# To the Existence of Non-Abelian Monopole: The Algebro-Geometric Approach

Victor Enolski
Institute of Magnetism, Kiev
&
ZARM, Bremen University

ICMS Workshop
Higher genus sigma function and applications
October 11-15, 2010
Edinburgh

## Publications

- Enolski Victor and Braden Harry. On the tetrahedrally symmetric monopole, *Commun. Math. Phys.*, 2010, **299**, no. 1, 255-282. arXiv: math-ph/0908.3449.
- Enolski Victor and Braden Harry. Finite-gap integration of the $SU(2)$ Bogomolny equations, *Glasgow Math.J.* 2009, **51** , Issue A, 25-41; arXiv: math-ph/ 0806.1807.

## What is the monopole?

Yang-Mills-Higgs Lagrangian density $L$ in Minkowski space of the **Georgi-Glashow Model, also Standard Model**

$$L = -\frac{1}{4}\mathrm{Tr}\, F_{ij}F^{ij} + \frac{1}{2}\mathrm{Tr}\, D_i\Phi D^i\Phi + V.$$

Here $F_{ij}$ Yang-Mills field strength

$$F_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j],$$

$a_i$ gauge field, $D_i$ covariant derivative acting on the Higgs field $\Phi$ by

$$D_i\Phi = \partial_i\Phi + [a_i, \Phi]$$

and $V$ -potential. The gauges and Higgs field take value in Lie algebra of the gauge group.

## Static solution

Gauges $a_i(\mathbf{x})$ and Higgs field $\Phi(\mathbf{x})$ are time-independent.

$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

The boundary conditions are supposed

$$\sqrt{-\frac{1}{2}\mathrm{Tr}\;\Phi(r)^2}\Bigg|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}),$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The positive integer $n \in \mathbb{N}$ is the first Chern number of the charge
Such static solution is called **non-abelian monopole of the charge** $n$ with $n \in \mathbb{N}$.

# Bogomolny equation

Suppose that (**i**) solution is static and (**ii**) potential $V = 0$ (BPS - Bogomolny-Prasad-Sommerfeld limit) but the above boundary condition remains unchanged.
Configurations that minimizing the energy of the system solve Bogomolny equations

$$D_i\Phi = \pm \sum_{j,k=1}^{3} \epsilon_{ijk} F_{jk}.$$

Moreover (**iii**) fix the gauge group as $SU(2)$
Our development deals with:

**static $SU(2)$ monopole in BPS limit**
**$\sim$ solutions of $SU(2)$ Bogomony equations**

In particular, Bogomolny equation for the gauge group $U(1)$ is Dirac equation $\equiv$ Abelian monopole.

$$U(1): \quad \mathbf{B} = \nabla\Phi, \quad \Phi = \frac{n}{2r}$$

# ADMHN theorem

The charge $n$ monopole solution is given

$$\begin{aligned}\Phi(\mathbf{x})_{\mu\nu} &= \imath \int_0^2 s\mathbf{v}_\mu^\dagger(\mathbf{x},s)\mathbf{v}_\nu(\mathbf{x},s)\mathrm{d}s, \\ a_i(\mathbf{x})_{\mu\nu} &= \imath \int_0^2 \mathbf{v}_\mu^\dagger(\mathbf{x},s)\frac{\partial}{\partial x_i}\mathbf{v}_\nu(\mathbf{x},s)\mathrm{d}s, \quad i=1,2,3,\end{aligned} \qquad \mu,\nu=1,2.$$

$\mathbf{v}_\mu(\mathbf{x},s)$ – two orthonormalizable solutions to the **Weyl equation**

$$\left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}s} + \sum_{j=1}^{3}(T_j(s)+\imath x_j 1_n)\otimes\sigma_j\right)\mathbf{v}(\mathbf{x},s)=0,$$

$n\times n$ matrices $T_j(s)$, $s\in(0,2)$ satisfy to the **Nahm equations**

$$\frac{\mathrm{d}T_i(s)}{\mathrm{d}s} = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j(s),T_k(s)],$$

$\mathrm{Res}_{s=0}T_i(s)$: irreducible $n$-dimensional representation of $su(2)$;
$T_i(s)=-T_i^\dagger(s),\ T_i(s)=T_i^\dagger(2-s)$.

