_{111}(\mathbf{u}_0) = -\frac{1}{2}\mu_1\,\sigma_2(\mathbf{u}_0) - 3\sqrt{\mu_0}\,\sigma_{12}(\mathbf{u}_0).$$

It follows, as it must, that $\varphi_2 \mathrm{d}u_1$ has zero residue here.

We now consider the function

$$\Psi(\mathbf{u}) = -\frac{1}{\mu_0}\frac{\sigma_{11}}{\sigma_1}(\mathbf{u})$$

for $\mathbf{u} \in \Theta_1$. This has a simple pole at $w_1 = 0$, so its derivative can be compared with our integrand. The (total) derivative of Ψ with respect to u_1 along Θ_1 is

$$\psi(\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}u_1}\left[-\frac{1}{\mu_0}\frac{\sigma_{11}}{\sigma_1}\right]$$

$$= -\frac{1}{\mu_0}\left[\sum_{i=1}^{g}(-1)^{i-1}\left(\frac{\sigma_1}{\sigma_2}\right)^{i-1}\left(\frac{\sigma_{11i}}{\sigma_1} - \frac{\sigma_{11}\,\sigma_{1i}}{{\sigma_1}^2}\right)\right].$$

Since only the first three terms in the sum contain negative powers of σ_1 we can rewrite $\psi(\mathbf{u})$ as

$$\psi(\mathbf{u}) = -\frac{1}{\mu_0}[(-\sigma_{11}^2\frac{1}{\sigma_1^2}$$

$$+(\sigma_{111} + \frac{\sigma_{11}\sigma_{12}}{\sigma_2})\frac{1}{\sigma_1} + \mathrm{O}(1)] \qquad (\forall g \geq 3).$$

Expanding near $\mathbf{u} = \mathbf{u}_0$, we find

$$\psi(\mathbf{u}) \simeq \left(\frac{1}{\mu_0}\right) {w_1^{-}}^2 + \mathrm{O}(1)$$

This has the same leading terms as φ_2. Neither $\psi \mathrm{d}u_1$ nor φ_2 has any other singularities, so they differ by at most holomorphic differentials.

We thus obtain, finally, an explicit expression for the Schwartz-Christoffel integral:

$$\lambda(p) = p + \int_{\infty}^{p} \Big[\varphi(p') - 1\Big] \, \mathrm{d}p'$$

$$= p_{2g+2} +$$

$$\Big\{k(\mathbf{A} + (-1)^{g-1}\mathbf{B})^{\top} \cdot \mathbf{u}\Big\} + \left\{(-1)^{g} \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1}(\mathbf{u})\right\} + \tilde{C}.$$

Here the coefficients $\mathbf{A}$ are given by

$$\sum_{i=0}^{g+1} \mathbf{A}_i \, t^i = \prod_{i=1}^{g+1} \Big[(p_{2g+2} - \widehat{p}_i)\, t - 1\Big],$$

$\mathbf{B}$ is known to vanish for the cases $g = 2$ and $g = 3$, and $\tilde{C}$ is given by

$$\tilde{C} = (-1)^{g} \sqrt{\mu_0}\, \mathbf{A}^{\top} \cdot \mathbf{u}_0 - \sqrt{\mu_0}\, \mathbf{B}^{\top} \cdot \mathbf{u}_0 + \frac{2}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_2}(\mathbf{u}_0) + \frac{1}{2}\frac{\mu_1}{\mu_0}.$$

A Trigonal Reduction

The mapping between the upper half p-plane and the slit upper half λ-plane shown above is given as a Schwartz-Christoffel mapping

$$\lambda(p) = p + \int_{\infty}^{p} (\varphi(p') - 1)\, \mathrm{d}p', \qquad (21)$$

where

$$\varphi(p) = \frac{\prod_{i=1}^{4} (p - \widehat{p}_i)}{\left[\prod_{i=1}^{6} (p - P_i)\right]^{2/3}}. \qquad (22)$$

The same approach can be used here, although there is much less known in detail about cyclic trigonal curves than hyperelliptic curves.

We found that the Schwartz-Christoffel mapping can be given explicitly. As before, rather than the coordinates (p, y) on the cyclic $(3, 6)$ curve

$$\Gamma = \left\{ (p, y) :\ y^3 = \prod_{i=1}^{6} (p - P_i) \right\},$$

we define new coordinates (t, s) by:

$$p = P_6 - \frac{1}{t}, \tag{23}$$

$$P_i = P_6 - \frac{1}{T_i} \qquad i = 1 \ldots 5, \tag{24}$$

$$s = yt^2 K, \tag{25}$$

$$K^3 = \prod_{i=1}^{5} (P_6 - P_i). \tag{26}$$

Define λ_i by the equation

$$\sum_{i=1}^{6} \lambda_i\, t^i = -\frac{\prod_{i=1}^{6} [(P_6 - P_i)t - 1]}{\prod_{i=1}^{5} (P_6 - P_1)}.$$

and set $\mathbf{A}^{\mathbf{T}} = (A_1, A_2, A_3, A_4)$ where the A_i are defined by

$$\sum_{i=1}^{4} A_i\, t^i = \prod_{i=1}^{4} [(P_6 - \widehat{p}_i)t - 1].$$

The image of Γ, $\mathbf{T}_4$, is then given by the cyclic $(3, 5)$ curve

$$\mathbf{T}_4 = \left\{ (t, s) : \; s^3 = t^5 + \sum_{i=0}^{4} \lambda_i t^i \right\},$$

