Cloaking by Change of Variables

Robert V. Kohn
Courant Institute, NYU

Edinburgh, June 2008

Points of contact with John

Some themes

- blowup for nonlinear heat eqns
- nonlinear elasticity
- quasiconvexity (existence)
- microstructure (nonexistence)
- martensitic phase transformation

Some memories

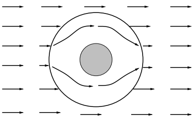
- MSRI, spring 1983?
- Heriot-Watt, summer 1986?
- Heriot-Watt, EU course with Nick Schryvers, 1990?

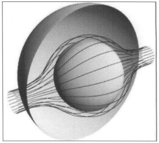
Some principles

- Our materials colleagues have interesting insights.
- Getting it right can provide additional insights, and can raise new mathematical issues.

Today's topic – cloaking – involves no elasticity, and no microstructure. But it definitely involves getting it right.

What is cloaking?





cloaked region can have any shape

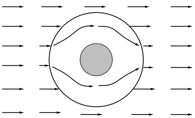
airplane?

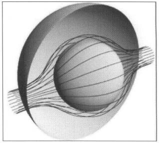
- the cloaked region should be invisible
- even the cloak itself should be invisible
- our cloaks will be coatings with heterogeneous, anisotropic dielectric properties

In what sense invisible?

- Most realistic: electromagnetic scattering, using pulses
- Easier to analyze: scattering at fixed frequency
- Easier still: frequency zero (impedance tomography).

What is cloaking?





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In the news

- 2003 Greenleaf, Lassas, Uhlmann, *Math Res Lett*. A region can be invisible. No press.
- 2006 Pendry, Schurig, Smith, *Science*. A region can be cloaked. MSNBC, BBC, and more.

- sounds like science fiction
- great press office
- Harry Potter

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Here's how to make an invisibility cloak

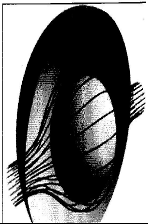
Theoretical cloaking device could soon become reality (sort of)

By Alan Boyle
Science editor
MSNBC
updated 2:03 p.m. ET, Thurs., May. 25, 2006

Alan Boyle
Science editor

- Profile
- E-mail

MSNBC Researchers say they are rapidly closing in on new types of materials that can throw a cloak of invisibility around objects, fulfilling a fantasy that is as old as ancient myths and as young as "Star Trek" and the Harry Potter novels.



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Plan for cloaking device unveiled

Researchers in the US and Britain have unveiled their blueprints for building a cloaking device.

So far, cloaking has been confined to science fiction; in Star Trek it is used to render spacecraft invisible.

Professor Sir John Pendry says a simple demonstration model that could work for radar might be possible within 18 months' time.

Outline

(1) Cloaking by change of variables at frequency zero

- Electric impedance tomography
- Cloaking by change of variables

(2) Cloaking by change of variables at finite frequency

- Finite frequency is similar, but different
- Recent results, for 2D and 3D Helmholtz

Collaborators: Onofrei, Shen, Vogelius, Weinstein
Inv Prob 2008, & in progress.
Explication of: Greenleaf, Lassas, Uhlmann, *Math Res Lett* 2003
Pendry, Schurig, Smith, *Science* 2006
Greenleaf, Kurylev, Lassas, Uhlmann, *CMP* 2008

Change-of-variable scheme is just one approach to cloaking. Others (worth separate talks) include:

- anomalous localized resonance (Milton, Nicorovici)
- optical conformal mapping (Leonhardt)

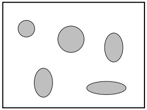
Impedance tomography

Impedance tomography uses electrostatics rather than scattering.

Sensing mechanism: currents and voltages at the *boundary*.

Goal: find conductivity in the *interior*.

