Some computational challenges in Abelian function theory at genus 3 and 4

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Outline

Basic idea - we associate multiply periodic functions with plane curves of genus g. These functions satisfy interesting integrable PDEs. How do we find these PDES, in particular what is the connection with τ-functions? How do we find the associated Kummer varieties, and what can we say about their structure?

σ and $\wp$ functions

Associated with a curve of genus g, there is an *entire* function σ of g variables which generalises the Weierstrass $\sigma(u)$ function

$$\sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \ldots, u_g).$$

Given this function, we can define generalized $\wp$ functions

$$\wp_{ij}(u_1, u_2, \ldots) \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u_1, u_2, \ldots),$$

(Note that in this notation the genus one, $\wp$ function becomes$\wp_{11}$). Higher derivatives are derived in a similar way

$$\wp_{ij\ldots}(u_1, u_2, \ldots) \equiv -\frac{\partial\partial\ldots}{\partial u_i \partial u_j \partial\ldots} \ln \sigma(u_1, u_2, \ldots),$$

Hyperelliptic case, $g = 2$

The general hyperelliptic curve takes the form

$$C: \quad y^2 = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0, \quad s > 4.$$

The simplest case, $s = 5$, has genus 2. It was considered in detail by Baker (1907). σ and $\wp$ are now functions of $g = 2$ variables, i.e.

$$\sigma = \sigma(u_1, u_2) = \sigma(\mathbf{u})$$

There are two differentials of the first kind, dx/y and $x\,dx/y$, and we have

$$u_1 = \int^{(x_1,y_1)} \frac{dx}{y} + \int^{(x_2,y_2)} \frac{dx}{y}, \quad u_2 = \int^{(x_1,y_1)} \frac{x\,dx}{y} + \int^{(x_2,y_2)} \frac{x\,dx}{y},$$

for two variable points (x_i, y_i) on C.

Hyperelliptic case, $g = 2$, PDEs

The classical method to derive the PDEs associated with a given curve involves a mix of two techniques. The first is to expand the "Klein formula"

$$\sum_{i,j=1}^{g} \wp_{ij}\left(\int_{P_0}^{(x,y)} \mathrm{d}\boldsymbol{u} - \sum_{k=1}^{g}\int_{P_0}^{(x_k,y_k)} \mathrm{d}\boldsymbol{u} + \boldsymbol{K}_{P_0}\right)\mathcal{U}_i(x,y)\mathcal{U}_j(x_k,y_k)$$
$$= \frac{\mathcal{F}(x,y;x_k,y_k)}{(x-x_k)^2}, \quad k = 1,\ldots,g.$$

The other is to use some of the equations following from this formula to derive terms in the expansion of the σ-function, then use this expansion itself in a "bootstrap" fashion to derive other equations.

Hyperelliptic case, $g = 2$, PDEs

In the genus 2 case this gives the five equations expressing the 4-index $\wp$ functions $\wp_{ijk\ell}$ in terms of the $\wp_{ij}$.

$$
\begin{aligned}
\wp_{2222} - 6\wp_{22}^2 &= \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12},\\
\wp_{1222} - 6\wp_{22}\wp_{12} &= \lambda_4\wp_{12} - 2\wp_{11},\\
\wp_{1122} - 2\wp_{22}\wp_{11} - 4\wp_{12}^2 &= \tfrac{1}{2}\lambda_3\wp_{12},\\
\wp_{1112} - 6\wp_{11}\wp_{12} &= \lambda_2\wp_{12} - \tfrac{1}{2}\lambda_1\wp_{22} - \lambda_0,\\
\wp_{1111} - 6\wp_{11}^2 &= \lambda_1\wp_{12} + \lambda_2\wp_{11} - 3\lambda_0\wp_{22} - \tfrac{1}{2}\lambda_0\lambda_4 - \tfrac{1}{8}\lambda_1\lambda_3.
\end{aligned}
$$

These are the generalization of $\wp'' - 6\wp^2 = -\frac{1}{2}g_2$ in genus 1.
We can *stratify* all the equations in the theory by assigning a weight to each term $u_1 = 3, u_2 = 1, \lambda_i = -2(5 - i)$. So the 4-index PDEs have homogeneous weights -4, -6, -8, -10, -12, respectively.

