

# Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms

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**Abstract.** In this Note we prove an a priori estimate and the existence of a solution for a class of nonlinear elliptic problems whose model is  $-div A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$ , when  $1 < q < 2$  and  $f \in L^m(\Omega)$  for some suitable  $m$ . The main interest of the result lies in the a priori estimate, the complete proof of which is given in the Note. © XXXX Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

*Existence et estimation a priori pour des problèmes elliptiques avec des termes sous quadratiques par rapport au gradient*

**Résumé.** Dans cette Note nous démontrons une estimation a priori et l'existence d'une solution pour une classe de problèmes non linéaires dont le modèle est  $-div A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$ , où  $1 < q < 2$  et où  $f \in L^m(\Omega)$  pour un  $m$  convenable. L'intérêt principal du résultat réside dans l'estimation a priori, dont la démonstration complète est donnée dans la Note. © XXXX Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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## Version française abrégée

Dans cette Note, nous démontrons l'existence d'une solution de

$$u \in H_0^1(\Omega), \quad -div A(x)Du + a(x)u = H(x, u, Du) \quad \text{in } \mathcal{D}'(\Omega), \quad (1)$$

quand  $\Omega$  est un ouvert borné de  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $A$  une matrice bornée coercive,  $a$  une fonction bornée non négative (voir les hypothèses (2), (3) de la version anglaise) et quand  $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  est une fonction de Carathéodory qui vérifie

$$|H(x, s, \xi)| \leq \gamma|\xi|^q + f(x) \quad \text{avec } \gamma \geq 0, 1 + \frac{2}{N} \leq q < 2, f \in L^m(\Omega) \text{ et } m = \frac{N}{q'}. \quad (4)$$

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Note présentée par XXX XXX

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Un exemple modèle de fonction  $H$  qui vérifie (4) est donné par  $H(x, u, Du) = c(x)Du + d(x)|Du|^r + f(x)$ , où  $c \in L^N(\Omega)^N$ ,  $0 \leq r \leq q < 2$ ,  $d \in L^s(\Omega)$  pour un  $s$  convenable et  $f \in L^p(\Omega)$  avec  $p \geq m = \frac{N}{q}$ .

Le problème (1) a fait l'objet de nombreux travaux quand  $H$  est à croissance quadratique ou linéaire par rapport à  $Du$ . Quand  $q = 2$ , la condition (4) devient  $f \in L^{\frac{N}{2}}(\Omega)$ ; sous cette hypothèse, l'existence d'une solution de (1) qui vérifie de plus la condition  $e^{\gamma|u|} - 1 \in H_0^1(\Omega)$  a été démontrée dans [10] si  $\alpha_0 = 0$  et dans [8] si  $\alpha_0 > 0$  (voir aussi [4], [5] dans le cas où  $q = 2$  et  $m > \frac{N}{2}$ ). Quant à lui, le cas  $q = 1$  est classique, même si l'opérateur est non coercif lorsque  $\gamma$  est grand; cette difficulté a été résolue dans [7]. Mais à notre connaissance, le cas  $1 < q < 2$  est resté ouvert jusqu'à maintenant, à l'exception de l'article récent [9] (voir aussi [6]).

Dans cette Note, nous nous restreignons au cas où  $q \geq 1 + \frac{2}{N}$ . En effet, dans ce cas,  $m$  vérifie  $m = \frac{N}{q} \geq \frac{2N}{N+2} = (2^*)'$ , et on a donc  $f \in H^{-1}(\Omega)$ , ce qui permet de chercher des solutions dans  $H_0^1(\Omega)$ . Dans notre futur article [11], nous traiterons aussi le cas  $1 < q < 1 + \frac{2}{N}$  par la méthode que nous utilisons ici; les résultats obtenus sont similaires même si les solutions, qui ne sont plus dans  $H_0^1(\Omega)$ , doivent être définies au sens des solutions renormalisées (ou d'entropie).

