Perturbative Symmetry Approach

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Symmetry Approach - basic definitions and facts

Suppose we have an evolutionary partial differential equation

$$u_t = F(u_n, ..., u_1, u)\,, \quad n \geq 2 \tag{1}$$

where

$$u = u(x,t),\ u_1 = u_x(x,t),\ u_2 = u_{xx}(x,t), ...\ .$$

In the Symmetry Approach it is assumed that all functions such as F depend on a finite number of variables and belong to a proper differential field $\mathcal{F}(u, D)$ generated by u and the derivation D:

$$D = u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \cdots\ ,$$

which represents the derivation in x.

Partial differential equation $u_t = F$ defines another derivation of the field $\mathcal{F}(u,D)$

$$D_t = F\frac{\partial}{\partial u_0} + F_1\frac{\partial}{\partial u_1} + F_2\frac{\partial}{\partial u_2} + \cdots$$

$$F_k = D^k(F) \in \mathcal{F}(u,D)$$

commuting with D.

A symmetry of equation $u_t = F$ can be defined as a derivation D_τ

$$D_\tau = G\frac{\partial}{\partial u_0} + G_1\frac{\partial}{\partial u_1} + G_2\frac{\partial}{\partial u_2} + \cdots$$

of the field $\mathcal{F}(u,D)$, where $G_k = D^k(G) \in \mathcal{F}(u,D)$, which commutes with derivations D and D_t.

This definition is equivalent to the following definition, which we will be using in what follows

Definition 1. *Function $G \in \mathcal{F}(u, D)$ generates a symmetry of the equation (1) if the differential equation*

$$D_\tau(u) = G$$

is compatible with (1).

The Frechét derivative of $a \in \mathcal{F}(u, D)$ is defined as a linear differential operator of the form

$$a_* = \sum_k \frac{\partial a}{\partial u_k} D^k \, .$$

Using the Frechét derivative one can express the derivation with respect to t as

$$D_t(a) = a_*(F), \quad a \in \mathcal{F}(u, D)$$

The Lie brackets for any two elements
$a, b \in \mathcal{F}(u, D)$ is defined as

$$[a, b] = a_*(b) - b_*(a)\,.$$

In these terms the definition of symmetry of equation (1) can be formulated as follows: function $G \in \mathcal{F}(u, D)$ generates a symmetry of equation (1) if

$$[F, G] = 0.$$

The order of the symmetry

$$\mathrm{ord}(G) = \deg(G_*)$$

Formal pseudo-differential series, which for simplicity we shall call formal series, are defined as

$$A = a_m D^m + a_{m-1} D^{m-1} + \cdots + a_0 + a_{-1} D^{-1} + \cdots .$$

The product of two formal series is defined by

$$aD^k \circ bD^m = a(bD^{m+k} + C_k^1 D(b) D^{k+m-1} + \cdots),$$

where $k, m \in \mathbb{Z}$ and the binomial coefficients are defined as

$$C_n^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}.$$

Definition 2. *The formal series*

$$\Lambda = l_m D^m + l_{m-1} D^{m-1} + \cdots + l_0 + l_{-1} D^{-1} + \cdots ,$$

where $l_k \in \mathcal{F}(u, D)$*, is called a formal recursion operator for equation (1) if*

$$D_t(\Lambda) = F_* \circ \Lambda - \Lambda \circ F_* .$$

The central result of the Symmetry Approach can be represented by the following Theorem:

Theorem 1. *If equation (1) possess an infinite hierarchy of symmetries of arbitrary high order, then there exists a formal recursion operator.*

The ring of differential polynomials $\mathcal{R}$.

The sequence of *dynamical variables* $\{u_0, u_1, u_2, \ldots\}$

$$u_0 = u, \quad u_n = \partial_x^n u, \quad n \in \mathbb{Z}_{\geq 0}.$$

We denote by $\mathcal{R}$ the ring of polynomials over $\mathbb{C}$ of infinite number of dynamical variables.

<u>Natural gradation:</u>

$$X = \sum_{k \geq 0} u_k \frac{\partial}{\partial u_k}$$

$$\mathcal{R} = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{R}^n, \quad \mathcal{R}^n \cdot \mathcal{R}^m \subset \mathcal{R}^{n+m}, \quad \mathcal{R}^n = \{f \in \mathcal{R} \,|\, Xf = nf\}.$$

Weighted gradation: Let μ be a positive rational number, which we call the weight of u and denote $W(u) = \mu$. We define the weights of the dynamical variables as

$$W(u_i) = \mu + i.$$

The weight of a monomial is defined as the sum of the weights of dynamical variables which contribute to the monomial including their multiplicities. We say that a polynomial $f \in \mathcal{R}$ is a homogeneous polynomial of weight λ $W(f) = \lambda$ if every its monomial is of the weight λ.