# Hitchin construction (1982,1983)

Nahm equations admit the **Lax form**:

$$\frac{\mathrm{d}A(s,\zeta)}{\mathrm{d}s} = [A(s,\zeta), M(s,\zeta)]$$
$$A(s,\zeta) = A_{-1}(s)\zeta^{-1} + A_0(s) + A_{+1}(s)\zeta,$$
$$M(s,\zeta) = \frac{1}{2}A_0(s) + \zeta A_{+1}(s),$$
$$A_{\pm 1}(s) = T_1(s) \pm \imath T_2(s), \quad A_0(s) = 2\imath T_3(s).$$

Condition

$$\det(A(s,\zeta) - \eta 1_n) = 0$$

yields the curve $\hat{\mathcal{C}} = (\eta, \zeta)$ of genus

$$g_{\widehat{\mathcal{C}}} = (n-1)^2$$

is the spectral curve of the $n$-charge of monopole

$$\eta^n + \alpha_1(\zeta)\eta^{n-1} + \ldots + \alpha_n(\zeta) = 0.$$

$a_k(\zeta)$- polynomials in $\zeta$ of degree $2k$

# Hitchin constraints

The curve $\hat{\mathcal{C}}$ is subjected to the constraints
**H1.** $\hat{\mathcal{C}}$ admits the involution

$$(\zeta,\eta)\to\left(-1/\overline{\zeta},-\overline{\eta}/\overline{\zeta}^2\right)$$

**H2.** $\mathfrak{b}$-periods of the second kind normalized differentials are half-integer

$$\gamma_\infty(P)_{P\to\infty_i}=\left(\frac{\rho_i}{\xi^2}+O(1)\right)\mathrm{d}\xi,\quad \oint_{\mathfrak{a}_k}\gamma_\infty=0,$$

$$\mathbf{U}=\frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty,\dots,\oint_{\mathfrak{b}_n}\gamma_\infty\right)^T=\frac{1}{2}\mathbf{n}+\frac{1}{2}\tau\mathbf{m},$$

$\mathbf{n},\mathbf{m}\in\mathbb{Z}^g$- **Ercolani-Sinha vectors** [Ercolani-Sinha (1989)].
**H3.** $\mathbf{U}s+\mathbf{K}$, $\mathbf{K}$- vector of Riemann constants, does not intersect theta-divisor, $\Theta$, i.e.:

$$\theta(\mathbf{U}s+\mathbf{K};\tau)\neq 0,\quad s\in(0,2).$$

## Result I: A charge 3 monopole curve

The most general charge 3 monopole curve, that respects $C_3$ symmetry,

$$(\eta, \zeta) \longrightarrow (\rho\eta, \rho\zeta), \quad \rho = e^{2\imath\pi/3}.$$

$$\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0,$$

where $\alpha, \beta, \gamma$ - real.

**Theorem [Braden & E, 2009 ] The class of the monopole curves**

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$$

**consists only two representatives,**

$$b = \pm 5\sqrt{2}, \qquad \chi = -\frac{1}{6}\frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}$$

**In other words there are no monopoles beyond tetrahedral symmetry.**

## Wellstein (1899), Matsumoto (2000)

The curve

$$w^3 = (z-\lambda_1)\dots(z-\lambda_6)$$

Holomorphic differentials

$$\frac{\mathrm{d}z}{w},\quad \frac{\mathrm{d}z}{w^2},\quad \frac{z\mathrm{d}z}{w^2},\quad \frac{z^2\mathrm{d}z}{w^2}.$$

Homology: $\{\mathfrak{a}_1,\dots,\mathfrak{a}_4;\mathfrak{b}_1,\dots,\mathfrak{b}_4\}$. Denote

$$\mathbf{X} = \left(\oint_{\mathfrak{a}_1}\frac{\mathrm{d}z}{w},\dots,\oint_{\mathfrak{a}_4}\frac{\mathrm{d}z}{w}\right).$$

Then the period matrix is of the form

$$\tau = \rho^2\left(H + (\rho^2-1)\frac{\mathbf{X}\mathbf{X}^T}{\mathbf{X}^T H\mathbf{X}}\right),$$

where $\rho = \exp(2\imath\pi/3)$, $H = \mathrm{diag}(1,1,1,-1)$.