Hyperelliptic case, $g = 2$, PDEs

In the genus 2 case we have 10 generalizations of $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ in genus 1. The first three are

$$
\begin{aligned}
\wp_{222}^2 &= 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + 4\wp_{11} + \lambda_4\wp_{22}^2 + \lambda_2, \\
\wp_{122}\wp_{222} &= 4\wp_{22}^2\wp_{12} + \lambda_4\wp_{22}\wp_{12} + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \tfrac{1}{2}\lambda_3\wp_{22} + \tfrac{1}{2}\lambda_1, \\
\wp_{122}^2 &= 4\wp_{12}^2\wp_{22} + \lambda_4\wp_{12}^2 + \lambda_4\wp_{12}^2 - \lambda_0, \\
\ldots &= \ldots
\end{aligned}
$$

There are also a set of relations which are *bilinear* in the 3-index and the 2-index $\wp$

$$
\begin{aligned}
&\wp_{122} + \wp_{22}\wp_{122} - \wp_{12}\wp_{222} = 0, \\
&2\wp_{11}\wp_{222} + 2\wp_{111} + (\tfrac{1}{2}\lambda_3 + 2\wp_{12})\wp_{122} - (\lambda_4 + 4\wp_{22})\wp_{112} = 0, \\
&\ldots
\end{aligned}
$$

Many results for the hyperelliptic case for *arbitrary* g, but *odd* s, have been developed by Buchstaber, Enolskii, and Leykin.

Kummer variety, $g = 2$

We can derive the formula for "Kummer's quadratic surface" in the genus two case by noting that

$$(\wp_{222}^2) \cdot (\wp_{122}^2) - (\wp_{122}\wp_{222})^2 = 0$$

We call such relations "Kummer Relations". This one gives the following quartic in three variables with weight -16.

$$\begin{aligned}
&\lambda_2\lambda_4 Y^2 + 4Z\lambda_4 Y^2 - 2X\lambda_3 YZ - \lambda_1 Y\lambda_4 X - 4ZY\lambda_2 + 4\lambda_0 XY + X\lambda_3\lambda_0 \\
&+ 2\lambda_1 XZ - \tfrac{1}{2}\lambda_1 Y\lambda_3 - 4YX^2\lambda_1 + \lambda_4 X^2\lambda_0 + 4\lambda_2 XY^2 + 8ZXY^2 + \lambda_2\lambda_0 \\
&+ 4\lambda_0 Z - 16YZ^2 - 2\lambda_1 Y^2 + 4X^3\lambda_0 - \tfrac{1}{4}Y^2\lambda_3^2 - 2Y^3\lambda_3 - 4X^2Z^2 - \tfrac{1}{4}\lambda_1^2 \\
&- 4Y^4 = 0, \qquad (K_2)
\end{aligned}$$

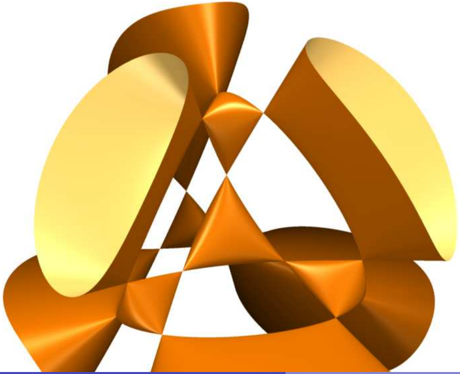
with $X = \wp_{22}, Y = \wp_{12}, Z = \wp_{11}$. This is not the only cross-product we can form, but all lower weight KRs, down to

$$\wp_{111}^2\wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0,$$

with weight -32, can be shown by direct calculation to factor into two or more terms, one of which being K_2.

