Liouville Theorems for the Navier-Stokes equations and applications

G. Koch

N. Nadirashvili

G. Seregin

V. Sverak

$$\left.\begin{aligned} u_t + u\nabla u + \nabla p - \Delta u &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,\infty)$$

$$u(x,0) = u_0(x)$$

Scaling symmetry

$$u(x,t) \longrightarrow u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \lambda > 0$$

Regularity criteria $\longleftrightarrow$ scale-invariant quantities

(regularity is invariant $\Rightarrow$ criteria should ideally also be invariant)

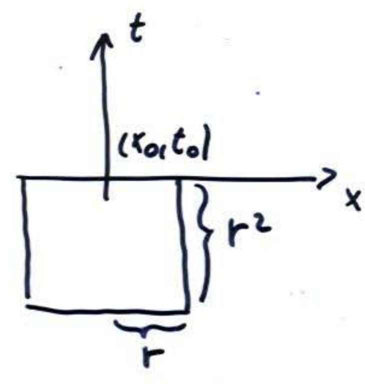
Notation:

$$B_{x_0,r} = \{x,\ |x - x_0| < r\}$$

$B_{x_0,r}$

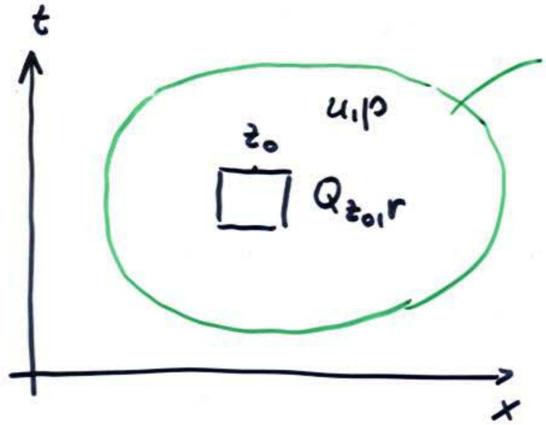
$$z = (x,t)$$

$$Q_{z_0,r} = B_{x_0,r} \times (t_0 - r^2, t_0]$$



$Q_{z_0,r}$

Set - up



$\mathcal{O} \subset \mathbb{R}^3 \times \mathbb{R}$ open set

(u,p) $\underbrace{\text{local solution in } \mathcal{O}}$

"suitable weak solution"
(allows for potential singularities)

$z_0 \in \mathcal{O}$ is <u>regular</u> if $u \in \mathcal{C}^\alpha$ near z_0

$\iff$ u smooth in x near z_0

Remark: For <u>local</u> solutions we do not expect smoothness in t. (Consider for example
$u(x,t) = b(t)$, $b : (t_1, t_2) \to \mathbb{R}^3$,
$p(x,t) = -b'(t) \cdot x$)

Scale-invariant quantities - examples

Global

$$\int_0^\infty \int_{\mathbb{R}^3} |u(x,t)|^5 \, dx\, dt = \int_0^\infty \int_{\mathbb{R}^n} |u_\lambda(x,t)|^5 \, dx\, dt \qquad (\lambda > 0)$$

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx = \operatorname*{ess\,sup}_{t\in(0,\infty)} \int_{\mathbb{R}^3} |u_\lambda(x,t)|^3 \, dx \qquad (\lambda > 0)$$

Local

$$\int_{Q_{z_0,R}} |u(x,t)|^5 \, dx\, dt = \int_{Q_{z_0',R_\lambda}} |u_\lambda(x,t)|^5 \, dx\, dt \qquad (\lambda > 0)$$

must scale R: $R \to R_\lambda = \frac{R}{\lambda}$

$$\frac{1}{R^{5-p}} \int_{Q_{z_0,R}} |u(x,t)|^p \, dx\, dt = \frac{1}{R_\lambda^{5-p}} \int_{Q_{z',R_\lambda}} |u_\lambda(x,t)|^p \, dx\, dt$$

$R_\lambda = \frac{R}{\lambda}$

enforces scale-invariance

Regularity criteria - examples

$$\int_{Q_{z_0,R}} |u|^5 \, dx\,dt < +\infty \Rightarrow z_0 \text{ is regular}$$