# Implementation of Wellstein's result

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0.$$

For a pair of relatively prime integers $(m, n)$ obtain a solution to **H1** and **H2**: First solve for $t$

$$\frac{2n - m}{m + n} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, 1 - t\right)}$$

Then

$$b = \frac{1 - 2t}{\sqrt{t(1 - t)}}$$

Ercolani-Sinha vectors and Riemann period matrix are

$$\mathbf{n} = \begin{pmatrix} n \\ m - n \\ -m \\ 2n - m \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} -m \\ n \\ m - n \\ 3n \end{pmatrix}$$

$$\widehat{\tau} = \rho^2 H - (\rho - \rho^2)\frac{(\mathbf{n} + \rho^2 H\mathbf{m})(\mathbf{n} + \rho^2 H\mathbf{m})^T}{(\mathbf{n} + \rho^2 H\mathbf{m})^T H(\mathbf{n} + \rho^2 H\mathbf{m})}.$$

## Strange equation

Compare our parametrization with Hitchin-Manton-Murray (1995) tetrahedral solution we conclude that at $n = 1$ and $m = 0$ should be:

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)} = 2,$$

$$t = \frac{1}{2} - \frac{5\sqrt{3}}{18}, \quad b = 5\sqrt{2}$$

In general: Do other **algebraic** numbers $t$ exist such that

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)} = \frac{p}{q} \in \mathbb{Q}, \quad t - \mathrm{algebraic}$$

## Ramanujan (1914)

Second Notebook: Let $r$ (signature) and $n \in \mathbb{N}$

$$\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} = n \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-y\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; y\right)}.$$

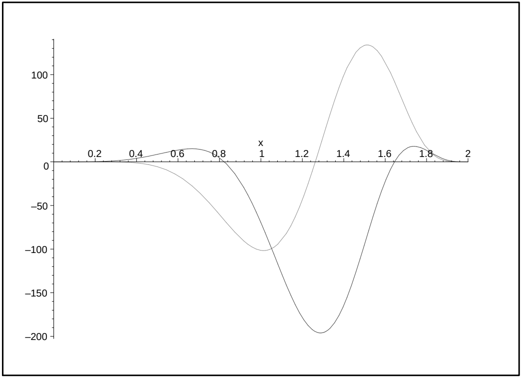
Then $\mathcal{P}(x, y) = 0$ is algebraic equation, find it!
**Ramanujan theory for signature 3**, $r = 3$, $n = 2$

$$(xy)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}(1-y)^{\frac{1}{3}} = 1$$

Set $y = \frac{1}{2}$ to obtain $b = 5\sqrt{2}$.
Other signatures: [Berndt & Bhargava & Garvan, 1995 ]

# Tetrahedral monopole exists

Value $b = 5\sqrt{2}$ corresponds to $n = 1, m = 0$

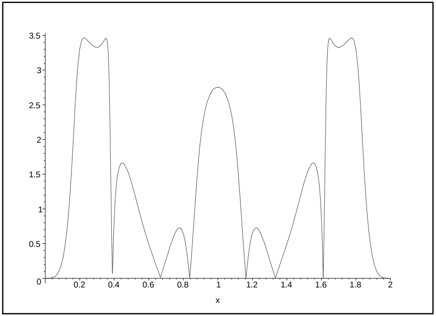



Plot of the real and imaginary parts of the function $\theta(\mathbf{U}s + \mathbf{K})$, $s \in [0, 2]$

The case $b = -5\sqrt{2}$ is given by $n = m = 1,\ b = -5\sqrt{2}$

# Conjecture: No monopoles at other (m,n)

Here $n = 4,\ m = -1$




Plot of $|\theta(\mathbf{U}s + \mathbf{K})|$ and $s \in [0, 2]$. There are 6 additional zeros.
**To make infinite number of plots at $(m, n) \in \mathbb{Z}^2$?**

## Unramified cover

[Schottky & Jung 1909, Fay 1973]
Our genus 4 curve $\widehat{\mathcal{C}}$ covers 3-sheetedly genus 2 curve $\mathcal{C}$.

$$\pi : \widehat{\mathcal{C}} \rightarrow \mathcal{C}$$

$$\widehat{\mathcal{C}} : \quad \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0,$$
$$\mathcal{C} := \nu^2 = (\mu^3 + b)^2 + 4$$

with $\nu = \zeta^3 + 1/\zeta^3, \mu = -\eta/\zeta$.
$\widehat{\mathcal{C}}$ admits automorphism: $\sigma : (\zeta, \eta) \rightarrow (\rho\zeta, \rho\eta)$
Riemann-Hurwitz formula,

$$2 - 2\widehat{g} = B + N(2 - g)$$

tells that the cover is unramified, $B = 0$.