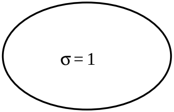
$$\sum \frac{\partial}{\partial x_i}\left(\sigma_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega,$$



Boundary measurements give us the Dirichlet-to-Neumann map

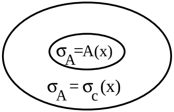
$$\Lambda_\sigma \;:\; u|_{\partial\Omega} \to (\sigma\nabla u)\cdot\nu|_{\partial\Omega}$$

Cloaking in this setting: $\sigma_c(x)$, defined on $\Omega \setminus D$, cloaks D if resulting bdry measurments "look uniform," regardless of content of D.



voltage f implies current flux g

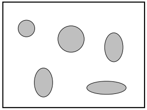
$$\sigma_A(x) = \begin{cases} \sigma_c(x) & \text{for } x \in \Omega \setminus D \\ A(x) & \text{for } x \in D \end{cases}$$



voltage f implies same current flux g

Getting used to the definitions

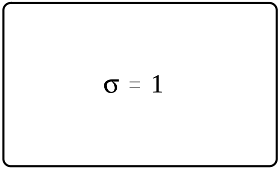
Impedance tomography seeks knowledge of interior conductivity, given voltage distrn at bdry assoc any applied current.



The measurements amount to Cauchy data for $\nabla \cdot (\sigma \nabla u) = 0$.

We say σ_c (defined in $\Omega \setminus D$) cloaks D if the Cauchy data at $\partial\Omega$ are (a) indep of content of D, and (b) same as the uniform case $\sigma = 1$.

Name is apt, since extn of σ_c by 1 to larger domain is also a cloak.



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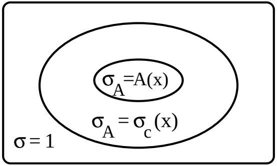


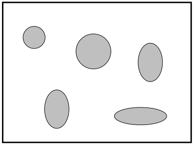
voltage f implies same current flux g

Is impedance tomography possible?

Do bdry meas determine the interior conductivity?

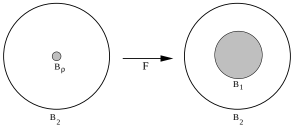
Ignoring "technicalities,"

- yes, if $\sigma(x)$ is known to be scalar-valued;
  Druskin, Kohn-Vogelius, Sylvester-Uhlmann, others
- no, if $\sigma(x)$ is allowed to be matrix-valued;
  Tartar



Latter is elementary: σ is at most determined "up to change of vars."

Basic observation: boundary measurements determine σ at most "up to change of variables."



If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial\Omega$ then σ and $F_*\sigma$ produce the same boundary measurements, where

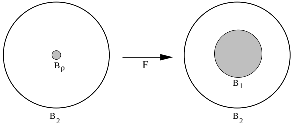
$$F_*\sigma(y) = \frac{1}{\det(DF)(x)} DF(x)\sigma(x)(DF(x))^T \quad \text{with } y = F(x).$$

Sketch:

- Change of vars: $\int_\Omega \langle \sigma(x)\nabla_x u, \nabla_x u\rangle \, dx = \int_\Omega \langle F_*\sigma(y)\nabla_y u, \nabla_y u\rangle \, dy$
- Variational principle: $\int_{\partial\Omega} f\Lambda_\sigma f = \min_{u=f \text{ at } \partial\Omega} \int_\Omega \langle \sigma(x)\nabla_x u, \nabla_x u\rangle \, dx$.
- Polarization: lin map Λ_σ determined by quadr form $\int_{\partial\Omega} f\Lambda_\sigma f$.

Invariance under change of variables

Basic observation: boundary measurements determine σ at most "up to change of variables."



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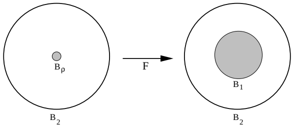

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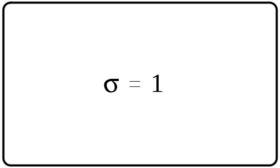
In 2D we know more

In 2D this is the only invariance Sylvester, Astala-Paivarinta-Lassas.
Bdry meas determine $\sigma_{ij}(x)$ up to change of vars. Sketch:

- There's a unique F such that $F_*\sigma$ is isotropic.
- An isotropic conductivity is determined by bdry meas.

Main "technicality:" σ should be uniformly elliptic and bounded.