Ladyzhenskaya
Prodi
Serrin

$$\operatorname*{ess\,sup}_{t\in(t_0-R^2,t_0)} \int_{B_{x_0,R}} |u(x,t)|^3 \, dx < +\infty \Rightarrow z_0 \text{ is regular} \quad (\text{E-S-S})$$

$$\left.\begin{array}{l} \frac{1}{r}\int_{Q_{z_0,r}} |\nabla u|^2 \, dx\,dt < \varepsilon^{1)} \\ r\in(0,r_0) \end{array}\right\} \Rightarrow z_0 \text{ is regular}$$

Caffarelli
Kohn
Nirenberg

$$\left.\begin{array}{l} \frac{1}{r^2}\int_{Q_{z_0,r}} |u|^3 \, dx\,dt < \varepsilon \\ r\in(0,r_0) \end{array}\right\} \Rightarrow z_0 \text{ is regular}$$

Tian - Xin
Gustafson - Kang - Tsai

1) ε is small

$$\operatorname*{ess\,sup}_{(x,t)\in Q_{z_0,R}} \sqrt{t_0-t}\,|u(x,t)| < \varepsilon \Rightarrow z_0 \text{ is regular}$$

(Leray)

$$\operatorname*{ess\,sup}_{(x,t)\in Q_{z_0,R}} |x-x_0|\,|u(x,t)| < \varepsilon \Rightarrow z_0 \text{ is regular}$$

?

Open problem:
$p < \frac{5}{2}$

$$\left.\begin{array}{l} \dfrac{1}{r^{5-p}} \displaystyle\int_{Q_{z_0,r}} |u|^p \,dx\,dt < \varepsilon \\ r \in (0, r_0) \end{array}\right\} \overset{?}{\Rightarrow} z_0 \text{ is regular}$$

True for $p \geq \frac{5}{2}$

$p = 2 < \frac{5}{2}$ ... "scaled average energy".

All the above conditions give (explicitly or implicitly) some <u>small quantities</u>.

$\downarrow$ enable us to prove regularity by perturbation techniques

Seemingly

$$\int_{Q_{z_0,R}} |u|^5 \, dx\, dt < +\infty \qquad 1)$$

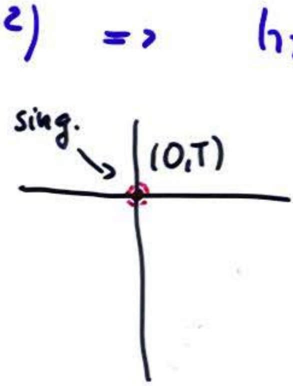
or

$$\operatorname*{ess\,sup}_{t \in (t_0 - r^2, t_0)} \int_{B_{x_0,r}} |u(x,t)|^3 \, dx < +\infty \qquad 2)$$

do not require "smallness", but

$$1) \Rightarrow \int_{Q_{z_0,R_1}} |u|^5 \, dx\, dt < \varepsilon \quad \text{for } R_1 \text{ small}$$

$$2) \Rightarrow \text{hidden small quantity}$$



$$\int_{B_R} |u(x,T)|^3 \, dx \longrightarrow 0 \quad \text{as } R \to 0$$

"blow-up profile" $\rightarrow$ gives a small quantity!

This talk

What if we have scale-invariant bounds which are only finite (and do not give smallness conditions)?

All the above statements with ε in the formulation become unknown once ε is not small.

An approach via Liouville

$$\begin{pmatrix}\text{Scale-invariant} \\ \text{bound}\end{pmatrix} + \begin{pmatrix}\text{Liouville} \\ \text{theorem}\end{pmatrix} \Rightarrow \text{regularity}$$

Remarks:

(i) For general 3d solutions no scale-invariant bounds are known

(ii) History of the "Liouville approach":

De Giorgi (minimal surfaces) Giga-Kohn (parabolic eq.)

Gidas-Spruck (elliptic eq.) Hamilton (geometric flows)

Liouville Theorems

Heat equation $\quad u_t - \Delta u = 0$

various forms of Liouville:

(a) A bounded entire solution (defined in $\mathbb{R}^n_x \times \mathbb{R}_t$) is constant

(b) A bounded solution defined in $\mathbb{R}^n \times (-\infty, 0)$ is constant

Solutions in $\mathbb{R}^n \times (-\infty, 0)$ ... "ancient solutions"
(terminology by R. Hamilton)

A suitable analogue of (b) for Navier-Stokes would imply regularity in the presence of practically any reasonable scale-invariant bound.