**THÉORÈME 1.** – *Supposons que l'on a, outre (2), (3) et (4), l'une des deux hypothèses suivantes :*

$$\text{ou bien } \alpha_0 > 0, \quad \text{ou bien } \alpha_0 = 0 \text{ et } \gamma^{\frac{1}{q-1}} \|f\|_{L^m(\Omega)} < C_0 \alpha^{\frac{q}{q-1}},$$

*où  $C_0$  est une constante qui dépend seulement de  $N$  et  $q$ .*

*Alors il existe au moins une solution  $u$  de (1) qui de plus vérifie*

$$|u|^\sigma \in H_0^1(\Omega) \quad \text{avec} \quad \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \quad (7)$$

*De plus, toute solution de (1) qui vérifie la condition de régularité (7) vérifie l'estimation a priori*

$$\|u\|_{H_0^1(\Omega)} + \||u|^\sigma\|_{H_0^1(\Omega)} \leq M, \quad (8)$$

*où  $M$  dépend seulement de  $N$ ,  $q$ ,  $|\Omega|$ ,  $\alpha$ ,  $\alpha_0$ ,  $\gamma$  et  $f$ .*

Quand  $\alpha_0 > 0$ , la constante  $M$  qui apparaît dans l'estimation a priori (8) dépend de la fonction  $f$ , non seulement par l'intermédiaire de sa norme  $\|f\|_{L^m(\Omega)}$ , mais aussi par l'intermédiaire du nombre  $k^*$  défini par (17). Cependant la constante  $M$  est bornée quand  $f$  varie dans un ensemble de fonctions qui sont bornées et équi-intégrables dans  $L^m(\Omega)$ .

La relation entre les paramètres  $q$  et  $m$  imposée dans (4) est naturelle, car les conditions nécessaires pour l'existence d'une solution de (1) démontrées dans [1], [12] conduisent dans le cadre adopté ici à  $m = \frac{N}{q}$ . De plus ces résultats montrent qu'une condition sur la taille des données est nécessaire pour l'existence d'une solution de (1) quand  $\alpha_0 = 0$ .

L'exposant  $\sigma$  défini par (7) vérifie  $\sigma \geq 1$  quand  $q \geq 1 + \frac{2}{N}$ .

Il existe des solutions de (1) qui ne vérifient pas la condition de régularité (7). Un exemple classique en est  $u(x) = C_q (|x|^{-\frac{2-q}{q-1}} - 1)$ , qui, lorsque  $q > 1 + \frac{2}{N}$ , vérifie, pour un choix convenable de  $C_q$ ,  $u \in H_0^1(\Omega)$ ,  $-\Delta u = |Du|^q$  in  $\mathcal{D}'(B_1)$  et  $|u|^\rho \in H_0^1(\Omega)$  pour tout  $\rho < \sigma$ , mais ne vérifie pas  $|u|^\sigma \in H_0^1(\Omega)$ . Par contre, des résultats d'unicité pour les solutions de (1) qui vérifient (7) ont été récemment démontrés dans [2].

Dans notre futur article [11], nous donnerons un certain nombre de généralisations du Théorème 1. Comme nous l'avons déjà dit, nous traiterons tout l'intervalle  $1 < q < 2$ . Nous étudierons aussi le cas des conditions aux limites de Neumann et de Fourier, le cas des ouverts non bornés, et celui de fonctions  $f$  appartenant à des espaces de Lorentz ou à des espaces de Sobolev négatifs. Tout cela sera présenté dans le cadre d'opérateurs non linéaires pseudomonotones de type Leray-Lions définis dans l'espace  $W^{1,p}(\Omega)$ . Nous considérerons plus tard l'analogie parabolique du problème (1).

### 1. Introduction, main result and comments

In this Note we prove the existence of a solution of

$$u \in H_0^1(\Omega), \quad -\operatorname{div} A(x)Du + a(x)u = H(x, u, Du) \quad \text{in } \mathcal{D}'(\Omega), \quad (1)$$

when  $\Omega$  is an open bounded set of  $\mathbb{R}^N$  ( $N \geq 3$ ),  $A$  a bounded coercive matrix,  $a$  a bounded nonnegative function, i.e.,

$$A \in L^\infty(\Omega)^{N \times N}, \quad A \geq \alpha I, \quad \alpha > 0, \quad (2)$$

$$a \in L^\infty(\Omega), \quad a \geq \alpha_0, \quad \alpha_0 \geq 0, \quad (3)$$

and when  $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function which satisfies

$$|H(x, s, \xi)| \leq \gamma |\xi|^q + f(x) \quad \text{with } \gamma \geq 0, \quad 1 + \frac{2}{N} \leq q < 2, \quad f \in L^m(\Omega) \text{ and } m = \frac{N}{q'}. \quad (4)$$

A model example of function  $H$  which satisfies (4) is  $H(x, u, Du) = c(x)Du + d(x)|Du|^r + f(x)$  with  $c \in L^N(\Omega)^N$ ,  $0 \leq r \leq q < 2$ ,  $d \in L^s(\Omega)$  for a suitable  $s$  and  $f \in L^p(\Omega)$  with  $p \geq m = \frac{N}{q'}$ .

