Critical magnetic fields for the magnetic Dirac-Coulomb operator

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– destabilize matter, distorting atoms and molecules and forming polymer-like chains,

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Earth's magnetic field = 1 Gauss

Maximal field on Earth (MRI) = 1 Tesla = 10^4 Gauss

Deatly field strength = 10^5 Tesla = 10^9 Gauss

In new magnetars one expectes fields of 10^{11} Tesla. In recent theories, up to 10^{16} Tesla in the heart of the magnetars.

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– hydrogenic atoms in a curved universe (Nowotny),

– intense electrostatic fields.... (..., Brodsky, Mohr,...),

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REMARK. Same problem but very different situation in nonrelativistic quantum mechanics (with the Schrödinger and the Pauli operators).

The free Dirac operator

$$\mathbf{H} = -\mathbf{i}\,\alpha \cdot \nabla + \beta\,, \qquad \alpha_1,\ \alpha_2,\ \alpha_3,\ \beta \in \mathcal{M}_{4\times 4}(\mathbb{C}) \qquad (c = m = \hbar = 1)$$

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QUESTION : Spectrum of $\mathbf{H} + V$?

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(actually also (recent result) if $\nu = 1$).

Its spectrum is given by:

$$\sigma(H_\nu) = (-\infty, -1] \ \cup \ \{\lambda_1^\nu, \lambda_2^\nu, \dots\} \cup [1, \infty)$$

$$0 < \lambda_1^\nu = \sqrt{1-\nu^2} \le \cdots \le \lambda_k^\nu \cdots < 1 \ .$$

and the fact that $\lambda_1(H + V_\nu)$ belongs to $(-1, 1)$ is a kind of "stability condition" for the electron.

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When a external magnetic field B is present, one considers a magnetic potential $\mathbf{A}_B$ s. t. $curl\, \mathbf{A}_B = \mathbf{B}$.

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Does $\lambda_1(B, V)$ ever leave the spectral gap $(-1, 1)$? and if yes, for which values of B ?

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Abstract min-max theorem (Dolbeault, E., Séré, 2000)

Let $\mathcal{H}$ be a Hilbert space and $A : F = D(A) \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator defined on $\mathcal{H}$. Let $\mathcal{H}_+$, $\mathcal{H}_-$ be two orthogonal subspaces of $\mathcal{H}$ satisfying: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Define $F_\pm := P_\pm F$.

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Let $$c_k = \inf_{\substack{V \text{ subspace of } F_+ \\ \dim V = k}} \ \sup_{x \in (V \oplus F_-) \setminus \{0\}} \frac{(x, Ax)}{||x||^2_{\mathcal{H}}} \,, \qquad k \geq 1.$$

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If (ii) $c_1 > a_-$, then c_k is the k-th eigenvalue of A in the interval (a_-, b), where $b = \inf\,(\sigma_{\text{ess}}(A) \cap (a_-, +\infty))$.

Application to magnetic Dirac operators I

$$\psi : \mathbb{R}^3 \to \mathbb{C}^4\,, \quad \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \chi \end{pmatrix}\,, \quad \varphi, \chi : \mathbb{R}^3 \to \mathbb{C}^2$$

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Suppose that V satisfies

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Then, for all $k \geq 1$,

$$\lambda_k(B,V) = \inf_{\substack{Y \text{ subspace of } C_o^\infty(\mathbb{R}^3,\mathbb{C}^2) \\ \dim Y = k}} \ \sup_{\varphi \in Y \setminus \{0\}} \ \sup_{\substack{\psi = \binom{\varphi}{\chi} \\ \chi \in C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)}} \frac{(\psi, (H_B + V)\psi)}{(\psi,\psi)}$$

Application to magnetic Dirac operators II

The first eigenvalue of $H_B + V$ in the spectral hole $(-1, 1)$ is given by

$$\lambda_1(B,V) := \inf_{\varphi \neq 0} \sup_{\chi} \frac{(\psi, (H_B + V)\psi)}{(\psi, \psi)}\,, \qquad \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

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$$\lambda^B(\varphi) := \sup_{\substack{\psi = \left(\begin{smallmatrix} \varphi \\ \chi \end{smallmatrix}\right) \\ \chi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)}} \frac{(\psi, (H_B + V)\psi)}{(\psi, \psi)}$$

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is the unique real number λ such that

$$\int_{\mathbb{R}^3} \Big(\frac{|\sigma \cdot \nabla_B \, \varphi|^2}{1 - V + \lambda} + (1 + V)|\varphi|^2 \Big) dx \; = \; \lambda \int_{\mathbb{R}^3} |\varphi|^2 dx \; .$$