Some technical points

Various possibilities for what is meant by "bounded":

1) $|u| \leq C$

2) $|u|, |p| \leq C$

3) $\|u(t)\|_{L^p_x} \leq C$

4) $|\mathrm{curl}\, u| \leq C$

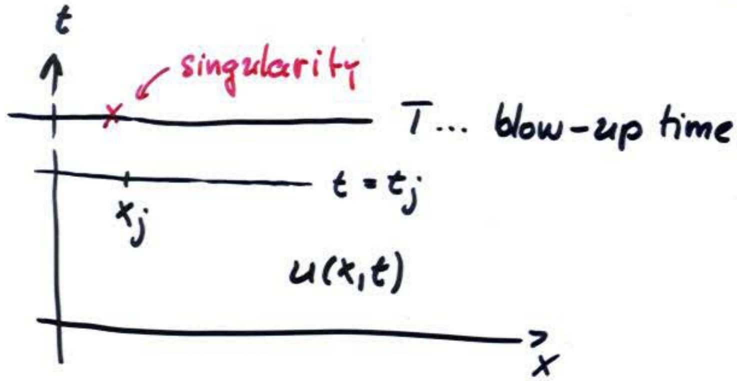
etc.

This talk

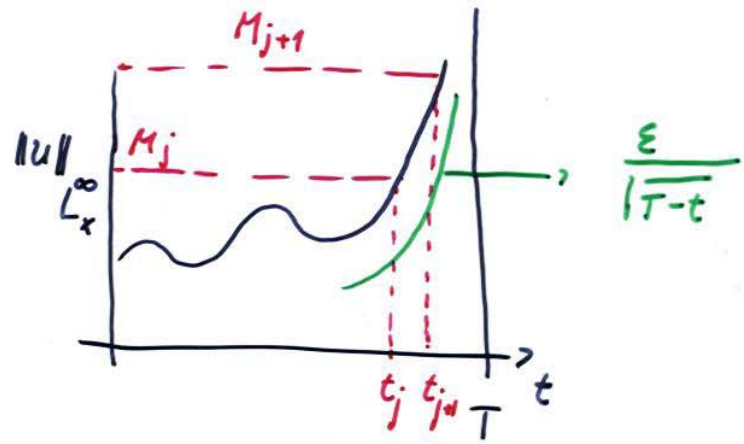
$|u| \leq C$ + $\underbrace{\text{a suitable class}}$ of solutions

If we "ignore p", we must be careful to define what exactly we mean by a solution

Generating ancient solutions by rescaling at a potential singularity



Assume u is global in x for simplicity

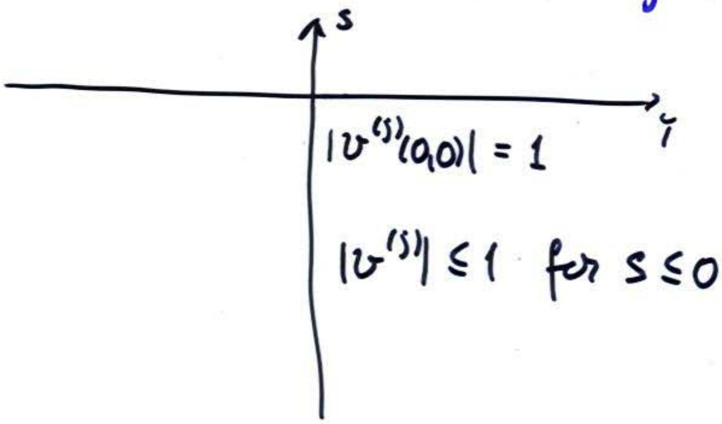


$$\|u(t)\|_{L^\infty_x} \geq \frac{\varepsilon}{\sqrt{T-t}} \quad \text{(Leray)}$$

$$M_j \nearrow +\infty, \quad |u(x_j, t_j)| = M_j$$

$$\|u(t)\|_{L^\infty_x} \leq M_j \text{ for } t \leq t_j$$

Re-scale: $$v^{(j)}(y,s) = \frac{1}{M_j}\, u\left(x_j + \frac{y}{M_j},\ t_j + \frac{s}{M_j^2}\right)$$