We then define the restriction to $\mathbf{T}_4$ of the

Abel map $\mathbf{u}$, with image $\Theta_1 \subset \mathrm{Jac}(\mathbf{T}_4)$, by

$$u_1 = \int_\infty^t \frac{\mathrm{d}t'}{3s'^2} \qquad u_2 = \int_\infty^t \frac{t'\mathrm{d}t'}{3s'^2}$$
$$u_3 = \int_\infty^t \frac{s'\mathrm{d}t'}{3s'^2} \qquad u_4 = \int_\infty^t \frac{t'^2\mathrm{d}t'}{3s'^2}$$

The inversion of these mappings is given by:

$$p = P_6 + \frac{\sigma_2}{\sigma_1}(\mathbf{u}).$$

Then, with $\mathbf{u} = (u_1, u_2, u_3, u_4) \in \Theta_1$ and $\sigma_1(\mathbf{u}_0) = 0$, we have:

$$\begin{aligned}\lambda(p) = \quad & 3\lambda_0^{2/3}\left(A_2 - \tfrac{1}{3}\tfrac{\lambda_2}{\lambda_0}\right)(u_1 - u_{0,1}) + \\ & A_3(u_2 - u_{0,2}) + A_4(u_4 - u_{0,4}) - \\ & \tfrac{1}{3}\tfrac{1}{\lambda_0^{1/3}}\tfrac{\sigma_{13}}{\sigma_1}(\mathbf{u}) + \tfrac{9}{\lambda_0^{1/3}}\tfrac{\sigma_{23}}{\sigma_2}(\mathbf{u}_0) + \tfrac{1}{3}\tfrac{\lambda_1}{\lambda_0}\end{aligned}$$

on the sheet of the Riemann surface

$$\left\{(p, y) :\ y^3 = \prod_{i=1}^{6}(p - P_i)\right\}$$

associated with the relation
$p \to +\infty \Leftrightarrow \mathbf{u} \to +\mathbf{u}_0$.

A Tetragonal Reduction

We may similarly look at a reduction in which the slits are at angles of $\pi/4$ to each other. The simplest non-trivial example is given by

$$\lambda(p) = p + \int_{\infty}^{p} [\varphi(p') - 1]dp' \qquad (27)$$

where

$$\varphi(p) = \frac{\prod_{i=1}^{6}(p - \widehat{v}_i)}{[\prod_{i-1}^{8}(p - \widehat{p}_i)]^{\frac{3}{4}}} = \frac{\prod_{i=1}^{6}(p - \widehat{v}_i)}{y^3} \quad (28)$$

where

$$y^4 = \prod_{i=1}^{8}(p - \widehat{p}_i). \qquad (29)$$

which, on mapping p_8 to ∞, we find is associated with the curve

$$s^4 \;= t^5 + \mu_4 t^4 + \mu_3 t^3 + \mu_2 t^2 + \mu_1 t + \mu_0,$$

This is a cyclic $(4,5)$ tetragonal curve, of genus six. Using a basis of holomorphic differentials

$$du_i(t,s) = \frac{g_i(t,s)}{4s^3}dt,$$

where

$$g_1(t,s) = 1, \quad g_2(t,s) = t, \quad g_3(t,s) = s,$$

$$g_4(t,s) = t^2, \quad g_5(t,s) = ts, \quad g_6(t,s) = s^2,$$

we proceed as before. Jorgenson's formula for Jacobi inversion on Θ_1 needs to be interpreted carefully - we find that *all* first derivatives of σ vanish on this stratum. Taking the limit as we approach this stratum, we get the inversion formula

$$t = -\frac{\sigma_{23}}{\sigma_{34}}. \tag{30}$$

The meromorphic part of our integrand then becomes

$$\varphi_2(t)dt = \left(\left(\frac{\sigma_{34}}{\sigma_{23}}\right)^2 - A_1\frac{\sigma_{34}}{\sigma_{23}}\right) du_1,$$

where A_1 is chosen so the residue vanishes.

Expanding everything about points $u_1^{(0)}$ where $\sigma_{23} = 0$, and using the known relations between derivatives holding on Θ_1, we find that this is (up to holomorphic terms), the derivative of

$$\Psi = -\frac{1}{4}\frac{1}{\mu_0}\frac{\sigma_{236}}{\sigma_{23}}.$$

Hence, after finding the holomorphic and constant terms as well, we obtain our conformal mapping explicitly as:

$$\lambda(p) = \hat{p}_8 + \frac{3}{8}\frac{\mu_1}{\mu_0}$$

$$+K\Big[-\frac{1}{4}\frac{1}{\mu_0}\Big[\frac{\sigma_{236}}{\sigma_{23}} - \frac{\sigma_{226}(u_0)}{\sigma_{22}(u_0)}\Big]$$

$$-\frac{1}{32}\frac{\mu_1}{\mu_0^{7/4}} + \frac{\mu_2 + 2A_2\mu_0}{2\mu_0}(u_1 - u_{1,0})$$

$$+\frac{4A_3\mu_0 + \mu_3}{4\mu_0}(u_2 - u_{2,0}) + A_4(u_4 - u_{4,0})\Big],$$

where the constants A_i and K are known. As with the trigonal and hyperelliptic cases, the key term is the quotient of σ-derivatives.

Further questions

- All examples studied so far in this way yield very similar expressions for the Schwartz-Christoffel mapping. Is there a general result of this form, applicable to much wider families of curves? This problem depends on understanding the order to which σ vanishes on the stratum Θ_1; if all the first derivatives vanish, Jorgenson's formula must be used carefully.

- Can these formulae be used to get a more detailed and explicit picture of the *differential*-geometric structure of these reductions? Andrea Raimondo has looked at the multiple Hamiltonian structures associated with general reductions - calculating these objects explicitly for Schwartz-Christoffel reductions may well yield useful insights.

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