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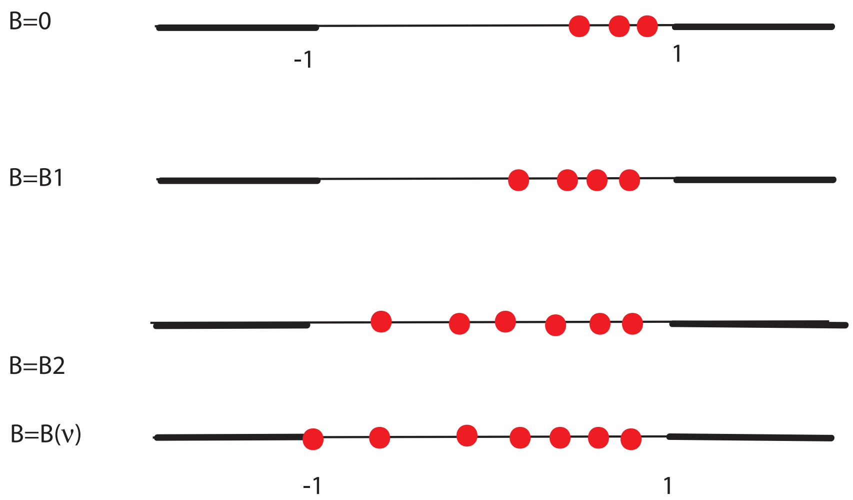
QUESTIONS : When do we have $\lambda_1(B,V)\in(-1,1)$?

If the eigenvalue $\lambda_1(B,V)$ leaves the interval $(-1,1)$, when ?

For a potential $V = V_\nu,\ \nu \in (0,1), \quad 0 < B1 < B2 < B(\nu)$:



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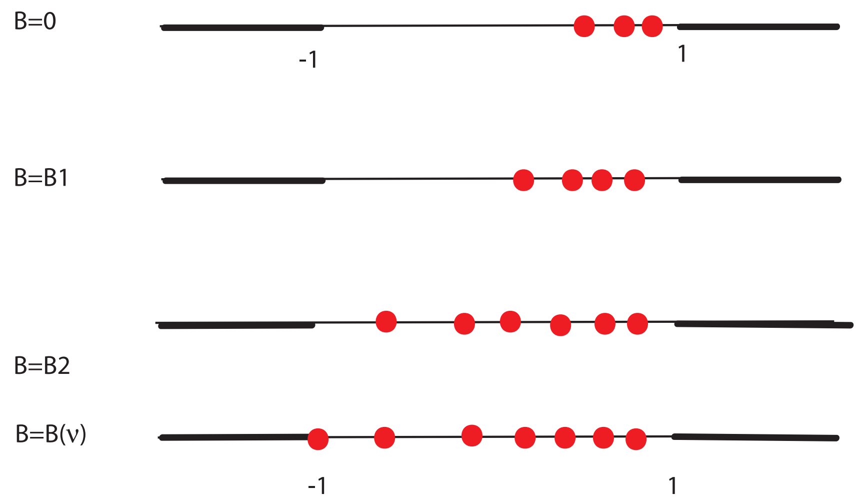

DEFINITION: $B(\nu) := \inf \left\{ B > 0 \ : \ \liminf_{b \nearrow B} \lambda_1(B,\nu) = -1 \right\}$.

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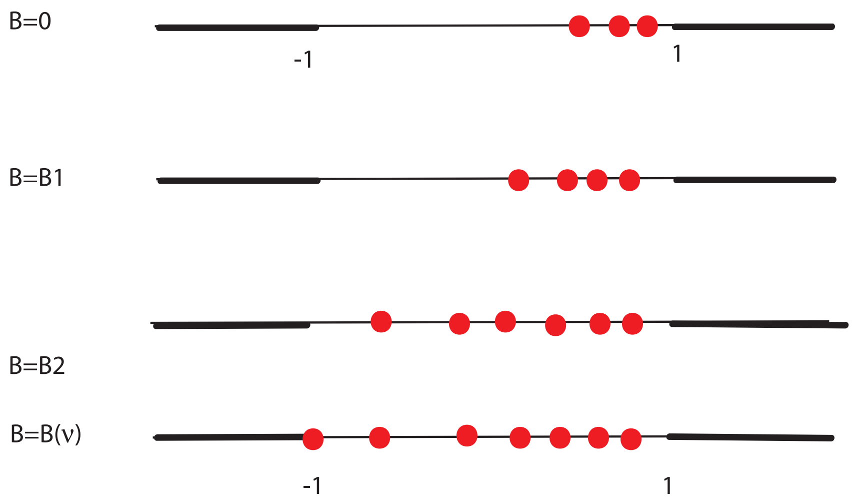