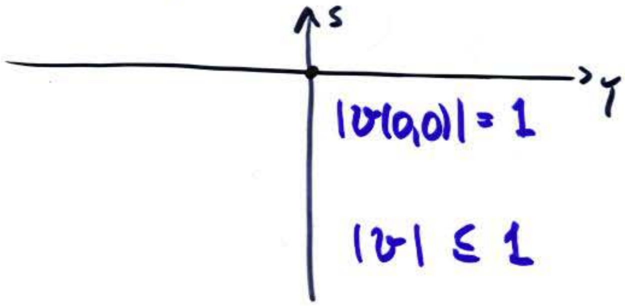
Defined for $s > -t_j M_j^2 \searrow -\infty$

Important point:

$v^{(j)}$ are uniformly Hölder-continuous in $\mathbb{R}^3 \times (-M_j^2 \frac{T}{2}, 0]$

(the bound depends only on $|v^{(j)}| \leq 1$ + suitable solution class)

$v^{(j)} \overset{\text{loc.}}{\rightrightarrows} v$ for a subsequence



$v : \mathbb{R}^n \times (-\infty, 0) \longrightarrow \mathbb{R}^n$

ancient solution

(for a suitable pressure field)

If we can show that $v \equiv \text{const.}$, we typically get a contradiction with a scale-inv. bound

For example:

$$\frac{1}{R^3} \int_{Q_R} |v|^2 \, dy \, ds \sim R^2 \quad (\text{because } |v| \equiv 1)$$

Technical points - formulating N-S without pressure, parasitic solutions

weak formulation

$$\iint [-v\varphi_t - v\Delta\varphi - v_i v_j \varphi_{i,j}]\,dx\,dt = 0$$

$\forall \varphi \in \mathcal{D}(\mathbb{R}^3 \times (-\infty, 0))$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$

$\operatorname{div} \varphi = 0$; also, $\operatorname{div} v \equiv 0$.

Exceercise in regularity theory

bouded weak sols. are smooth in x

Parasitic solutions allowed by the weak formulation:

$v(x,t) = b(t)$, $q(t) = -b'(t)\cdot x$

$b : (-\infty, 0) \to \mathbb{R}^n$ arbitrary bounded measurable

$\longrightarrow$ not good for Liouville applications!

Mild solutions

$$u_t - \Delta u + \nabla p = \partial_j f_j$$
$$\operatorname{div} u = 0$$
$$u(x,0) = u_0(x)$$

Helmholtz

$$\partial_j \left(\Gamma(t-s) P \right)$$

Representation formula

$$u(t) = \Gamma(t) * u_0 + \int_0^t \underbrace{K_j(t-s)}_{\text{integrable in } x \ (\forall\, t-s>0)} * f_j(s)\, ds$$

L^1_x

Def.: Mild solutions of N-S

$$u(t) = \Gamma(t-t_0) * u(t_0) - \int_{t_0}^t K_j(t-s) * \left(u_j(s) u(s) \right) ds$$

1) Well defined for $|u| \le C$

2) Good formulation in the Liouville context $\longrightarrow$ rules out the parasitic solutions!

Important point:

The re-scaling procedure generates ancient mild solutions. This is true even if we apply the procedure to (reasonable) local solutions.

(Obvious in the global case. The local case requires some work.)

Excercise in regularity theory:
a bounded mild solution of N-S in $\mathbb{R}^3 \times (t_1, t_2)$
is smooth in x, t.

The most optimistic "Liouville conjecture"
(consistent with all we know)

Conjecture:
A bounded mild ancient solution
of N-S is constant

Remarks:

(i) The case $n=3$ seems to be completely
out of reach of existing methods

(ii) Even when $\frac{\partial u}{\partial t} \equiv 0$ (steady-state case)
is open. In fact, if we assume
in addition that $\int_{\mathbb{R}^3} |\nabla u|^2 < +\infty$,
it is still open

Examples

(i) $u_t + u u_x = u_{xx}$ (viscous Burgers)

Liouville fails completely (travelling waves and their generalizations)

(ii) $u_t + u\nabla u + \frac{1}{2} u \operatorname{div} u = \Delta u$, $n \geq 1$

(modified Burgers, with energy identity for all $n \geq 1$)

For $n = 3$ it has non-trivial radial steady states $u(x) = v(r)\frac{x}{r}$,

$|u(x)| \sim |x|^{-\frac{2}{3}}$ as $x \to +\infty$

$|\nabla u(x)| \sim |x|^{-\frac{2}{3}-1}$ —"— $\Rightarrow |\nabla u| \in L^2_x$

etc. ...