"Little oh":

$$f = o(\mathcal{R}^n) \text{ iff } f \in \bigoplus_{k>n} \mathcal{R}^k$$

Symbolic representation $\hat{\mathcal{R}}$ of differential ring $\mathcal{R}$.

The symbolic representation is a simplified form of notations and rules for formal Fourier images of dynamical variables u_n, differential polynomials and formal series with coefficients from the ring $\mathcal{R} \oplus \mathbb{C}$.

Let $\hat{u}(\kappa, t)$ denotes a Fourier image of $u(x, t)$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\kappa, t) \exp(i\kappa x)\, d\kappa\,.$$

Then we have the following correspondences:

$$u_0 \to \ \hat{u},\ \ u_1 \to i\kappa\hat{u},\ \ \ldots\ \ u_m \to (i\kappa)^m \hat{u}.$$

The Fourier image of a monomial $u_n u_m$ can obviously be represented as

$$u_n u_m = \iiint \delta(\kappa_1 + \kappa_2 - \kappa) \frac{[(i\kappa_1)^n (i\kappa_2)^m + (i\kappa_2)^n (i\kappa_1)^m]}{2}$$

$$\widehat{u}(\kappa_1, t)\widehat{u}(\kappa_2, t)\, \mathrm{exp}(i\kappa x)\, d\kappa_1\, d\kappa_2\, d\kappa\,,$$

Therefore

$$u_n u_m \rightarrow$$

$$\iint \delta(\kappa_1 + \kappa_2 - \kappa) \frac{[(i\kappa_1)^n (i\kappa_2)^m + (i\kappa_2)^n (i\kappa_1)^m]}{2} \widehat{u}(\kappa_1, t)\widehat{u}(\kappa_2, t)\, d\kappa_1\, d\kappa_2\,.$$

We shall simplify notations further omitting the delta function, integrations, replacing $i\kappa_n$ by ξ_n and $\widehat{u}(\kappa_1, t)\widehat{u}(\kappa_2, t)$ by u^2. Thus we shall represent the monomial $u_n u_m$ by a symbol

$$u_n u_m \rightarrow u^2 \frac{[\xi_1^n \xi_2^m + \xi_2^n \xi_1^m]}{2}$$

Following this rule we shall represent any differential monomial $u_0^{n_0}u_1^{n_1}u_2^{n_2}\cdots$
of degree

$$m = n_0 + n_1 + \cdots + n_q$$

by the symbol

$$u_0^{n_0}u_1^{n_1}\cdots u_q^{n_q} \to u^m\langle \xi_1^0\cdots\xi_{n_0}^0\xi_{n_0+1}^1\cdots\xi_{n_0+n_1}^1\xi_{n_0+n_1+1}^2\cdots\xi_m^q\rangle$$

where $m = n_0 + n_1 + \cdots + n_q$ and the brackets $\langle\rangle$ mean the symmetrisation over the group of permutation of m elements (i.e. permutation of all arguments ξ_j)

$$\langle f(\xi_1, ..., \xi_m)\rangle = \frac{1}{m!}\sum_{\sigma\in\Sigma_m} f(\sigma(\xi_1), ..., \sigma(\xi_m))\,.$$

For example

$$u_n \to u\xi_1^n\,, \quad u_3^2 \to u^2\xi_1^3\xi_2^3, \quad u^3u_2 \to u^4\frac{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2}{4}\,.$$

The symbolic representation $\hat{\mathcal{R}}$ of the differential ring $\mathcal{R}$ can be defined as follows.