Does this mean cloaking is impossible?



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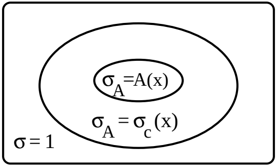

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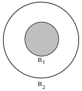
No: technicalities can be important.

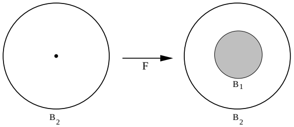
The change-of-variable-based cloak

Radial version, for simplicity only:
domain is B_2, cloaked region is B_1.



Choose conductivity of the cloak to be $\sigma_c = F_* 1$, where F “blows up” the origin to B_1:

$$F(x) = \left(1 + \tfrac{1}{2}|x|\right) \frac{x}{|x|}$$



Formally B_1 is cloaked. In fact, if

$$\sigma_A(y) = \begin{cases} \sigma_c(y) & \text{for } y \in B_2 \setminus B_1 \\ A(y) & \text{for } y \in B_1 \end{cases}$$

we have, using F^{-1} as our change of variable,

$$\int_{B_2} \langle \sigma_A(y) \nabla_y u, \nabla_y u \rangle \, dy = \int_{B_2} |\nabla_x u|^2 \, dx$$

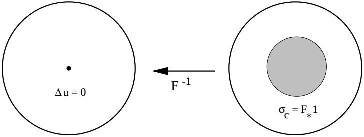
since F^{-1} shrinks the region being cloaked to a point.

Is this correct? (F and F^{-1} are very singular.)

Analysis of the singular cloak

Argument is correct at frequency 0, using removability of singularities for Laplace's eqn.

$\sigma_c(y) = F_* 1$ on $B_2 \setminus B_1$, where F takes
$x \in B_2 \setminus \{0\}$ to $y = F(x) \in B_2 \setminus B_1$.
notation: potential is $v(y) = u(x)$



Outside B_1:

- By change of vars, $\nabla \cdot (\sigma_c(y) \nabla_y v) = 0$ in $B_2 \setminus B_1$ iff $\Delta_x u = 0$ in $B_2 \setminus \{0\}$, where $u(x) = v(y)$.
- Potential uniformly bounded $\Longrightarrow$ sing'y of u is removable. Thus $v(y) = u(x)$ where $\Delta u = 0$ in B_2.
- Bdry meas are invariant under change of vars. So $\Lambda_\sigma = \Lambda_1$.

Inside B_1:

- Potential is const at ∂B_1, since $v(y) \to u(0)$ as $|y| \to 1$.
- So potential is constant throughout B_1, since $\nabla \cdot (\sigma \nabla_y v) = 0$.

B_1 is cloaked, because $\nabla v = 0$ there, regardless of bdry data.

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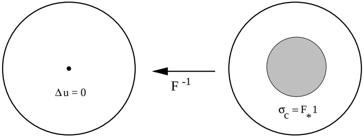

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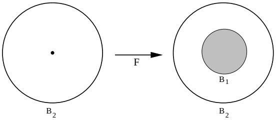
- Potential is const at ∂B_1, since $v(y) \to u(0)$ as $|y| \to 1$.
- So potential is constant throughout B_1, since $\nabla \cdot (\sigma \nabla_y v) = 0$.

B_1 is cloaked, because $\nabla v = 0$ there, regardless of bdry data.

Remarks on the singular cloak

- This scheme requires exotic materials. Recall that

  $$\sigma_c(y) = F_* 1 \quad \text{at } y = F(x)$$

  where F blows up a point to the region being cloaked. In 2D, σ_c is both singular and degenerate; in 3D and higher it is just degenerate: as $|y| \downarrow 1$, $\sigma_c(y)$ has

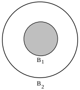
  - radial eigenvector with eigenvalue $\sim (|y| - 1)^{n-1}$
  - tangential eigenspace with eigenvalue $\sim (|y| - 1)^{n-3}$.