There is a wide literature concerning problem (1) when  $H$  has a quadratic or a linear growth with respect to  $Du$ . When  $q = 2$ , condition (4) becomes  $f \in L^{\frac{N}{2}}(\Omega)$ ; in this case existence of a solution of (1) which satisfies the further regularity  $e^{\gamma|u|} - 1 \in H_0^1(\Omega)$  has been proved in [10] if  $\alpha_0 = 0$  and in [8] if  $\alpha_0 > 0$  (previous references for the case where  $q = 2$  and  $m > \frac{N}{2}$  are, e.g., [4], [5]). On the other hand, the case  $q = 1$  is classical, but exhibits an important difficulty when  $\gamma$  is large, due to the fact that the operator is then non coercive; this problem has been solved in [7]. As far as we know, the case  $1 < q < 2$  has been left open until now, except for the very recent paper [9] (see also [6]).

In the present Note, we restrict ourselves to the case where  $q \geq 1 + \frac{2}{N}$ . Indeed in this case  $m$  satisfies  $m = \frac{N}{q'} \geq \frac{2N}{N+2} = (2^*)'$ , hence  $f \in H^{-1}(\Omega)$ . This allows us to look for solutions which belong to  $H_0^1(\Omega)$ . In our forthcoming paper [11] we will also treat the case  $1 < q < 1 + \frac{2}{N}$  by the method used in the present Note and obtain very similar results, except for the fact that the solution is no more in  $H_0^1(\Omega)$  and has to be defined as a renormalized (or entropy) solution.

**Theorem 1.** – Assume (2), (3), (4), and one of the two following hypotheses:

$$\text{either } \alpha_0 > 0, \quad (5)$$

$$\text{or } \alpha_0 = 0 \quad \text{and} \quad \gamma^{\frac{1}{q-1}} \|f\|_{L^m(\Omega)} < C_0 \alpha^{\frac{q}{q-1}}, \quad (6)$$

where  $C_0$  is a constant which depends only on  $N$  and  $q$ .

Then there exists at least one solution  $u$  of (1) which further satisfies

$$|u|^\sigma \in H_0^1(\Omega) \quad \text{with} \quad \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \quad (7)$$

Moreover, every solution of (1) which satisfies the regularity requirement (7) satisfies the estimate

$$\|u\|_{H_0^1(\Omega)} + \||u|^\sigma\|_{H_0^1(\Omega)} \leq M, \quad (8)$$

where  $M$  depends only on  $N$ ,  $q$ ,  $|\Omega|$ ,  $\alpha$ ,  $\alpha_0$ ,  $\gamma$  and  $f$ .

**Remark 2.** – When  $\alpha_0 > 0$ , the constant  $M$  which appears in the a priori estimate (8) depends on the function  $f$ , not only through its norm  $\|f\|_{L^m(\Omega)}$ , but also through the number  $k^*$  defined by (17) below. However, the constant  $M$  is bounded when  $f$  varies in a set of functions which are bounded and equi-integrable in  $L^m(\Omega)$ .

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**Remark 3.** – The link imposed in (4) between the parameters  $q$  and  $m$  is natural. Consider indeed the model problem  $u \in H_0^1(\Omega)$ ,  $-\Delta u + u = \gamma|Du|^q + f(x)$  in  $\mathcal{D}'(\Omega)$ , with  $1 < q < 2$  and  $f \in L^m(\Omega)$  for some  $m$ . Since  $f \in L^m(\Omega)$ , one expects, in view of the  $W^{2,p}$  regularity result, that  $u \in W^{2,m}(\Omega)$ , which in turns implies, by reading the equation, that  $|Du|^q \in L^m(\Omega)$ . But  $W^{2,m}(\Omega) \subset W^{1,m^*}(\Omega)$  implies  $|Du| \in L^{m^*}(\Omega)$ . This leads to  $qm = m^*$ , namely  $m = \frac{N}{q'}$  as required. On the other hand, the necessary conditions obtained in [1], [12] for the existence of a solution of (1), when specialized to the present setting, lead to the condition  $m = \frac{N}{q'}$ . Moreover, these papers show that a condition like (6) on the size of the data is necessary in order to have the existence of a solution of (1) when  $\alpha_0 = 0$ .