Kummer variety, $g = 2$, matrix version

$$K_2 \equiv \det \begin{bmatrix} 2\,\lambda_0 & \lambda_1 & -\wp_{11} & -\wp_{12} \\ \lambda_1 & 2\,\wp_{11} + 2\,\lambda_2 & \wp_{12} + \lambda_3 & -\wp_{22} \\ -\wp_{11} & \wp_{12} + \lambda_3 & 2\,\wp_{22} + 2\,\lambda_4 & 1 \\ -\wp_{12} & -\wp_{22} & 1 & 0 \end{bmatrix} = 0$$



τ- and σ- functions

The 'algebro-geometric τ-function' AGT of a genus g curve X_g is defined (Fay83) as a function of the 'times' $\boldsymbol{t} = (t_1, \ldots, t_g, t_{g+1}, \ldots)$, a point $\boldsymbol{u} \in \mathrm{Jac}(X_g)$, as well as a point $P \in X_g$;

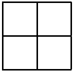
$$\tau(\boldsymbol{t}; \boldsymbol{u}, P) = \theta\left(\sum_{k=1}^{\infty} \boldsymbol{U}_k(P) t_k + \frac{1}{2}\omega^{-1}\boldsymbol{u}\right) \exp\left\{\frac{1}{2}\sum_{m,n\geq 1} \omega_{mn}(P) t_m t_n\right\}.$$

Here the "winding vectors" $\boldsymbol{U}_k(P)$ appear in the expansion of the normalized holomorphic integral $\boldsymbol{v}$, the quantities $\omega_{mn}(P)$ define the holomorphic part of the expansion of the fundamental differential of the second kind $\omega(Q, S)$ near the point P.
We then introduce the τ-function by the formula:

$$\frac{\tau(\boldsymbol{t}; \boldsymbol{u}, P)}{\tau(\boldsymbol{0}; \boldsymbol{u}, P)} = \frac{\sigma\left(\sum_{k=1}^{\infty} \mathcal{A}^{-1}\boldsymbol{U}_k(P) t_k + \boldsymbol{u}\right)}{\sigma(\boldsymbol{u})} \exp\left\{\frac{1}{2}\sum_{k,l=0}^{\infty} \omega_{k,l}^{\mathrm{alg}}(P) t_k t_l\right\}.$$

Young tableau and PDEs in genus 2

The first non-trivial Plücker relation corresponds to the partition $\lambda = (2,2)$, with Young diagram

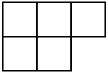
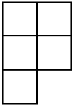


In this case we obtain, after simplification, our previous result

$$\mathrm{KdV}_4 : \qquad \wp_{2222}(\boldsymbol{u}) = 6\wp_{22}^2(\boldsymbol{u}) + 4\wp_{12}(\boldsymbol{u}) + \lambda_4\wp_{22}(\boldsymbol{u}) + \tfrac{1}{2}\lambda_3$$

The weight of the tableau is 4 in this case, the same as $(-)$ the weight of the equation.

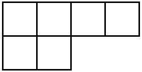
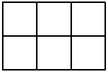
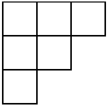
The tableau of weight 5 with 2×2 centres are

 and 

and both give the derivative of the result above.

Young tableau and PDEs in genus 2

At weight 6 we have three independent Young tableaux with $(2,2)$ centres

, , and .

(and transposes, which give the same results in the hyperelliptic case.) These three independent tableaux give an overdetermined system of three equations. After substituting for derivatives of the relation above, we can solve for the two unknowns $\wp_{1222}$ and $\wp_{222}^2$ to get

$$\begin{aligned}
\mathrm{KdV}_6: \quad & \wp_{1222} = 6\wp_{12}\wp_{22} - 2\wp_{11} + \lambda_4\wp_{12} \\
\mathrm{Jac}_6: \quad & \wp_{222}^2 = 4\wp_{22}^3 + \lambda_3\wp_{22} + \lambda_4\wp_{22}^2 + 4\wp_{12}\wp_{22} + \lambda_2 + 4\wp_{11}
\end{aligned}$$

each tableaux gives mix of 4-index and quadratic 3-index equations (and Kummer Relations if appropriate).