Theorem 1: (KNSS)

The Liouville conjecture for N-S is true for $n = 2$.

Corollary:

No travelling waves for $n=2$

(Also gives 2d regularity -- energy est. is scale-invariant for $n=2$)

Proof of Theorem 1:

$\omega = \operatorname{curl} u$, $|\omega| \leq C$ (by regularity)

$$(*) \quad \omega_t + u\nabla\omega = \Delta\omega \quad \text{in } \mathbb{R}^2 \times (-\infty, 0]$$

?? Linear Liouville for (*) ?? $\longrightarrow$ fails!

using only $\operatorname{div} u = 0$, but not $\omega = \operatorname{curl} u$

Have to work with $\omega = \operatorname{curl} u$

Remarks on the linear Liouville theorem

1) Classical $\left.\begin{array}{l}\Delta u = 0 \text{ in } \mathbb{R}^n \\ |u| \le C\end{array}\right\} \Rightarrow u \equiv \text{const.}$

2) $\left.\begin{array}{l}-\Delta u + \overset{a_j = a_j(x)}{a_j} \dfrac{\partial u}{\partial x_j} = 0 \text{ in } \mathbb{R}^n \\ |u| \le C\end{array}\right\} \not\Rightarrow u \equiv \text{const.}$

fails even for ODE ($n=1$) and a smooth, comp. supp.

3) $\left.\begin{array}{l}-\Delta u + a_j \dfrac{\partial u}{\partial x_j} = 0 \text{ in } \mathbb{R}^n \\ \operatorname{div} a = 0 \\ |u| \le C\end{array}\right\} \overset{?}{\Rightarrow} u \equiv \text{const.}$

fails (for $n \geq 2$) for $|\nabla^k a| \le C_k$ in $\mathbb{R}^n$

true (for $n \geq 2$) when a is rapidly decaying

Example : $n = 2$, $\operatorname{div} a = 0 \Leftrightarrow a = \nabla^\perp \psi$

If ψ is bd. (in fact VMO is enough)

$\Rightarrow$ 3) is true (S-S)

back to the 2d NS Liouville

"Remnants of the linear Liouville":

$$M = \sup_{x,t} \omega$$

Harnack $\Rightarrow$ $\exists$ arbitrarily large parabolic balls $Q_{z_0,R}$ where $\omega \geq M - \varepsilon$

$t \in (t_0 - R^2, t_0)$

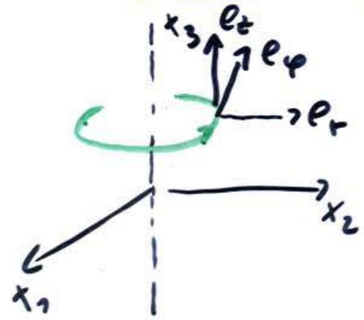
Calculation at a fixed t:

$$\text{(i)} \quad \int_{B_{x_0,R}} \omega \geq 2\pi R^2 (M-\varepsilon)$$

$$\text{(ii)} \quad \int_{B_{x_0,R}} \omega = \int_{\partial B_{x_0,R}} (u_2 n_1 - u_1 n_2) = O(R)$$

$\rightarrow$ contradiction for large R.

Axi - symmetric solutions in 3d



$$u(Rx) = R\,u(x) \qquad R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$u = u^{(r)}(r,z)\,e_r + u^{(\varphi)}(r,z)\,e_\varphi + u^{(z)}(r,z)\,e_z$$

<u>Remark:</u> $u^{(\varphi)} \equiv 0$ ... "no swirl" $\rightarrow$ full regularity known (Yudovich, Ladyzhenskaya)

$u^{(\varphi)} \not\equiv 0$ open problem

<u>Theorem 2</u> (KNSS)

The Liouville conjecture is true for 3d axi-symmetric solution (possibly with swirl) if the following decay condition is satisfied:

$$|u(x,t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}}$$

$\leftarrow$ can be verified in applications to singularities

Remarks:

(i) The conjecture is also true for axi-sym. solutions with no swirl (no further cond. necessary)

(ii) If we allow nonzero swirl and drop the decay cond., the conjecture is open even for steady-state sols. $(\frac{\partial u}{\partial t} = 0)$

Implications for potential singularities:

Theorem 3 (u, p) axi-symm. suitable weak sol.