- The PDE holds weakly, i.e. $\nabla \cdot (\sigma \nabla u) = 0$ even across $|y| = 1$, since $(\sigma_c \nabla u) \cdot \nu = 0$ at ∂B_1

- The singular cloak makes me uncomfortable. We usually deal with singularities by smoothing them. Why not here?

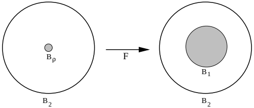
A regular near-cloak

Same idea, with more regular F.
Domain still B_2, cloaked region still B_1.



Near-cloak uses $\sigma_c = F_* 1$, where $F = F_\rho$ is less singular:

- F is cont's and piecewise smooth
- it expands B_ρ to B_1 while preserving B_2
- $F(x) = x$ at the outer bdry $|x| = 2$.



Impact of contents of B_1 on bndry data becomes, via change of vars, effect of small inclusion with uncontrolled properties. In fact, if

$$\sigma_A(y) = \begin{cases} \sigma_c(y) & \text{for } y \in B_2 \setminus B_1 \\ A(y) & \text{for } y \in B_1 \end{cases}$$

then, using F^{-1} as change of variable,

$$\int_{B_2} \langle \sigma_A(y) \nabla_y u, \nabla_y u \rangle \, dy = \int_{B_2 \setminus B_\rho} |\nabla_x u|^2 \, dx + \int_{B_\rho} \langle F_*^{-1}(A) \nabla_x u, \nabla_x u \rangle \, dx.$$

Claim: effect of a small inclusion is small, regardless of its contents.

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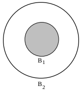

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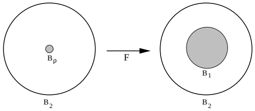

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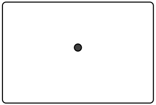
Claim: effect of a small inclusion is small, regardless of its contents.

A small inclusion is almost invisible

> Theorem: *If* $\sigma \equiv 1$ *outside* B_ρ*, then*
> $$\|\Lambda_\sigma - \Lambda_1\| \leq C\rho^n \quad \textit{in space dim } n.$$



Sketch:

- Use operator norm, $\Lambda_\sigma \; : \; H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, where
$$\|f\|_{H^{1/2}(\partial\Omega)} = \min_{u=f \text{ at } \partial\Omega} \int_\Omega |\nabla u|^2.$$
Natural choice, since finite-energy solutions of $\nabla \cdot (\sigma \nabla u) = 0$ have Dirichlet data in $H^{1/2}$ and Neumann data in $H^{-1/2}$.
- Estimate is well-known when inclusion has constant conductivity – even for the extreme cases, when $\sigma = 0$ or $\sigma = \infty$ in B_ρ.
- Variational principle says effect of any inclusion is bracketed by effect of extreme inclusions:
$$\Lambda_{\text{incl cond} = 0} \leq \Lambda_\sigma \leq \Lambda_{\text{incl cond} = \infty}$$

So: our "regularized near-cloak" almost cloaks B_1, if ρ is small.

Consistency of cloaking with uniqueness theorem

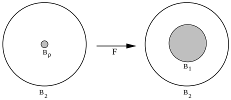
Recall paradox: in 2D (at least), bdry measurements determine $\sigma(x)$ up to change of variables. Doesn't this prohibit cloaking?

Recall argument: for σ uniformly elliptic and bounded,

(1) there's a unique G such that $G_*\sigma$ is isotropic;
(2) an isotropic conductivity is determined by bdry meas.

Resolution: Theorem applies to our near-cloak (using $F = F_\rho$).

- In fact $G = F^{-1}$.
- But as $\rho \to 0$, G loses invertibility.

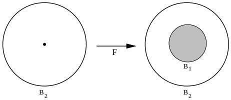


But thm doesn't apply when $\rho = 0$, since σ_c is degenerate and/or singular near ∂B_1.

Transition to finite frequency

Cloaking by change-of-variables. Simple idea:

- Suppose $F : \Omega \to \Omega$ "blows up" a point to a region D, with $F(x) = x$ at $\partial\Omega$;
- then $\sigma_c = F_* 1$ cloaks D.



Correct for electrostatics. Essence of analysis via regularized near-cloak: a small inclusion can have little effect on bdry meas (regardless of its contents).