Hyperelliptic case, $g = 3$, PDEs

The genus 3 hyperelliptic curve is the $(2,7)$ curve

$$C: \quad y^2 = x^7 + \lambda_6 x^6 + \cdots + \lambda_0,$$

We can again grade the expressions by weights. These are

	x	y	u_1	u_2	u_3	λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0
wt.	-2	-7	5	3	1	-2	-4	-6	-8	-10	-12	-14

The first few 4-index PDEs are

$$\begin{aligned}
\wp_{3333} - 6\wp_{33}^2 &= \lambda_6\wp_{33} + 4\wp_{23} + \tfrac{1}{2}\lambda_5, \\
\wp_{2333} - 6\wp_{23}\wp_{33} &= 6\wp_{13} - 2\wp_{22} + \lambda_6\wp_{23}, \\
\ldots &= \ldots \\
\wp_{2222} - 6\wp_{22}^2 &= -12\Delta - 3\lambda_6\wp_{11} + \lambda_5\wp_{12} + \lambda_4\wp_{22} + \lambda_3\wp_{23} \\
&\quad + \tfrac{1}{8}\lambda_5\lambda_3 - \tfrac{1}{2}\lambda_6\lambda_2 - 3\lambda_2\wp_{33} - \tfrac{3}{2}\lambda_1, \\
\ldots &= \ldots
\end{aligned}$$

where $\Delta = \wp_{11}\wp_{33} - \wp_{12}\wp_{23} - \wp_{13}^2 + \wp_{13}\wp_{22}$. (Baker, Athorne).

Hyperelliptic case, $g = 3$, Kummer relations

For the (2,7) curve we have 55 different quadratic 3-index relations, the first three being

$$\begin{aligned} \wp_{333}^2 &= 4\,\wp_{33}^3 + 4\,\wp_{33}\wp_{23} + 4\,\lambda_5\wp_{33} + 4\,\lambda_6\wp_{33}^2 + 4\,\lambda_4 - 4\,\wp_{13} + 4\,\wp_{22}, \\ \wp_{233}\wp_{333} &= 4\,\wp_{33}^2\wp_{23} + 2\,\wp_{23}^2 + 2\,\lambda_5\wp_{23} + 4\,\wp_{33}\lambda_6\wp_{23} + 4\,\wp_{33}\wp_{13} \\ &\quad - 2\,\wp_{22}\wp_{33} + 2\,\wp_{12} + 2\,\lambda_3, \\ \wp_{233}^2 &= 4\,\wp_{11} + 8\,\wp_{23}\wp_{13} - 4\,\wp_{22}\wp_{23} + 4\,\wp_{33}\wp_{23}^2 + 4\,\lambda_2 + 4\,\lambda_6\wp_{23}^2. \end{aligned}$$

Defining a *Kummer relation* (KR) as

$$(\wp_{ijk}\wp_{\ell mn}) \cdot (\wp_{opq}\wp_{rst}) - (\wp_{ijk}\wp_{rst}) \cdot (\wp_{opq}\wp_{\ell mn}) = 0$$

for $1 \leq i, \ldots, t \leq g$. Each KR is at most *sextic* in the six variables $\wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$. We have a *large* number of these.

The set of KRs form the *Kummer variety* for this curve. How many KRs do we need to characterise the Kummer variety?