(i) $\operatorname*{ess\,sup}_{(x,t) \in Q_{z_0,R}} \sqrt{t_0 - t}\,|u(x,t)| < +\infty \Rightarrow z_0$ is regular

(ii) $\operatorname*{ess\,sup}_{(x,t) \in Q_{z_0,R}} |x - x_0|\,|u(x,t)| < +\infty \Rightarrow z_0$ is regular

This seems to be out of reach of the usual perturbation techniques (no small quantities).

Similar results were obtained independently by a different method by Chen, Strain, Tsai and Yau.

For N-S, the self-similar (scale-invariant) blow-up rate would be

$$\|u(t)\|_{L^\infty} \sim \frac{C}{\sqrt{T-t}} \qquad \text{"Type I" blow-up}$$

(Leray proved $\|u(t)\|_{L^\infty} \geq \frac{\varepsilon}{\sqrt{T-t}}$ at a potential singularity)

"Type II" blow-up ... anything not of Type I

example: $\|u(t)\|_{L^\infty} \sim \frac{C}{(T-t)^{\frac{1}{2}+\varepsilon}}$ ("slow blow-up")

A potential axi-symmetric singularity for N-S must be of Type II.

<u>Remark</u>: Other equations:

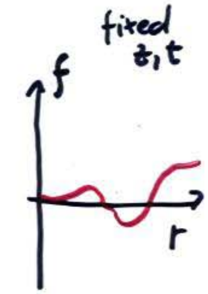
(i) $u_t = \Delta u + u^3$, $n=3$, $u \geq 0$ always Type I (Kohn-Giga)

(ii) $u_t = \Delta u + |\nabla u|^2 u$ $n=2$, $u: \mathbb{R}^2 \times (t_1,t_2) \to S^2$ always Type II

(iii) $u_t + u u_x = u_{xx}$, $u: \mathbb{R} \times (t_1,t_2) \to \mathbb{C}$ (complex-valued viscous Burgers) always Type II axi-symm. Liouville

Proof of Theorem 2:

2 important quantities for axi-sym. flows



① $r u^{(r)} = f$ note: $f(r,z,t)\big|_{r=0} \equiv 0$

② $\dfrac{\omega^{(\varphi)}}{r} = \eta$ (with $\omega = \mathrm{curl}(u)$)

Equation for f:

flux from the x_3-axis which spreads the influence of $f\big|_{r=0} \equiv 0$

$$(*) \quad f_t + u\nabla f = \Delta f - \underbrace{\frac{2}{r}\frac{\partial f}{\partial r}}$$

Important: no pressure term in the eq. for f $\left(\dfrac{\partial p}{r\,\partial\varphi} = 0\right)$

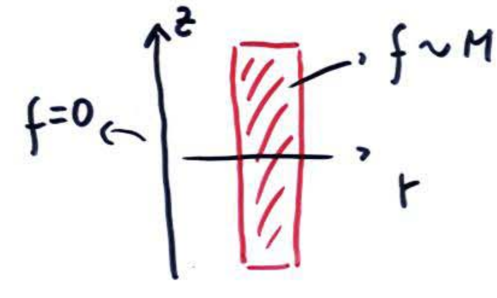
Remark:

Euler $\longrightarrow$ $f_t + u\nabla f = 0$ $\longleftrightarrow$ $\dfrac{d}{dt}\displaystyle\int_{\Gamma_t} u_i\,dx_i = 0$

Main point:

$|u| \le \dfrac{C}{\sqrt{x_1^2 + x_2^2}}$ $\implies$ Linear Liouville for f.

Harnack for f
+ scaling
+ $\sup f \sim M > 0$ $\longrightarrow$



lasts for a long time

nothing in (*) can balance $\frac{2}{r}\frac{\partial f}{\partial r}$

Equation for $\eta = \frac{\omega^{(\varphi)}}{r}$:

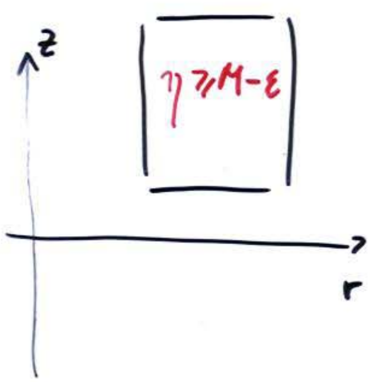
$$(**) \qquad \frac{\partial \eta}{\partial t} + u \nabla \eta = \overbrace{\Delta \eta + \frac{2}{r}\frac{\partial \eta}{\partial r}}^{\Delta_5 f} + \overbrace{\frac{2 f f_{,z}}{r^4}}^{\text{coupling to the } f\text{-equation}}$$