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TOOL TO ESTIMATE IT :

$$\int_{\mathbb{R}^3} \frac{|(\sigma\cdot\nabla_B)\varphi|^2}{1+\lambda_1(B,V)-V}\,dx + (1-\lambda_1(B,V))\int_{\mathbb{R}^3}|\varphi|^2\,dx \ \geq\ \int_{\mathbb{R}^3} V|\varphi|^2\,dx$$

First results to estimate $\lambda_1(B, V)$

For a given $\overline{C}$, we have $\lambda_1(B,V) < \overline{C}$ if we can find a function $\bar{\varphi}$ s. t.

$$(*) \qquad \int_{\mathbb{R}^3} \frac{|(\sigma \cdot \nabla_B)\bar{\varphi}|^2}{1 + \overline{C} - V}\, dx + (1 - \overline{C}) \int_{\mathbb{R}^3} |\bar{\varphi}|^2\, dx + \int_{\mathbb{R}^3} V |\bar{\varphi}|^2\, dx < 0$$

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And to verify that $\lambda_1(B,V) > \underline{C}$ it is enough to prove that for all φ

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and of course we are intersted in the case $\overline{C} = 1$ in (*), and $\underline{C} = -1$ in (**).

Some results (with J. Dolbeault and M. Loss)

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- $\lim_{\nu\to 1} B(\nu) > 0\,,$ $\qquad \lim_{\nu\to 0} \nu \log B(\nu) = \pi$

Some results (with J. Dolbeault and M. Loss)

$$V_\nu = -\frac{\nu}{|x|}, \quad \mathbf{A}_B(x) := \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \mathbf{B}(x) := \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \; ; \; \nu \in (0,1), \;\; B \geq 0$$

- For $B = 0$, $\lambda_1(0, V_\nu) = \sqrt{1-\nu^2} \in (-1, 1)$
- For all $B \geq 0$, $\lambda_1(B, V_\nu) < 1$
- For all $\nu \in (0,1)$ there exists a critical field strength $B(\nu)$ such that

$$\lambda_1(B, V_\nu) \leq -1 \quad \text{if} \quad B \geq B(\nu)$$

- $\lim_{\nu \to 1} B(\nu) > 0\,,$ $\qquad \displaystyle\lim_{\nu \to 0} \nu \log B(\nu) = \pi$
- For ν small, the asymptotics of $B(\nu)$ can be calculated by an approximation in the first relativistic "Landau level".

A better method to determine $B(\nu)$

$$\int_{\mathbb{R}^3} \frac{|(\sigma \cdot \nabla_B)\varphi|^2}{1 + \lambda_1(B, V_\nu) + \frac{\nu}{|x|}}\, dx + (1 - \lambda_1(B, V_\nu)) \int_{\mathbb{R}^3} |\varphi|^2\, dx \geq \int_{\mathbb{R}^3} \frac{\nu}{|x|}\, |\varphi|^2\, dx$$

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and we are looking for $B_n \longrightarrow B(\nu)$ such that $\lambda_1(B_n, \nu) \longrightarrow -1$.

If everything were compact, we would be able to pass to the limit and obtain

$$\int_{\mathbb{R}^3} \frac{|x|\,|(\sigma \cdot \nabla_B)\varphi|^2}{\nu} - \int_{\mathbb{R}^3} \frac{\nu}{|x|}\, |\varphi|^2\, dx + 2 \int_{\mathbb{R}^3} |\varphi|^2\, dx \geq 0$$

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Now define the functional

$$\mathcal{E}_{B,\nu}[\phi] := \int_{\mathbb{R}^3} \frac{|x|}{\nu}\, |(\sigma \cdot \nabla_B)\, \phi|^2\, dx - \int_{\mathbb{R}^3} \frac{\nu}{|x|}\, |\phi|^2\, dx\ ,$$

If everything were compact and “nice",

$$\mu_{B,\nu} + 2 = 0\ ; \qquad \mu_{B,\nu} := \inf \left\{ \mathcal{E}_{B,\nu}[\phi]\ ;\quad \int_{\mathbb{R}^3} |\varphi|^2\, dx = 1 \right\}.$$

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A very nice property is that the scaling $\phi_B := B^{3/4}\,\phi\left(B^{1/2}\,x\right)$ preserves the L^2 norm and $\mathcal{E}_{B,\nu}[\phi_B] = \sqrt{B}\,\mathcal{E}_{1,\nu}[\phi]\;.$

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Now we would like to estimate $B(\nu)$. This can be done analytically or/and numerically.

As we said before, analytically we have some estimates for ν close to 0 and to 1.