# Schottky-Jung proportionality

In the case of unramified cover

$$\pi : \hat{\mathcal{C}}(\eta,\zeta) \longrightarrow \mathcal{C}(x,y)$$

exists a basis in homology group

$$H(\hat{\mathcal{C}},\mathbb{Z}), \quad (\mathfrak{a}_1,\ldots,\mathfrak{a}_4;\mathfrak{b}_1,\ldots,\mathfrak{b}_4)$$

admitting automorphism $\sigma$,

$$\sigma\circ\mathfrak{a}_k=\mathfrak{a}_{k+1},\quad \sigma\circ\mathfrak{b}_k=\mathfrak{b}_{k+1},\quad k=1,2,3$$
$$\sigma\circ\mathfrak{b}_0=\mathfrak{b}_0.$$

Associated period matrices

$$\hat{\tau}=\begin{pmatrix} a & b & b & b\\ b & c & d & d\\ b & d & c & d\\ b & d & d & c\end{pmatrix} \qquad \tau=\begin{pmatrix}\frac{1}{3}a & b\\ b & c+2d\end{pmatrix}.$$

## Factorization of the $\theta$-function

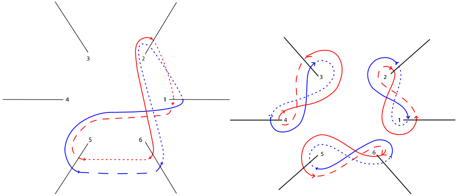
At the above conditions the associated $\theta$-function admits remarkable factorization [ Fay-Accola theorem, Fay-73, Eq.67]

$$\frac{\theta(3z_1, z_2, z_2, z_2; \widehat{\tau})}{\theta(z_1, z_2; \tau)\theta(z_1 + 1/3, z_2; \tau)\theta(z_1 - 1/3, z_2; \tau)} = c.$$

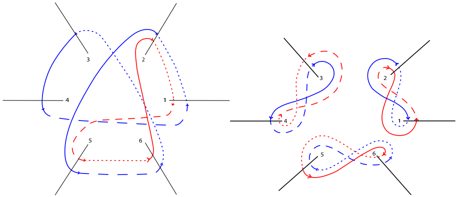
Here $c$ independent of $z_1, z_2$

# Homology transformation

Wellstein basis

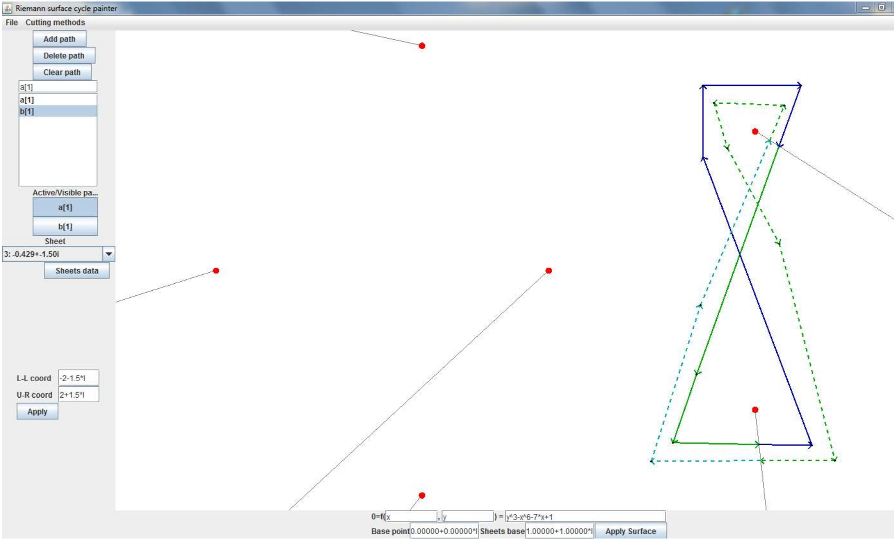



Schottky-Jung basis



# Transformation between homology bases

**T. Northower program** http://gitorious.org/riemanncycles



## Humbert variety

Krazer, Lehrbuch der Thetafunktionen, (1903), Belokolos et al., Springer (1994).
If period matrix $\tau$ of genus two curve $\mathcal{C}$ satisfies

$$q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0;$$

$$q_i \in \mathbb{Z}, \quad q_3^2 - 4(q_1q_5 + q_2q_4) = h^2, \quad h \in \mathbb{N}.$$