**Remark 4.** – The exponent  $\sigma$  defined by (7) satisfies  $\sigma \geq 1$  when  $q \geq 1 + \frac{2}{N}$ .

**Remark 5.** – There exist solutions of (1) which do not satisfy the regularity (7). A well-known example is the function  $u(x) = C_q (|x|^{-\frac{2-q}{q-1}} - 1)$ , which, when  $q > 1 + \frac{2}{N}$ , satisfies, for a suitable choice of  $C_q$ ,  $u \in H_0^1(\Omega)$ ,  $-\Delta u = |Du|^q$  in  $\mathcal{D}'(B_1)$  and  $|u|^\rho \in H_0^1(\Omega)$  for every  $\rho < \sigma$ , but does not satisfy  $|u|^\sigma \in H_0^1(\Omega)$ . In contrast, uniqueness results for the solutions of (1) which satisfy the regularity requirement (7) have been recently proved in [2].

**Remark 6.** – In our forthcoming paper [11], we will present many extensions of Theorem 1. As said before, we will consider, as far as the growth of  $H(x, u, Du)$  with respect to  $|Du|$  is concerned, the full range  $1 < q < 2$ . We will also treat the case of Neumann's and Robin's boundary conditions, the case of unbounded domains and the case of functions  $f$  in some Lorentz spaces and in some negative Sobolev spaces. All of this will be done for general nonlinear pseudomonotone operators of Leray-Lions type defined in  $W^{1,p}(\Omega)$ . We will consider the parabolic analogue of problem (1) later.

## 2. Proof of Theorem 1

In this proof,  $C_0, C_1, C_2, C_3$  and the generic constant  $C$  will denote different positive constants which depend only on  $N$  and  $q$ .

We first prove the second part of Theorem 1, namely the a priori estimate (8).

Let  $u$  be any solution of (1) which satisfies the regularity requirement (7). In equation (1) we can take as test function any function  $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , but also  $v = |u|^{2\sigma-2}u$  (recall that  $\sigma \geq 1$ ): to prove this assertion, take  $v = |T_n(u)|^{2\sigma-2}T_n(u)$ , where  $T_n$  is the truncation at height  $n$ , and pass to the limit using (4) and (7). We actually use a slight modification of the latest test function: given  $k > 0$ , we choose as test function in (1)  $v = |G_k(u)|^{2\sigma-2}G_k(u)$ , where  $G_k(s) = s - T_k(s)$ . We obtain

$$\begin{aligned} \alpha(2\sigma - 1) \int_{\Omega} |G_k(u)|^{2\sigma-2} |DG_k(u)|^2 dx + \alpha_0 \int_{\Omega} |u| |G_k(u)|^{2\sigma-1} dx &\leq \\ &\leq \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx + \int_{\Omega} |f| |G_k(u)|^{2\sigma-1} dx. \end{aligned} \quad (9)$$

The first term of the left-hand side of (9) is nothing but

$$\alpha(2\sigma - 1) \int_{\Omega} |G_k(u)|^{2\sigma-2} |DG_k(u)|^2 dx = \alpha C \int_{\Omega} |D(|G_k(u)|^\sigma)|^2 dx. \quad (10)$$

We then estimate the first term of the right-hand side of (9); Hölder's inequality and the same computation as in (10) yield

$$\begin{aligned} \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx &= \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{\frac{2\sigma-2}{2}q} |G_k(u)|^{2\sigma-1-(\sigma-1)q} dx \leq \\ &\leq \gamma C \left( \int_{\Omega} |D(|G_k(u)|^\sigma)|^2 dx \right)^{\frac{q}{2}} \left( \int_{\Omega} |G_k(u)|^{(2\sigma-1-(\sigma-1)q)\frac{2}{2-q}} dx \right)^{1-\frac{q}{2}}. \end{aligned}$$

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But  $(2\sigma - 1 - (\sigma - 1)q) \frac{2}{2-q} = \sigma 2^*$ , and Sobolev's embedding yields

$$\gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx \leq \gamma C \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}^{q+(1-\frac{q}{2})2^*}. \quad (11)$$

We finally estimate the second term of the right-hand side of (9) by (note that  $|u| > k$  when  $G_k(u) \neq 0$ )

$$\begin{aligned} \int_{\Omega} |f| |G_k(u)|^{2\sigma-1} dx &= \int_{\{|f| \leq \alpha_0 |u|\}} |f| |G_k(u)|^{2\sigma-1} dx + \int_{\{|f| > \alpha_0 |u|\}} |f| |G_k(u)|^{2\sigma-1} dx \leq \\ &\leq \alpha_0 \int_{\Omega} |u| |G_k(u)|^{2\sigma-1} dx + \int_{\{|f| > \alpha_0 k\}} |f| |G_k(u)|^{2\sigma-1} dx. \end{aligned} \quad (12)$$