1. The sum of differential monomials is represented by the sum of the corresponding symbols.

2. To the multiplication of monomials f and g with symbols

$$f \to u^p a(\xi_1, ..., \xi_p)\,, \;\; g \to u^q b(\xi_1, ..., \xi_q)$$

corresponds the symbol

$$fg \to u^{p+q} \langle a(\xi_1, ..., \xi_p) b(\xi_{p+1}, ..., \xi_{p+q}) \rangle \,.$$

3. The derivative $D(f)$ of a monomial f with the symbol $u^p a(\xi_1, ..., \xi_p)$ is represented by

$$D(f) \to u^p (\xi_1 + \xi_2 + \cdots + \xi_p) a(\xi_1, ..., \xi_p)\,.$$

Symmetry Approach in symbolic representation

Consider an evolutionary equation

$$u_t = F(u_n, u_{n-1}, \ldots, u_1, u) \in \mathcal{R}$$

We can always represent F as

$$F = F_1[u] + F_2[u] + \ldots + F_s[u], \qquad F_i[u] \in \mathcal{R}^i,\ i = 1, \ldots, s$$

In the symbolic representation it can be written as

$$u_t = u\omega(\xi_1) + \frac{u^2}{2} a_2(\xi_1, \xi_2) + \cdots = u\omega(\xi_1) + \sum_{i=2}^{s} \frac{u^i}{i} a_i(\xi_1, \ldots, \xi_i) = \widehat{F}\,, \quad (2)$$

where $\omega(\xi_1), a_i(\xi_1, ..., \xi_i)$ are symmetrical polynomials. We will also assume that $\deg \omega(\xi_1) \geq 2$.

Symmetries of equation (2), if they exist, can be found recursively:

Proposition 1. *Expression*

$$u_\tau = u\Omega(\xi_1) + \sum_{j\geq 2} \frac{u^j}{j} A_j(\xi_1, ..., \xi_j) = G \tag{3}$$

is a symmetry of (2) if and only if functions $A_j(\xi_1, ..., \xi_j)$ *determined as follows are polynomials in* $\xi_1, \ldots, \xi_j$

$$A_2(\xi_1, \xi_2) = \frac{\Omega(\xi_1 + \xi_2) - \Omega(\xi_1) - \Omega(\xi_2)}{\omega(\xi_1 + \xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_2(\xi_1, \xi_2),$$

$$A_3(\xi_1, \xi_2, \xi_3) = \frac{\Omega(\xi_1 + \xi_2 + \xi_3) - \Omega(\xi_1) - \Omega(\xi_2) - \Omega(\xi_3)}{\omega(\xi_1 + \xi_2 + \xi_3) - \omega(\xi_1) - \omega(\xi_2) - \omega(\xi_3)} a_3(\xi_1, \xi_2, \xi_3) +$$

$$+ \frac{3}{2} \frac{\langle A_2(\xi_1, \xi_2 + \xi_3) a_2(\xi_2, \xi_3) - a_2(\xi_1, \xi_2 + \xi_3) A_2(\xi_2, \xi_3) \rangle}{\omega(\xi_1 + \xi_2 + \xi_3) - \omega(\xi_1) - \omega(\xi_2) - \omega(\xi_3)}$$

$$A_{m+1}(\xi_1,...,\xi_{m+1}) = \frac{G^{\Omega}(\xi_1,...,\xi_{m+1})}{G^{\omega}(\xi_1,...,\xi_{m+1})} a_{m+1}(\xi_1,...,\xi_{m+1}) +$$

$$G^{\omega}(\xi_1,...,\xi_{m+1})^{-1} \cdot \Big[$$

$$\langle \sum_{j=1}^{m-1} \frac{m+1}{m-j+1} A_{j+1}(\xi_1,...,\xi_j, \sum_{k=j+1}^{m+1} \xi_k) a_{m-j+1}(\xi_{j+1},...,\xi_{m+1}) -$$

$$- \sum_{j=1}^{m-1} \frac{m+1}{j+1} a_{m-j+1}(\xi_1,...,\xi_{m-j}, \sum_{k=m-j+1}^{m+1} \xi_k) \cdot A_{j+1}(\xi_{m-j+1},...,\xi_{m+1}) \rangle \Big]$$

where

$$G^{\omega}(\xi_1,...,\xi_m) = \omega(\sum_{n=1}^{m} \xi_n) - \sum_{n=1}^{m} \omega(\xi_n), G^{\Omega}(\xi_1,...,\xi_m) = \Omega(\sum_{n=1}^{m} \xi_n) - \sum_{n=1}^{m} \Omega(\xi_n)$$