Hyperelliptic case, $g = 3$, Kummer relations

The Kummer variety as defined above contains *all* (?) the polynomial relations satisfied by the 2-index $\wp_{ij}$ functions in the (2,7) case.
We can grade the Kummer relations (KR) by weight, from

$$K_{16} \equiv \wp_{333}^2 \cdot \wp_{233}^2 - (\wp_{233}\wp_{333})^2 = 0$$

at weight -16 to

$$K_{56} \equiv \wp_{111}^2 \cdot \wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0$$

at weight -56. Note we may have two or more KRs with the same weight. How many are independent?
We can use Groebner base theory to answer this question. Start from K_{16} and keep adding lower weight KRs. At each stage test the lhs of new addition using a Groebner base of the lhs of the set of KRs obtained previously.

Groebner basis for the Kummer variety, (2,7) case

Algorithm: Start from K_{16}. Check the next KR in the list against a Groebner basis (GB) for the existing KRs in the list to see if it is independent. If so add to the list and generate a new GB. Repeat.

In addition to K_{16}, we find one of the weight -18 KRs is independent, two of the weight -20 KRs, one from weight 22, and one from weight 24. All these are formed from quadratic terms of the type $\wp_{igg}\wp_{jgg}$ (singled out in the BEL98 approach). *All* the others belong to the ideal generated by these *six* relations.

The second weight 20 and the weight 22 Jac KR lie in the radical ideal generated by the first three, but it is not clear if the weight 24 KR does (big calculation). Is it possible to prove that the Kummer variety can be generated by the first three KRs?.

Can we visualise the Kummer variety in any way?

Visualisation of the (2,7) Kummer variety

Take the KRs $\{K_{16}, K_{18}, K_{20a}\}$, three quartics in six variables. Take resultants of two pairs to eliminate one variable, say $\wp_{11}$. take a further resultant with respect to another, say $\wp_{12}$. We now have a single relation in four variables - a surface in 4D. If we take one of the variables to be "time", we can display this as an animation in 3D space.

What should we be looking for - or are the properties of the (2,7) Kummer variety completely understood and in no need of explicit calculations?

Movie of (2,7) Kummer variety

firstframe

Matrix theory, (2,7) case

All the quadratic 3-index $\wp_{ijk}\wp_{\ell mn}$ relations can be derived with the aid of a 5×5 matrix (BEL97, Athorne)

$$\begin{bmatrix} 4\lambda_0 & 2\lambda_1 & -2\wp_{11} & -2\wp_{12} & -2\wp_{13} \\ 2\lambda_1 & 4\wp_{11}+4\lambda_2 & 2\wp_{12}+2\lambda_3 & 4\wp_{13}-2\wp_{22} & -2\wp_{23} \\ -2\wp_{11} & 2\wp_{12}+2\lambda_3 & 4\wp_{22}-4\wp_{13}+4\lambda_4 & 2\wp_{23}+2\lambda_5 & -2\wp_{33} \\ -2\wp_{12} & 4\wp_{13}-2\wp_{22} & 2\wp_{23}+2\lambda_5 & 4\wp_{33}+4\lambda_6 & 2 \\ -2\wp_{13} & -2\wp_{23} & -2\wp_{33} & 2 & 0 \end{bmatrix}$$

This matrix is of rank 3 and the 4×4 minors of this matrix can all shown to belong to the Kummer variety.

$(3, 4)$ trigonal curve

The simplest trigonal curve is the *strictly* trigonal (3,4) curve

$$C: \quad y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0,$$

which has genus 3. Now all functions are functions of $\mathbf{u} = (u_1, u_2, u_3)$. We can again grade the expressions by weights. These are