Then exists a symplectic transformation $\mathfrak{S}$

$$\mathfrak{S} : \tau \to \begin{pmatrix} T_1 & \frac{1}{h} \\ \frac{1}{h} & T_2 \end{pmatrix}, \quad h \in \mathbb{N}.$$

Here $h$ - degree of the cover $\mathcal{C}$ over elliptic curve $\mathcal{E}$

$$\pi : \mathcal{C} \to \mathcal{E}.$$

# Outline of theta-transformations

$$\widehat{\tau} = \rho^2 H - (\rho - \rho^2)\frac{(\mathbf{n}+\rho^2 H\mathbf{m})(\mathbf{n}+\rho^2 H\mathbf{m})^T}{(\mathbf{n}+\rho^2 H\mathbf{m})^T H(\mathbf{n}+\rho^2 H\mathbf{m})}. \qquad \text{Wellstein}$$

$$\Downarrow$$

$$\begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\Downarrow \qquad \text{Fay-Accola}$$

$$\begin{pmatrix} \frac{1}{3}a & b \\ b & c+2d \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} T & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{12T} \end{pmatrix} \qquad \text{Bolza} \quad D_6$$

## H3 condition is reduced to

**Proposition [Braden & E, 2009 ]**

$$\theta(\mathbf{U}s+\mathbf{K};\tau)=0 \quad \text{at} \quad s\in(0,2)$$

iff one from the following **3** conditions satisfies

$$\frac{\vartheta_3}{\vartheta_2}\left(y\sqrt{-3}+\varepsilon\frac{T}{3}|T\right)+(-1)^{\varepsilon}\frac{\vartheta_2}{\vartheta_3}\left(y+\varepsilon\frac{1}{3}|\frac{T}{3}\right)=0$$

$$\varepsilon=0,\pm1, \quad y=\frac{1}{3}s(n+m), \quad T=\frac{2\sqrt{-3}(n+m)}{2n-m}$$

The solution $y=y(T)$ provides the answer.
We reduced problem in $(n,m)\in\mathbb{Z}^2$ to one variable $T$

## A new $\theta$-constant relation ?

$$\frac{\vartheta_3}{\vartheta_2}\left(\frac{\tau}{3}|\tau\right) = \frac{\vartheta_2}{\vartheta_3}\left(\frac{1}{3}|\frac{\tau}{3}\right)$$

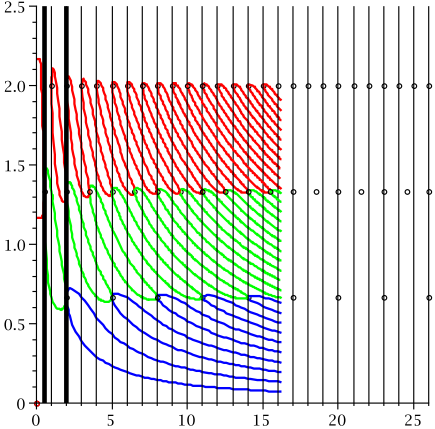
$$\vartheta_4^3(0|\tau)\imath\sqrt{3}\frac{\vartheta_1\left(\frac{\tau}{3}|\tau\right)\vartheta_4\left(\frac{\tau}{3}|\tau\right)}{\vartheta_2\left(\frac{\tau}{3}|\tau\right)^2} + \vartheta_4^2\left(0|\frac{\tau}{3}\right)\frac{\vartheta_1\left(\frac{1}{3}|\frac{\tau}{3}\right)\vartheta_4\left(\frac{1}{3}|\frac{\tau}{3}\right)}{\vartheta_3\left(\frac{1}{3}|\frac{\tau}{3}\right)^2} = 0$$

We are able to prove that using Ramanujan third order transformation of Jacobian moduli

$$k(\tau) \equiv \frac{\vartheta_2(0|\tau)^2}{\vartheta_3(0|\tau)^2} = \frac{(p+1)^3(3-p)}{16p},$$

$$k(\tau/3) \equiv \frac{\vartheta_2(0|\tau/3)^2}{\vartheta_3(0|\tau/3)^2} = \frac{(p+1)(3-p)^3}{16p^3}$$