Definition 3. *We will call $G \in \mathcal{R}$ an approximate symmetry of degree p if*

$$[G, F] = o(\mathcal{R}^p)$$

Theorem 2. *(Sanders–Wang) Consider two equations of the form*

$$(A) \quad u_t = u\xi_1^n + \sum_{i>1} \frac{u^i}{i} a_i(\xi_1, \dots, \xi_i)$$

and

$$(B) \quad u_t = u\xi_1^n + \sum_{i>1} \frac{u^i}{i} b_i(\xi_1, \dots, \xi_i)$$

Suppose that

- $\deg(a_i) < n$, $\deg(b_i) < n$,
- $a_2(\xi_1, \xi_2) \equiv b_2(\xi_1, \xi_2)$
- $a_3(\xi_1, \xi_2, \xi_3) = b_3(\xi_1, \xi_2, \xi_3)$

Then if equations (A) and (B) have symmetries of the form

$$u_\tau = u\xi_1^m + \sum_{i>1} \frac{u^i}{i} A_i(\xi_1, \ldots, \xi_i),$$

and

$$u_\tau = u\xi_1^m + \sum_{i>1} \frac{u^i}{i} B_i(\xi_1, \ldots, \xi_i),$$

then

$$a_i(\xi_1, \ldots, \xi_i) \equiv b_i(\xi_1, \ldots, \xi_i),\ A_i(\xi_1, \ldots, \xi_i) \equiv B_i(\xi_1, \ldots, \xi_i),$$

$$i = 4, 5, \ldots .$$

Proof From Proposition 2 we have

$$A_2(\xi_1,\xi_2) = \frac{(\xi_1+\xi_2)^m - \xi_1^m - \xi_2^m}{(\xi_1+\xi_2)^n - \xi_1^n - \xi_2^n} a_2(\xi_1,\xi_2),$$

$$B_2(\xi_1,\xi_2) = \frac{(\xi_1+\xi_2)^m - \xi_1^m - \xi_2^m}{(\xi_1+\xi_2)^n - \xi_1^n - \xi_2^n} b_2(\xi_1,\xi_2).$$

Therefore $A_2(\xi_1,\xi_2) \equiv B_2(\xi_1,\xi_2)$. For cubic terms we obtain

$$\begin{aligned} A_3(\xi_1,\xi_2,\xi_3) &= \frac{(\xi_1+\xi_2+\xi_3)^m - \xi_1^m - \xi_2^m - \xi_3^m}{(\xi_1+\xi_2+\xi_3)^n - \xi_1^n - \xi_2^n - \xi_3} a_3(\xi_1,\xi_2,\xi_3) + \\ &\quad + L^{(3)}(\xi_1,\xi_2,\xi_3) \end{aligned}$$

$$\begin{aligned} B_3(\xi_1,\xi_2,\xi_3) &= \frac{(\xi_1+\xi_2+\xi_3)^m - \xi_1^m - \xi_2^m - \xi_3^m}{(\xi_1+\xi_2+\xi_3)^n - \xi_1^n - \xi_2^n - \xi_3} b_3(\xi_1,\xi_2,\xi_3) + \\ &\quad + \tilde{L}^{(3)}(\xi_1,\xi_2,\xi_3) \end{aligned}$$

Terms $L^{(3)}(\xi_1,\xi_2,\xi_3)$ and $\tilde{L}^{(3)}(\xi_1,\xi_2,\xi_3)$ depend only on terms of degree 1 and 2 and therefore $\tilde{L}^{(3)}(\xi_1,\xi_2,\xi_3) = L^{(3)}(\xi_1,\xi_2,\xi_3)$

and hence $A_3(\xi_1,\xi_2,\xi_3) = B_3(\xi_1,\xi_2,\xi_3)$. Consider now 4th degree terms:

$$A_4(\xi_1,\dots,\xi_4) = \frac{(\xi_1+\dots+\xi_4)^m-\xi_1^m-\dots-\xi_4^m}{(\xi_1+\dots+\xi_4)^n-\xi_1^n-\dots-\xi_4}a_4(\xi_1,\dots,\xi_4)+ +L^{(4)}(\xi_1,\dots,\xi_4)$$

$$B_4(\xi_1,\dots,\xi_4) = \frac{(\xi_1+\dots+\xi_4)^m-\xi_1^m-\dots-\xi_4^m}{(\xi_1+\dots+\xi_4)^n-\xi_1^n-\dots-\xi_4}b_4(\xi_1,\dots,\xi_4)+ +\tilde{L}^{(4)}(\xi_1,\dots,\xi_4)$$

Terms $L^{(4)}(\xi_1,\xi_2,\xi_3,\xi_4)$ and $\tilde{L}^{(4)}(\xi_1,\xi_2,\xi_3,\xi_4)$ are equal. Therefore

$$A_4-B_4 = \frac{(\xi_1+\dots+\xi_4)^m-\xi_1^m-\dots-\xi_4^m}{(\xi_1+\dots+\xi_4)^n-\xi_1^n-\dots-\xi_4}(a_4-b_4)$$

Lemma 1. *(F. Beukers) Polynomials*

$$g_n^{(s)} = (\xi_1+\dots+\xi_s)^n-\xi_1^n-\dots-\xi_s^n$$

are irreducible over $\mathbf{C}$ *if* $s \geq 4$ *and* $n>1$.