	x	y	u_1	u_2	u_3	λ_3	λ_2	λ_1	λ_0
wt.	-3	-4	5	2	1	-3	-6	-9	-12

There are 14 four-index relations, the first 5 are

$$\begin{aligned}
\wp_{3333} - 6\,\wp_{33}^2 &= -3\,\wp_{22}, \qquad \text{(Boussinesq eqn)}\\
\wp_{2333} - 6\,\wp_{23}\wp_{33} &= 3\,\lambda_3\wp_{33},\\
\wp_{2233} - 2\,\wp_{33}\wp_{22} - 4\,\wp_{23}^2 &= 3\,\lambda_3\wp_{23} + 4\,\wp_{13} + 2\,\lambda_2,\\
\wp_{2223} - 6\,\wp_{22}\wp_{23} &= 3\,\lambda_3\wp_{22},\\
\wp_{2222} - 6\wp_{22}^2 &= 12\wp_{33}\lambda_2 - 3\wp_{33}\lambda_3^2 - 4Q_{1333},\\
\ldots &= \ldots
\end{aligned}$$

Note basis requires $Q_{1333} = \wp_{1333} - 6\wp_{33}\wp_{13}$, as well as the six $\wp_{ij}$.

Quadratic relations for the (3,4) curve

We have calculated all the 55 *quadratic* 3-index relations, first few are

$$\begin{aligned}
\wp_{333}^2 &= 4\wp_{33}^3 + \wp_{23}^2 + 4\wp_{13} - 4\wp_{33}\wp_{22}, \quad [-6] \\
\wp_{233}\wp_{333} &= 4\wp_{23}\wp_{33}^2 - \wp_{22}\wp_{23} + 2\wp_{33}^2\lambda_3 - 2\wp_{12}, \quad [-7] \\
\wp_{233}^2 &= 4\wp_{33}\wp_{23}^2 + 4\wp_{33}\lambda_3\wp_{23} + \wp_{22}^2 + 4\wp_{33}\lambda_2 - \tfrac{4}{3}Q_{1333}, \quad [-8]
\end{aligned}$$

A full list would go down to $\wp_{111}^2$ of weight -30.

Again we have a large number of *Kummer relations* (KR) in the form

$$(\wp_{ijk}\wp_{\ell mn}) \cdot (\wp_{opq}\wp_{rst}) - (\wp_{ijk}\wp_{rst}) \cdot (\wp_{opq}\wp_{\ell mn}) = 0,$$

but now we have *seven* variables $\wp_{11}, Q_{1333}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$.

We can again grade the KRs by weight, from

$$K_{14} \equiv \wp_{333}^2 \cdot \wp_{233}^2 - (\wp_{233}\wp_{333})^2 = 0 \qquad [-14]$$

to

$$K_{54} \equiv \wp_{111}^2 \cdot \wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0 \qquad [-54]$$

Kummer variety, (3,4) curve?

One difference from the (2,7) case is that the set of all KRs does *not* contain all the polynomial relationships involving the seven variables $\wp_{11}, Q_{1333}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$, which we will rewrite as $X_{10}, X_8, X_7, X_6, X_4, X_3, X_2$.

Calculations show an *extra* relationship, which we will call K_{12}. It is quartic and of weight -12

$$\begin{aligned} K_{12} \equiv & -2\lambda_0 - 4X_2X_3X_7 + \lambda_3^2X_2^3 - 4\lambda_2X_2^3 - X_3^2\lambda_2 - 2X_6\lambda_2 - 8X_6X_2^3 \\ & - 4X_6X_3^2 + 4X_2X_4\lambda_2 + 6X_2X_4X_6 - 2X_6^2 - X_4X_8 - X_2^2X_4^2 \\ & + 2X_4X_2X_3^2 + 3X_4X_2\lambda_3X_3 + X_4^3 - 3X_2\lambda_3X_7 - 3X_3X_6\lambda_3 - \lambda_3X_3^3 \\ & - X_3^4 + \tfrac{4}{3}X_2^2X_8 \; - \lambda_1X_3 + 2X_2X_{10} = 0, \end{aligned}$$

On weight grounds this *cannot* belong to the set of KRs.

Where does this extra relation come from? Does it have any deeper significance?