<u>Remark</u>: Euler for axi-sym. fields with no swirl:

$\eta_t + u \nabla \eta = 0 \quad \longleftrightarrow$ Helmholtz law for $\omega = \eta r e_\varphi$

Linear Liouville for f removes the difficult coupling term to the f-equation $\longrightarrow$ we get

$$\frac{\partial \eta}{\partial t} + u \nabla \eta = \Delta_5 \eta$$



Assume $M = \sup \eta > 0$

Harnack $\longrightarrow$ large areas with $\eta \geq M - \varepsilon$

$\Downarrow$

Contradiction with $|r\eta| \leq C$

Summary of the proof of Theorem 2 :

Step 1 : establish regularity of u (standard methods)

Step 2 : Linear Liouville for f ← needs $|u| \le \frac{C}{\sqrt{x_1^2+x_2^2}}$. The linear result fails without this

Step 3 : (non-linear) Liouville for $\omega^{(\varphi)}$

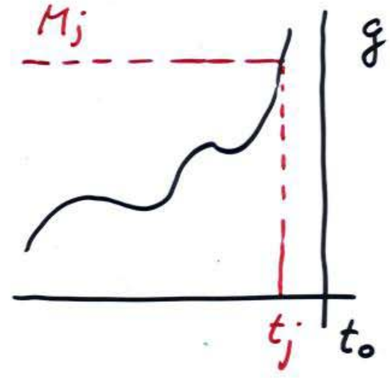
but not hard

<u>Key point</u> : Step 2 is decoupled from Step 3. In the general axi-sym. situation (without the decay condition) one would need to deal with both the f-eq. and the η-eq. at once.

Proof of Theorem 3 (no type II axi-symm. singularities)

Turn $\sqrt{t_0 - t}\ |u(x,t)| \leq C$

into $\sqrt{x_1^2 + x_2^2}\ |u(x,t)| \leq C$

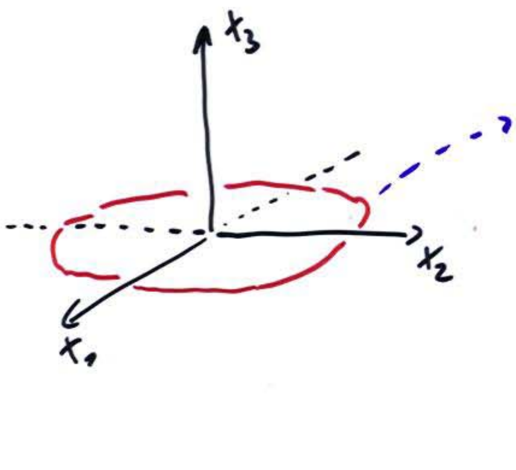


$$f(x,t) = \sqrt{x_1^2 + x_2^2}\ |u(x,t)|$$

$$g(t) = \sup_x f(x,t)$$

Rescale $\longrightarrow$ Can assume

$$f(x_j', 0, t_j) = M_j \ , \quad |x_j'| = |(x_{1j}, x_{2j})| = 1$$



sequence of solutions
with a uniform bound
$$|u^{(j)}(x,t)| \leq \frac{C}{\sqrt{t_0 - t}}$$
blowing up along a circle

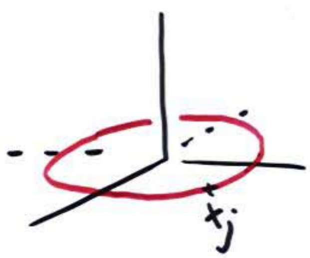
contradiction

2 ways to get a contradiction

(i) $|u^{(j)}| \leq \frac{C}{\sqrt{t_0 - t}}$ => local energy estimate for $u^{(j)}$

$\longrightarrow$ contradiction with blow-up along a circle (C-K-N)

(ii) Another blow-up procedure at the circle



Re-scale around $x_j \to \bar{x} = (1,0,0)$.

$\longrightarrow$ contradiction with 2d Liouville