Therefore $a_4=b_4, A_4=B_4$. Applying now the above arguments inductively we prove the theorem. $\diamond$

As an example let us consider equation

$$u_t = u_n + uu_1, \quad n = 2, 3, \ldots.$$

This is a homogeneous equation with $W(u) = n - 1$ and total weight $2n - 1$. In the symbolic representation it can be rewritten as

$$u_t = u\xi_1^n + \frac{u^2}{2}(\xi_1 + \xi_2).$$

Without loss of generality let us suppose that it possess a higher symmetry of the form

$$u_\tau = u\xi_1^m + \frac{u^2}{2}A_2(\xi_1, \xi_2) + \cdots .$$

Then for $A_2(\xi_1, \xi_2)$ we obtain

$$A_2(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^m - \xi_1^m - \xi_2^m}{(\xi_1 + \xi_2)^n - \xi_1^n - \xi_2^n}(\xi_1 + \xi_2)$$

Let us define $h_n(x,y) = (x+y)^n - x^n - y^n$.

Theorem 3. *(Lech-Mahler) Let* $c_1, c_2, \dots c_k$ *and* $C_1, C_2, \dots, C_k$ *be non-zero complex numbers. Suppose that none of the ratios* C_i/C_j, $i \neq j$ *is a root of unity. Then the equation*

$$c_1 C_1^n + c_2 C_2^n + \cdots + c_k C_k^n = 0$$

in the unknown integer n *has finitely many solutions.*

It is convenient to introduce an affine coordinate $q = x/y$. Applying the Lech-Mahler theorem to the equation

$$(1+q)^n - q^n - 1 = 0$$

we find that this equation possess infinitely many solutions in integer n if and only if

- $q = 0$, any n,
- $q = -1$, odd n,
- $1 + q + q^2 = 0$, $n = 5 \bmod 6$,
- $1 + q + q^2 = 0$ (double roots), $n = 1 \bmod 6$

Theorem 4. *(F. Beukers.) Polynomials* $h_n(x,y)$ *can be factorized as* $h_n(x,y) = t_n(x,y) g_n(x,y)$, *where* $(g_n(x,y), g_l(x,y)) = 1,\ l \neq n$ *and*

$$\begin{aligned} t_n(x,y) &= xy, \quad \forall n \\ &= xy(x+y), \quad n = 1 \ \ mod\ 2, \\ &= xy(x+y)(x^2+xy+y^2), \quad n = 5 \ \ mod\ 6 \\ &= xy(x+y)(x^2+xy+y^2)^2, \quad n = 1 \ \ mod\ 6 \end{aligned}$$

From this theorem it follows that if the equation possess the approximate symmetry of degree 2 then $n = 2, 3, 5$ or 7. Then using the conditions of the existence of approximate symmetries of degree 3 one can prove that the equation is integrable for $n = 2, 3$ and not integrable for any other n.

Classification theorem of scalar homogeneous evolutionary equations

Theorem 5. *(Sanders–Wang) If a homogeneous equation with* $W(u) > 0$

$$u_t = u_n + F[u]$$

possess an infinite hierarchy of higher symmetries then it is up to re-scaling one of the following

$$\begin{aligned}
u_t &= u_2 + uu_1, \\
u_t &= u_3 + uu_1, \\
u_t &= u_3 + u_1^2, \\
u_t &= u_3 + u^2u_1, \\
u_t &= u_3 + 9uu_1^2 + 3u^2u_2 + 3u^4u_1, \\
u_t &= u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \\
u_t &= u_5 + 5u_1u_3 + \frac{5}{3}u_1^3, \\
u_t &= u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1, \\
u_t &= u_5 + 5u_1u_3 + \frac{15}{4}u_2^2 + \frac{5}{3}u_1^3, \\
u_t &= u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1
\end{aligned}$$

Non-evolutionary equations.