What about finite frequency? Brief summary:

- Change-of-variable scheme works formally at any frequency.
- But finite frequency is different. Small inclusions are not necessarily negligible.
- Near-cloaking still possible (sort of).

Time-harmonic Maxwell

$$\nabla \times H = (\sigma - i\omega\varepsilon)E, \quad \nabla \times E = i\omega\mu H$$

$\sigma =$ conductivity, $\varepsilon =$ dielectric permittivity, $\mu =$ magnetic permeability

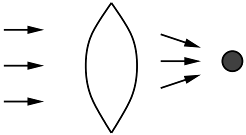
Physical fields $\Re(Ee^{-i\omega t})$ and $\Re(He^{-i\omega t})$; recover electrostatics if $\omega = 0$.

Admittance is analogue of Λ_σ. It takes $E_{\tan}|_{\partial\Omega}$ to $H_{\tan}|_{\partial\Omega}$. Same admittance $\Leftrightarrow$ indistinguishable by EM measurements.

Invariance by change of vars extends: If $F : \Omega \to \Omega$ has $F(x) = x$ at $\partial\Omega$ then $(\sigma, \varepsilon, \mu)$ and $(F_*\sigma, F_*\varepsilon, F_*\mu)$ have same admittance.

So change-of-vars-based scheme works formally at any frequency. But can this be right? Danger sign:

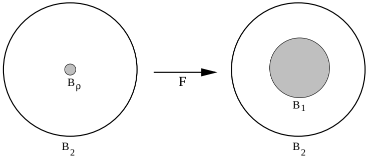
- Small inclusions can be important, at least in geometrical optics limit



Regularized near-cloak at finite frequency

Recall the construction:

- Suppose $F_\rho : \Omega \to \Omega$ "blows up" B_ρ to D, with $F_\rho(x) = x$ at $\partial\Omega$.
- Fill $\Omega \setminus D$ by $(F_\rho)_*$ of uniform space. Is D nearly cloaked?
- Changing vars by F_ρ^{-1}, effect of D on boundary measurements becomes effect of small inclusion with uncontrolled contents.



So the essential task is to identify sense in which a small inclusion with uncontrolled content has small effect on bdry meas.

Our idea (Kohn-Onofrei-Vogelius-Weinstein): include a lossy layer at the inner edge of the cloak.

2D scalar version

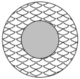
Consider 2D Helmholtz-type eqn:

$$\nabla \cdot (\gamma \nabla u) + \omega^2 q u = 0 \quad \text{in } \Omega \subset R^2$$

Reduction of Maxwell, for TE waves: $E = (0, 0, u)$, $\gamma = \mu^{-1}$, $q = \varepsilon + i\frac{\sigma}{\omega}$. Uniform space is $\gamma = 1, q = 1$. Lossless if γ, q are real.

Consider small inclusion, with arbitrary core coated by lossy shell:

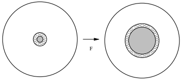
$$\begin{cases} \gamma = 1, q = 1 + i\rho^{-2} & \text{for } \rho < |x| < 2\rho \\ \text{any real, pos values} & \text{for } |x| < \rho. \end{cases}$$



Theorem. *When embedded in a uniform medium ($\gamma = 1$, $q = 1$), such inclusions have little effect on bndry meas:*

$$\|\Lambda_{\gamma,q} - \Lambda_{1,1}\| \le C_\omega / |\log \rho|.$$

Pushing forward by F_ρ, we get a finite-frequency near-cloak of B_1. But note slow decay of $|\log \rho|^{-1}$.



Why is finite frequency different?