# No charge 3 monopoles beside tetrahedral




Three branches of the function $y$ plotted against $(n+m)/(2n-m)$

Only two cases $(n+m)/(2n-m) = 2$ and
$(n+m)/(2n-m) = 1/2$ satisfy **H3**

## Result II: Explicit integration of the Weyl equation in the ADHMN construction

Let $\widehat{\mathcal{C}}$ - monopole curve of genus $g=(n-1)^2$

$$\eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta) = 0$$

satisfying Hitchin constraint **H1**, **H2**, **H3**. Then monopole fields $\Phi(\mathbf{x})$ and $a_j(\mathbf{x})$ are expressible in terms of values of **Baker-Akhiezer function**

$$\mathbf{\Psi}(\zeta, z) = \mathbf{\Psi}(P_k(\mathbf{x}), \pm 1)$$

at the boundaries of the interval $z = \pm 1$ and algebraic functions of $\mathbf{x}$, $P_k(\mathbf{x})$ that are solutions of $2n$ algebraic equation, so called **Atiyah-Ward constraint**.

# Nahm Ansatz

**Weyl equation**:

$$\left(\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} - \sum_{j=1}^{3}(T_j(z) + \imath x_j 1_n)\otimes\sigma_j\right)\mathbf{v}(\mathbf{x},z) = 0.$$

**Construction equation**:

$$\left(\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{j=1}^{3}(T_j(z) + \imath x_j 1_n)\otimes\sigma_j\right)\mathbf{V}(\mathbf{x},z) = 0.$$

Fundamental solutions of the Weyl and Construction equations

$$v = \left(\mathbf{v}^{(1)}(\mathbf{x},z),\ldots,\mathbf{v}^{(2n)}(\mathbf{x},z)\right)$$
$$V = \left(\mathbf{V}^{(1)}(\mathbf{x},z),\ldots,\mathbf{V}^{(2n)}(\mathbf{x},z)\right)$$

are related as

$$v(\mathbf{x},z) = V(\mathbf{x},z)^{-1\dagger}$$

# Reduction to $n$-the order ODE

Any column vector of the fundamental solution $V$ is presented in the form – **Nahm Ansatz**

$$\mathbf{V} = \left[ 1_2 + \sum_{k=1}^{3} u_k(\zeta)\sigma_k \right] |s> \otimes \mathbf{\Psi}(z,\zeta),$$

where $\zeta$- is certain parameter and the real vector,

$$\mathbf{u} = (u_1, u_2, u_3), \quad u_1^2 + u_2^2 + u_3^2 = 1$$

is constructed in terms of the vector $\mathbf{y}$,

$$\mathbf{y} = \left( \frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2}, -\zeta \right), \quad \mathbf{y} \cdot \mathbf{y} = 0$$

$$\mathbf{u} = \imath \frac{\mathbf{y} \times \mathbf{y}}{\mathbf{y} \cdot \overline{\mathbf{y}}}$$

Substitution to the Construction equation leads

$$(A(\zeta)-\eta)\,\boldsymbol{\Psi}(z,\zeta)=0,$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}z}+M(\zeta)\right)\boldsymbol{\Psi}(z,\zeta)=0,$$

where $A(\zeta)$ and $M(\zeta)$ are precisely Hitchin operators in the Lax representation of Nahm equations

$$A(z,\zeta)=A_{-1}(z)\zeta^{-1}+A_0(z)+A_{+1}(z)\zeta,$$
$$M(z,\zeta)=\frac{1}{2}A_0(z)+\zeta A_{+1}(z),$$
$$A_{\pm 1}(z)=T_1(z)\pm\imath T_2(z),\quad A_0(z)=2\imath T_3(z)$$

with the constraint: – **Atiyah-Ward constraint:**

$$\eta=2\mathbf{y}\cdot\mathbf{x},$$

that is $2n$-th order algebraic equation

$$\det(L(\zeta)-2\mathbf{y}\cdot\mathbf{x})=0.$$

The vector function $\boldsymbol{\Psi}(z,\zeta)$ is the Baker-Akhiezer function appearing at the integration of the Nahm equation.