$$u_{tt} = \alpha_1 \partial_x^p u + \alpha_2 \partial_x^q u_t + f(u, u_x, \ldots, \partial_x^{p-1} u, u_t, u_{tx}, \ldots, \partial_x^{q-1} u_t)$$

$p > q, \quad \alpha_1, \alpha_2 \in \mathbb{C}$

Example – the Boussinesq equation

$$u_{tt} = \partial_x^4 u + (u^2)_{xx}$$

Every non-evolutionary equation can always be replaced by a system of two evolutionary equations

$$\begin{cases} u_t = v, \\ v_t = \alpha_1 \partial_x^p u + \alpha_2 \partial_x^q v + f(u, u_x, \ldots, \partial_x^{p-1} u, v, v_x, \ldots, \partial_x^{q-1} v) \end{cases}$$

If $f = D_x(\tilde{f})$, then the system

$$\begin{cases} u_t = v_x, \\ v_t = \alpha_1 \partial_x^{p-1} u + \alpha_2 \partial_x^q v + \tilde{f} \end{cases}$$

also represents our non-evolutionary equation.

For example, the Boussinesq equation $u_{tt} = \partial_x^4 u + (u^2)_{xx}$ can be represented by

(A) $u_t = v, \quad v_t = \partial_x^4, u + (u^2)_{xx},$

(B) $u_t = v_x, \quad v_t = \partial_x^3 u + (u^2)_x,$

(C) $u_t = v_{xx}, \quad v_t = \partial_x^2 u + u^2,$

We restrict our attention to the systems of the form:

Even order equations:

$$\begin{cases} u_t = v_x, \\ v_t = \alpha_1 \partial_x^{2n-1} u + \alpha_2 \partial_x^n v + f(u, u_x, \ldots, \partial_x^{2n-2} u, v, v_x, \ldots, \partial_x^{n-1} v) \end{cases}$$

Odd order equations:

$$\begin{cases} u_t = \partial_x^r v, \\ v_t = \partial_x^{2n+1-r} u + f(u, u_x, \ldots, \partial_x^{2n-r} u, v, v_x, \ldots, \partial_x^{2n-r} v), \end{cases}$$

$r = 0, 1, \ldots, n$.

Approximate Symmetries in the Symbolic Representation. Necessary Integrability Conditions.

Our system in the symbolic representation takes the form

$$\begin{cases} u_t = v\zeta_1, \\ v_t = \alpha_1 u\xi_1^{2n-1} + \alpha_2 v\zeta_1^n + \\ \qquad + \sum_{s\geq 2}\sum_{i=0}^{s} u^i v^{s-i} a_{i,s-i}(\xi_1,\ldots,\xi_i,\zeta_1,\ldots,\zeta_{s-i}) \end{cases}$$

Symmetry

$$u_\tau \;=\; \beta_1 u\xi_1^m + \beta_2 v\zeta_1^{m-n+1} + \sum_{s\geq 2}\sum_{i=0}^{s} u^i v^{s-i} A_{i,s-i}(\xi_1,\ldots,\xi_i,\zeta_1,\ldots,\zeta_{s-i})$$

"Generic case":

$$4\alpha_1 \neq \alpha_2^2, \quad \Longrightarrow \alpha_1 = \frac{\alpha^2-1}{4}, \alpha \neq 0, \pm 1, \quad \alpha_2 = 1$$

and $\beta_1 = \frac{\beta}{2}, \quad \beta_2 = 1.$

Affine coordinate q: $\xi_1 = q,\ \xi_2 = 1$.

$$\begin{aligned}
\overrightarrow{A} &= (A_{2,0}(q,1), A_{0,2}(q,1), A_{1,1}(q,1), A_{1,1}(1,q))^T, \\
\overrightarrow{a} &= (a_{2,0}(q,1), a_{0,2}(q,1), a_{1,1}(q,1), a_{1,1}(1,q))^T.
\end{aligned}$$