Groebner basis for the Kummer variety, (3,4) case

We can use the same agorithm as before, building up a list of KRs together with the corresponding Groebner basis. But now we have two choices. The first is to use K_{12} as a starting point, then add known KRs starting with K_{14}. We find we need *seven* relations in total, $K_{12}, K_{14}, K_{15}, K_{16a}, K_{16b}, K_{17}, K_{18}$.

Another approach: A tentative matrix theory (BEL00) gives prominence to KR's involving only 3-index $\wp$ of the form $\wp_{igg}\wp_{jgg}$. This theory does not result in any equations involving $\wp_{11}$ (X_{10}). If we restrict ourselves to this subset, which excludes K_{12}, we can form a consistent variety from the *six* KRs: $K_{14}, K_{17}, K_{18}, K_{20}, K_{21}, K_{22}$. Some of these KRs are quintic.

It has not been possible so far to characterise the radical ideal involved in these two cases. Is it possible to say something about the number of generators required from general theory?

Visualisation of the (3,4) Kummer variety

We have seen that we have two “different” varieties arising from the (3,4) curve, in seven and six variables respectively. We can procede as in the (2,7) case to eliminate variables using resultants to give a single expression. The perhaps surprizing result is that we end up with the *same* single expression in four variables in both cases. Of course to achieve this, one of the variables which is eliminated must be $X_{10} = \wp_{11}$.

The resulting expression has 1,506 terms if all the λ_i are nonzero. The case of the more general (3,4) curve (still with a branch point at infinity) also goes through with some very long computations - the resulting expression in four variables has 56,653 terms.

Questions: What should we be looking for? Is there anything known in the literature about this variety? Can we get a full matrix theory for (3,4)?

Genus 4

In genus 4 we again have a hyperelliptic curve $y^2 = x^9 + \ldots$ and a trigonal curve, the simplest of which is $y^3 = x^5 + \ldots$. Most but not all of the PDEs are known, enough to begin to look at the Kummer problem. Unfortunately the computations rapidly become unmageable, using much time (weeks) or too much memory ($>$8GB) so not much progress has been made so far. We have established that there are *two* independent polynomial relations in the (3,5) case which cannor be written as KRs.

Movie of (3,4) Kummer variety

firstframe

Further reading

- H. F. Baker,
  Multiply Periodic Functions.
  Cambridge Univ. Press, Cambridge, 1907.
- H. F. Baker,
  Note on Hyperelliptic Serfaces and certain Kummer surfaces
  Mathematics. Proc. Camb. Phil. Soc. **32**: 342–354 **1936**,
- Sadie Baldwin, J. C. Eilbeck, John Gibbons, and Y. Ônishi.
  Abelian functions for purely trigonal curves of genus four.
  J. Geom. Phys., **58**, 450–467, 2008
- V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin.
  Kleinian functions, hyperelliptic Jacobians and applications.
  Rev. in Math. and Math. Phys., 10:1–125, 1997.
- V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin. *Uniformisation of Jacobi varieties of trigonal curves and nonlinear differential equations*,
  Func. Anal. Appl. **34** (2000), no. 3, 159–171.
- J. C. Eilbeck, M. England and Y. Onishi,
  Abelian functions associated with genus three algebraic curves J. C. Eilbeck, M. England and Y. Onishi, `arXiv:1008.0289` (2010)

Further reading 2

- J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato,
  Abelian Functions for Trigonal Curves of Genus Three,
  International Mathematics Research Notices **2007**, Art.ID rnm140 (2007).
- J. C. Eilbeck, S. Matsutani, and Y. Ônishi,
  Some addition formulae for Abelian functions for elliptic and hyperelliptic curves of cyclotomic type,
  arxiv:0803.3899, 2008
- M. England and J. C. Eilbeck,
  Abelian functions associated with a cyclic tetragonal curve of genus six,
  J. Phys. A: Math. Theor., **42**, 095210, 2009.
- R. W. H. Hudson,
  Kummer's quadratic surface,
  CUP 1905 (reprinted 1990).