The first term of the right-hand side of (12) is absorbed by the second term of the left-hand side of (9). Using Hölder's inequality, and since  $(2\sigma - 1)m' = \sigma 2^*$ , we estimate the second term by

$$\begin{aligned} \int_{\{|f| > \alpha_0 k\}} |f| |G_k(u)|^{2\sigma-1} dx &\leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} \| |G_k(u)|^{2\sigma-1} \|_{L^{m'}(\Omega)} \leq \\ &\leq C \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}^{\frac{2^*}{m'}}. \end{aligned} \quad (13)$$

Set

$$Y_k = \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}.$$

Using (10), (11), (12) and (13) in (9) yields, for two positive constants  $C_1$  and  $C_2$

$$\alpha C_1 Y_k^2 \leq \gamma C_2 Y_k^{q+(1-\frac{1}{q})2^*} + \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} Y_k^{\frac{2^*}{m'}}.$$

Dividing by  $Y_k^{\frac{2^*}{m'}}$ , and using  $2 - \frac{2^*}{m'} = \frac{2^*}{N}(q' - 2)$  and  $q + (1 - \frac{q}{2})2^* - \frac{2^*}{m'} = \frac{2^*}{N}(q' - q)$ , we finally obtain

$$\forall k \geq 0, \quad \alpha C_1 Y_k^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y_k^{\frac{2^*}{N}(q'-q)} \leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)}. \quad (14)$$

Define the function  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$F(Y) = \alpha C_1 Y^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y^{\frac{2^*}{N}(q'-q)}.$$

Then (14) is equivalent to

$$\forall k \geq 0, \quad F(Y_k) \leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)}. \quad (15)$$

Since  $q < 2$ ,  $F$  is a concave function with a unique maximizer  $Z^*$  and maximum  $F^*$ , where  $Z^*$  and  $F^*$  are given by

$$Z^* = C_3 \left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}} \quad \text{and} \quad F^* = F(Z^*) = C_0 \frac{\alpha^{\frac{q}{q-1}}}{\gamma^{\frac{1}{q-1}}}.$$

Inequality (15) is non trivial only if its right-hand side is strictly smaller than  $F^*$ .

Here we split the proof into two cases.

(i) If  $\alpha_0 = 0$ , hypothesis (6) is nothing but  $\|f\|_{L^m(\Omega)} < F^*$ . Then equation  $F(Y) = \|f\|_{L^m(\Omega)}$  has two roots  $Z_0^-$  and  $Z_0^+$ , with  $0 < Z_0^- < Z^* < Z_0^+$ , and inequality (15) is equivalent to

$$\forall k \geq 0, \quad \text{either } Y_k \leq Z_0^- \quad \text{or} \quad Y_k \geq Z_0^+. \quad (16)$$

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But since  $|u|^\sigma \in H_0^1(\Omega)$ , the function  $k \rightarrow Y_k = \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}$  is continuous and tends to zero when  $k$  tends to infinity. The alternative (16) then implies that  $Y_k \leq Z_0^-$  for every  $k$ ; in particular, one has

$$Y_0 = \|D(|u|^\sigma)\|_{L^2(\Omega)} \leq Z_0^- < Z^*$$

(ii) If  $\alpha_0 > 0$ , we define  $k^*$  as

$$k^* = \inf \{k > 0 : \|f\chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} < F^*\}. \quad (17)$$

For every  $\delta > 0$ , one has  $\|f\chi_{\{|f| > \alpha_0(k^* + \delta)\}}\|_{L^m(\Omega)} < F^*$ , and equation  $F(Y) = \|f\chi_{\{|f| > \alpha_0(k^* + \delta)\}}\|_{L^m(\Omega)}$  has two roots  $Z_{k^* + \delta}^-$  and  $Z_{k^* + \delta}^+$ , with  $0 < Z_{k^* + \delta}^- < Z^* < Z_{k^* + \delta}^+$ . Inequality (15) implies that for every  $k \geq k^* + \delta$ , either  $Y_k \leq Z_{k^* + \delta}^-$ , or  $Y_k \geq Z_{k^* + \delta}^+$ . But the function  $k \rightarrow Y_k$  is continuous and tends to zero when  $k$  tends to infinity. We conclude that for every  $k \geq k^* + \delta$ , one has  $Y_k \leq Z_{k^* + \delta}^-$ , and in particular that  $Y_{k^* + \delta} = \|D(|G_{k^* + \delta}(u)|^\sigma)\|_{L^2(\Omega)} \leq Z_{k^* + \delta}^- < Z^*$ . We then let  $\delta$  tend to zero.