Let us introduce the following polynomials

$$S_1(\alpha;q) = (1+\alpha)(1+q)^n - (1-\alpha)(1+q^n)$$

$$S_2(\alpha;q) = (1-\alpha)(1+q)^n - (1+\alpha)q^n - 1 + \alpha$$

$$S_3(\alpha;q) = (1-\alpha)(1+q)^n - (1-\alpha)q^n - 1 - \alpha$$

$$S_4(q) = (1+q)^n - 1 - q^n$$

and

$$M_1(\alpha,\beta;q) = (1+\alpha+\beta)(1+q)^n - (1-\alpha+\beta)(1+q^n)$$

$$M_2(\alpha,\beta;q) = (1-\alpha+\beta)(1+q)^n - (1+\alpha+\beta)q^n - 1 + \alpha - \beta$$

$$M_3(\alpha,\beta;q) = (1-\alpha+\beta)(1+q)^n - (1-\alpha+\beta)q^n - 1 - \alpha - \beta$$

$$M_4(\alpha,\beta;q) = (1-\alpha+\beta)\left((1+q)^n - 1 - q^n\right)$$

Proposition 2. *Expression*

$$u_\tau = \frac{\beta}{2}u\xi_1^m + v\zeta_1^{m-n+1} + \\ + \sum_{s\geq 2}\sum_{i=0}^{s} u^i v^{s-i} A_{i,s-i}(\xi_1,\ldots,\xi_i,\zeta_1,\ldots,\zeta_{s-i})$$

is an approximate symmetry of degree 2 of system

$$\begin{cases} u_t = v\zeta_1, \\ v_t = \frac{\alpha^2-1}{4}u\xi_1^{2n-1} + v\zeta_1^n + \\ \qquad + \sum_{s\geq 2}\sum_{i=0}^{s} u^i v^{s-i} a_{i,s-i}(\xi_1,\ldots,\xi_i,\zeta_1,\ldots,\zeta_{s-i}) \end{cases}$$

if and only if functions $A_{i,2-i}(q,1)$, $i = 0,1,2$, *determined as follows are polynomials in* q*:*

$$\overrightarrow{A} = T^{-1}(F_1(\alpha;q), F_1(-\alpha;q), F_2(\alpha;q), F_2(-\alpha;q))$$

$$\overrightarrow{a} = T^{-1}(f_1(\alpha;q), f_1(-\alpha;q), f_2(\alpha;q), f_3(\alpha;q))$$

$$T = \begin{pmatrix} 4 & (1-\alpha)^2 q^{n-1} & 1-\alpha & (1-\alpha)q^{n-1} \\ 4 & (1+\alpha)^2 q^{n-1} & 1+\alpha & (1+\alpha)q^{n-1} \\ 4 & (1-\alpha^2) q^{n-1} & 1-\alpha & (1+\alpha)q^{n-1} \\ 4 & (1-\alpha^2) q^{n-1} & 1+\alpha & (1-\alpha)q^{n-1} \end{pmatrix}$$

$$F_1(\alpha,\beta;q) = \frac{2\Big[(1-\alpha)Z_1(\alpha,\beta;q) - Z_4(\alpha,\beta,q)\Big]}{\alpha(1-\alpha)(1+q)^{n-1}},$$

$$F_2(\alpha,\beta;q) = \frac{2\Big[Z_3(-\alpha,\beta;q) - Z_2(\alpha,\beta;q)\Big]}{\alpha(1+q)^{n-1}},$$

where

$$Z_i(\alpha,\beta;q) = \frac{M_i(\alpha,\beta;q)}{S_i(\alpha;q)} f_i(\alpha;q), \quad i = 1,2,3$$

$$Z_4(\alpha,\beta;q) = \frac{M_4(\alpha,\beta;q)}{S_4(q)} f_1(\alpha;q),$$

In particular, functions $Z_i(\pm\alpha,\beta;q)$ must be polynomials in q.

“Degenerate” dispersion relations

Functions

$$Z_i(\pm\alpha,\beta;q) = \frac{M_i(\pm\alpha,\beta;q)}{S_i(\pm\alpha;q)} f_i(\pm\alpha;q)$$

must be polynomials in q.

Consider two of these conditions $Z_1(\pm\alpha,\beta;q)$. Recall that

$$S_1(\alpha;q) = (1+\alpha)(1+q)^n - (1-\alpha)(1+q^n)$$

$$M_1(\alpha,\beta;q) = (1+\alpha+\beta)(1+q)^n - (1-\alpha+\beta)(1+q^n)$$

Suppose that $p, s \neq 0, -1$ and the values of α, β are such that

$$M_1(\alpha,\beta;p) = 0, \quad S_1(\alpha;p) = 0$$

$$M_1(-\alpha,\beta;s) = 0, \quad S_1(-\alpha;s) = 0$$

Then

$$1 + p^n + s^n + (ps)^n - ((1+p)(1+s))^n = 0,$$
$$1 + p^m + s^m + (ps)^m - ((1+p)(1+s))^m = 0 \qquad (4)$$

Applying the Lech-Mahler theorem we obtain that if equation (4) has infinitely many solutions in m then

p, s and $(1+p)(1+s)$ are roots of unity.