In singular (perfect-cloaking) limit we expect weak solution of PDE:

$$\nabla \cdot (\sigma \nabla u) = 0 \qquad \text{vs} \qquad \nabla \cdot (\gamma \nabla u) + \omega^2 q u = 0.$$

For impedance tomography,

$$\text{perfect cloaking} \Leftrightarrow \nabla u = 0 \text{ in cloaked region}$$

At finite frequency,

$$\text{perfect cloaking} \Leftrightarrow u = 0 \text{ in cloaked region}$$

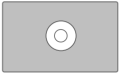
Remark: A different treatment of the finite-frequency case was given by Greenleaf, Kurylev, Lassas, Uhlmann (CMP, 2008):

- focus is on perfect (singular) cloaks, not approx (regular) cloaks
- they change bc at edge of cloak, rather than lossy layer
- they cloak even active sources, using a “double-coating” constrn

Key steps for 2D Helmholtz

I. Compare Helmholtz in shell $\Omega \setminus B_{2\rho}$ to Helmholtz in Ω. Consider

$$\begin{aligned} \Delta u_0 + \omega^2 u_0 &= 0 \text{ in } \Omega \\ \Delta u_\rho + \omega^2 u_\rho &= 0 \text{ in } \Omega \setminus B_{2\rho} \end{aligned}$$



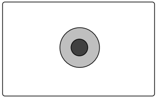
with same Neumann data ψ at $\partial\Omega$, and Dir data ϕ for u_ρ at $\partial B_{2\rho}$. Then

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le \frac{C}{|\log \rho|} \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|\phi(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \right)$$

II. Control u_ρ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (\gamma \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

$$\begin{cases} \gamma = 1, q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ \gamma = 1, q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\ \text{any real, pos values} & \text{for } |x| < \rho. \end{cases}$$



using Neumann data ψ at $\partial\Omega$. Then

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \le \frac{C}{\omega}(1 + \omega^2(1+\beta)\rho^2)\frac{1}{\rho\sqrt{\beta}} \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)} \right)$$

3D is better

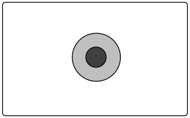
For 2D Helmholtz, cloaking error was $C/|\log \rho|$.
Linked to fund soln of Laplacian.

For 3D Helmholtz, obvious guess is $C\rho$.
But our method gives only $C\sqrt{\rho}$: for

$$\nabla \cdot (\gamma \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \subset R^3$$

with

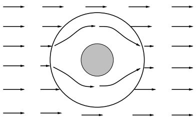
$$\begin{cases} \gamma = 1, q = 1 & \text{in } \Omega \setminus B_{2\rho} \\ \gamma = 1, q = 1 + i\rho^{-2} & \text{in } B_{2\rho} \setminus B_\rho \\ \text{arbitrary real, positive} & \text{in } B_\rho. \end{cases}$$

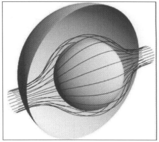


we have

$$\|\Lambda_{\gamma,q} - \Lambda_{1,1}\| \leq C_\omega \sqrt{\rho}.$$

Stepping back

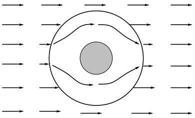


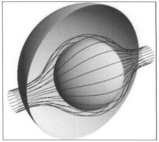


Is cloaking possible using the change-of-variable-based scheme?

- Yes, at frequency zero (impedance tomography).
- Yes, in geometrical optics limit (Pendry-Schurig-Smith)
- Somewhat, at finite frequency (there's much more to do).
- Radial geometry was not important. Method works for any region. Cloak can be as thin as desired.

Stepping back





Is cloaking practical using the change-of-variable-based scheme?

- Major problem in 2D: slow decay of $1/|\log \rho|$. (Comes from fund'l soln. So 3D is better.)
- Another problem: scheme requires highly anisotropic dielectrics, degenerate and/or singular at edge of cloak. Perhaps achievable at particular frequencies using metamaterials.
- Construction is not frequency dependent. But dielectric properties of materials *are* frequency-dependent. So cloaking from pulses is not so easy.

Conclusions

VIEWPOINT MATTERS

- Greenleaf-Lassas-Uhlmann ’03: a region can be invisible
- Pendry-Schurig-Smith ’06: a region can be made invisible

FREQUENCY MATTERS

- Change-of-variable-based scheme works well at frequency 0, but not nearly as well at finite frequency

HAPPY BIRTHDAY, JOHN!