In both cases, we have proved that

$$\|D(|G_{k^*}(u)|^\sigma)\|_{L^2(\Omega)} \leq Z^* = C_3 \left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}}, \quad (18)$$

where  $k^* = 0$  when  $\alpha_0 = 0$ , and where  $k^*$  is defined by (17) when  $\alpha_0 > 0$ . When  $k^* = 0$ , inequality (18) is nothing but the second part of the a priori estimate (8). Note that the constant  $Z^*$ , which plays here a role similar to the constant  $M$ , depends only on  $N$ ,  $q$ ,  $\alpha$  and  $\gamma$ , but that  $k^*$  depends on the function  $f$  itself.

We now prove the first part of the a priori estimate (8). Since one has  $DG_{k_1^*}(u) = \chi_{\{x: |u| \geq k_1^*\}} Du$  and  $|D(|G_{k^*}(u)|^\sigma)| = \sigma |G_{k^*}(u)|^{\sigma-1} |DG_{k^*}(u)|$ , estimate (18) provides an estimate of  $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$  for  $k_1^* = k^* + 1$ . We then use  $v = T_{k_1^*}(u)$  in (1) and we get

$$\begin{aligned} \alpha \int_{\Omega} |DT_{k_1^*}(u)|^2 dx &\leq \int_{\Omega} (\gamma |Du|^q + f) |T_{k_1^*}(u)| dx \leq \gamma k_1^* \int_{\Omega} |Du|^q dx + k_1^* \|f\|_{L^1(\Omega)} \leq \\ &\leq \gamma k_1^* \int_{\Omega} |DT_{k_1^*}(u)|^q dx + \gamma k_1^* \int_{\Omega} |DG_{k_1^*}(u)|^q dx + k_1^* \|f\|_{L^1(\Omega)}, \end{aligned}$$

from which we deduce, using  $q < 2$  and the estimate on  $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$ , an estimate on  $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$ , and thus the first part of the a priori estimate (8), with a constant which depends on  $k^*$ .

Finally combining the estimates on  $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$  and  $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$  and estimate (18) completes the proof of the second part of the a priori estimate (8). The constant which appears in this estimate depends on  $k^*$ .

We now pass to the proof of the existence of a solution of (1) which satisfies the regularity requirement (7). This proof is classical. One considers the approximation of (1) by the problem in which the function  $H$  is replaced by the function  $H_\varepsilon = T_{\frac{1}{\varepsilon}}(H)$ ; this function  $H_\varepsilon$  satisfies (4) for every  $\varepsilon > 0$ . This equation has at least one solution  $u_\varepsilon$  which, by the weak maximum principle, belongs to  $H_0^1(\Omega) \cap L^\infty(\Omega)$ . Therefore  $u_\varepsilon$  satisfies the regularity requirement (7), and the a priori estimate (8) ensures that  $u_\varepsilon$  and  $|u_\varepsilon|^\sigma$  are bounded in  $H_0^1(\Omega)$ . In view of the growth condition (4),  $H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$  is then bounded in  $L^{\frac{2}{q}}(\Omega)$ . Let  $u \in H_0^1(\Omega)$  be such that a subsequence (still denoted by  $\varepsilon$ )  $u_\varepsilon$  weakly converges to  $u$  in  $H_0^1(\Omega)$ . The bound of  $u_\varepsilon$  in  $H_0^1(\Omega)$  implies that  $-\operatorname{div} A(x) Du_\varepsilon$  is bounded both in  $H^{-1}(\Omega)$  and in  $L^{\frac{2}{q}}(\Omega)$ . These bounds imply (see, e.g., [3]) that, extracting if necessary a new subsequence,  $Du_\varepsilon$  converges to  $Du$  almost everywhere in  $\Omega$ , which in turn implies that  $H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$  converges to  $H(x, u, Du)$  strongly in  $L^s(\Omega)$  for every  $s < \frac{2}{q}$ . This result easily allows one to pass to the limit in the approximate equation, which proves the existence of a solution of (1) which satisfies (7).

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