# Spectral problem

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + Q(z)\right)\mathbf{\Psi} = -\zeta\mathrm{diag}(\rho_1,\ldots,\rho_n)\mathbf{\Psi}$$

is solved in $\theta$-functions of the curve $\widehat{\mathcal{C}}$
Dubrovin (1977): N-dimensional Euler top,
Ercolani-Sinha (1989): **Krichever method**:

$$\Psi_j(z,P) = G(P)\exp\left\{z\int_{P_0}^{P}\gamma_\infty - \nu_j z\right\}$$
$$\times\frac{\theta(\phi(P)-\phi(\infty_j)+\mathbf{U}(z+1)+\mathbf{K};\tau)\theta(\mathbf{U}+\mathbf{K};\tau)}{\theta(\mathbf{U}(z+1)+\mathbf{K};\tau)\theta(\mathbf{U}(z+1)+\mathbf{K};\tau)},$$

$$Q(z)_{j,l} = q_{j,l}\exp\{z(\nu_l-\nu_j)\}$$
$$\times\frac{\theta(\phi(\infty_l)-\phi(\infty_j)+\mathbf{U}(z+1)+\mathbf{K};\tau)}{\theta((z+1)\mathbf{U}+\mathbf{K};\tau)},$$

where $G(P)$ is a given function and $\nu_j$, $q_{j,l}$, $\mathbf{K}$ are given constants.

# Monopole fields

$$\Phi(\mathbf{x})_{\mu\nu} = \imath \int_{-1}^{1} z \mathbf{v}_\mu^\dagger(\mathbf{x}, z)\mathbf{v}_\nu(\mathbf{x}, z)\mathrm{d}z,$$

$$a_i(\mathbf{x})_{\mu\nu} = \imath \int_{-1}^{1} \mathbf{v}_\mu^\dagger(\mathbf{x}, z)\frac{\partial}{\partial x_i}\mathbf{v}_\nu(\mathbf{x}, z)\mathrm{d}z, \quad i = 1, 2, 3 \qquad \mu, \nu = 1, 2.$$

Antiderivatives in these expressions are computed in closed form by Panagopoulos (1983):

$$\int \mathbf{v}_\mu^\dagger(\mathbf{x}, z)\mathbf{v}_\nu(\mathbf{x}, z)\mathrm{d}z = \mathbf{v}_\mu^\dagger(\mathbf{x}, z)\mathcal{F}^{-1}(\mathbf{x}, z)\mathbf{v}_\nu(\mathbf{x}, z).$$

$$\mathcal{F}(\mathbf{x}, z) = \frac{1}{r^2}\mathcal{H}(\mathbf{x})\mathcal{T}(z)\mathcal{H}(\mathbf{x}) - \mathcal{T}(z),$$

$$\mathcal{T}(z) = \sum_{i=1}^{3} \sigma_i \otimes T_i(z), \quad \mathcal{H} = \sum_{i=1}^{3} x_i \sigma_i \otimes 1_n.$$

Also

$$\int z\mathbf{v}_\mu^\dagger(\mathbf{x},z)\mathbf{v}_\nu(\mathbf{x},z)\mathrm{d}z = \mathbf{v}_\mu^\dagger(\mathbf{x},z)\mathcal{F}^{-1}(\mathbf{x},z)\left[z+2\mathcal{H}(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}r^2}\right]\mathbf{v}_\nu(\mathbf{x},z),$$

$$\int \mathbf{v}_\mu^\dagger(\mathbf{x},z)\frac{\partial}{\partial x_i}\mathbf{v}_\nu(\mathbf{x},z)\mathrm{d}z = \mathbf{v}_\mu^\dagger(\mathbf{x},z)\mathcal{F}^{-1}(\mathbf{x},z) \times\left[\frac{\partial}{\partial x_i}+\frac{1}{r^2}\mathcal{H}(\mathbf{x})(zx_i+\imath(\mathbf{x}\times\nabla)_i)\right]\mathbf{v}_\nu(\mathbf{x},z).$$

**Conclusion:** Monopole fields are expressible in terms of

$$\mathrm{Res}|_{z=\pm 1}\,\frac{\theta(\phi(P_i)-\phi(\infty_j)+\mathbf{U}(z+1)+\mathbf{K};\tau)}{\theta(\mathbf{U}(z+1)+\mathbf{K};\tau).}$$

with $P_j$ solutions of the algebraic equation - Atiyah-Ward constraint. Realization of the construction in particular cases $n=2$, $n=3$ - in progress