The Landau level approximation

Consider the class of functions $\mathcal{A}(B,\nu)$:

$$\phi_\ell := \frac{B}{\sqrt{2\,\pi\,2^\ell\,\ell!}}\,(x_2 + i\,x_1)^\ell\,e^{-B\,s^2/4}\binom{1}{0}, \quad \ell \in \mathbb{N}, \quad s^2 = x_1^2 + x_2^2\,,$$

where the coefficients depend only on x_3, *i.e.,*

$$\phi(x) = \sum_\ell f_\ell(x_3)\,\phi_\ell(x_1, x_2)\,, \qquad z := x_3\,.$$

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Now, we shall restrict the functional $\mathcal{E}_{B,\nu}$ to the first Landau level. In this framework, that we shall call the Landau level ansatz, we also define a critical field by

$$B_{\mathcal{L}}(\nu) := \inf\left\{B > 0 \;:\; \liminf_{b \nearrow B} \lambda_1^{\mathcal{L}}(b,\nu) = -1\right\}\,,$$

where

$$\lambda_1^{\mathcal{L}}(b,\nu) := \inf_{\phi \in \mathcal{A}(B,\nu)\,,\;\Pi^\perp \phi = 0} \lambda[\phi,b,\nu]\,.$$

THEOREM : $B^{\mathcal{L}}(\nu) = \frac{4}{\mu^{\mathcal{L}}(\nu)^2}$, where

$$\mu^{\mathcal{L}}(\nu) := \inf_f \quad \frac{\mathcal{L}_\nu[f]}{\|f\|^2_{L^2(\mathbb{R}^+)}} < 0 \, .$$

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COROLLARY. $\mu(\nu) \le \mu^{\mathcal{L}}(\nu) < 0 \quad \Longrightarrow \quad B(\nu) \le B^{\mathcal{L}}(\nu)\,.$

Final results

THEOREM. For $\nu \in (0, \nu_0)$, $$B^{\mathcal{L}}(\nu + \nu^{3/2}) \leq B(\nu) \leq B^{\mathcal{L}}(\nu)$$

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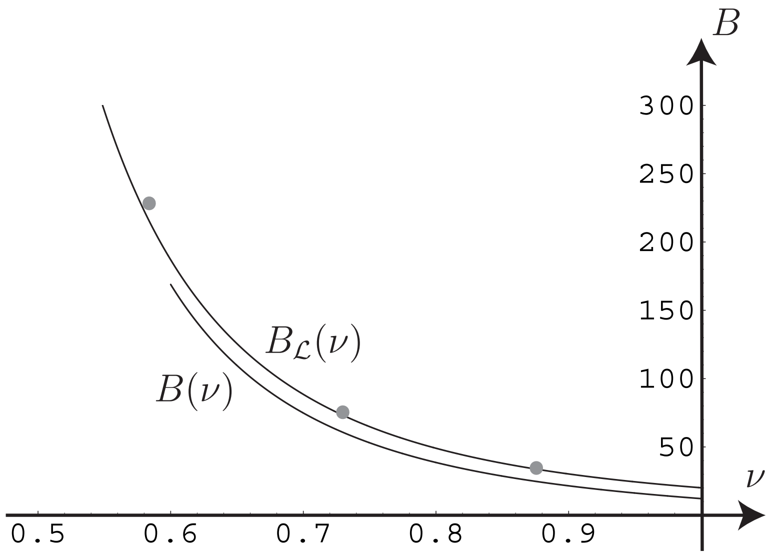

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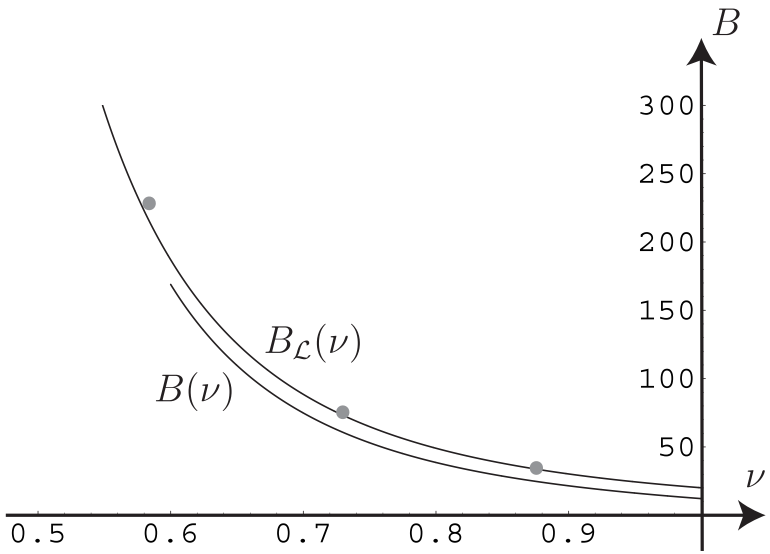

NUMERICAL OBSERVATION. For ν near 1, $B(\nu)$ is below $B^{\mathcal{L}}(\nu)$ by 30%.

Dear John :

even if late ,

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even if late ,

BON ANNIVERSAIRE !

HAPPY BIRTHDAY !