Therefore

$$p^2s^2 + 2sp^2 + 2ps^2 + p^2 + s^2 + 3ps + 2s + 2p + 1 = 0$$

Applying the Smyth's algorithm we obtain (up to the change $p \longrightarrow \frac{1}{p}$, $s \longrightarrow \frac{1}{s}$, $p \longrightarrow s$, $s \longrightarrow p$) the following solutions:

1) $p = e^{\frac{\pi i}{6}},\ s = e^{\frac{5\pi i}{6}}, \quad n, m = 1, 5, 7, 11 \,\mathrm{mod} 12,$

2) $p = e^{\frac{2\pi i}{5}},\ s = e^{\frac{4\pi i}{5}}, \quad n, m = 1, 3, 7, 9 \,\mathrm{mod} 10,$

For $\underline{n=3}$ the possible common root is $p = e^{\frac{2\pi i}{5}}$. Substituting this into $S_1(\alpha; p) = 0$ we find

$$\alpha = -\frac{3}{\sqrt{5}}, \quad \frac{\alpha^2 - 1}{4} = \frac{1}{5}$$

For $\underline{n=5}$ the possible common root is $p = e^{\frac{\pi i}{6}}$. For α we find

$$\alpha = -\frac{5}{3\sqrt{3}}, \quad \frac{\alpha^2 - 1}{4} = -\frac{1}{54}.$$

Odd order non-evolutionary equations.

Theorem 6. *If a homogeneous system with* $W(u) > 0$

$$\begin{cases} u_t = v_r, \\ v_t = u_{2n+1-r} + f[u,v], \;\; r = 0,1,\ldots,n, \;\; n = 1,2,3,\ldots, \end{cases}$$

possess an infinite hierarchy of higher symmetries then it is (up to re-scaling $u \to \alpha u, v \to \beta v, t \to \gamma t, x \to \delta x,\ \alpha,\beta,\gamma,\delta = const$*)*

$$\begin{cases} u_t = v_1, \\ v_t = u_2 + 3uv_1 + vu_1 - 3u^2 u_1, \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = (D_x + u)^{2n}(u) - v^2, \quad n = 1,2,3,\ldots. \end{cases}$$

Even order non-evolutionary equations for $n = 2, 3, 5$

We remind the form of the system in consideration

$$\begin{aligned} u_t &= v_1, \\ v_t &= \alpha_1 u_{2n-1} + \alpha_2 v_n + f[u,v] := F[u,v] \in \mathcal{R}_{w+2n-1} \end{aligned} \tag{5}$$

Case $n = 2$.

It is easy to show that if system (5) is homogeneous and has non-zero quadratic terms, then $W(u) = w = 1, 2, 3$. There are no homogeneous integrable systems (5) with $n = 2$, $w = 3$ and non-zero quadratic terms.

The most general form of the system (5) in the case of $w = 2$ is

$$\begin{aligned} &u_t = v_1 \\ &v_t = \alpha_1 u_3 + \alpha_2 v_2 + c_1 u u_1 + c_2 u v, \end{aligned} \tag{6}$$

Proposition 3. *System (6) possess an infinite hierarchy of higher symmetries if and only if $\alpha_2 = c_2 = 0$. By obvious re-scaling it can be put in the form*

$$\begin{aligned} &u_t = v_1 \\ &v_t = u_3 + 2 u u_1 \end{aligned}$$

The most general form of the system (5) in the case of $w = 1$ is

$$\begin{aligned} u_t &= v_1 \\ v_t &= \alpha_1 u_3 + \alpha_2 v_2 + c_1 u u_2 + c_2 u_1^2 + c_3 u_1 v + c_4 u v_1 + \\ &+ c_5 v^2 + c_6 u^2 u_1 + c_7 u^2 v + c_8 u^4 \end{aligned} \tag{7}$$

Proposition 4. *System (7) posses an infinite hierarchy of higher sym-*

metries if and only if (up to a re-scaling $u \to \alpha u$, $x \to \beta x$, $t \to \gamma t$)

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + u_1^2 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 2u_1 v + 2u^2 u_1 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 2u_1 v + 4uv_1 - 6u^2 u_1 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 4uu_2 + 3u_1^2 - v^2 + 6u^2 u_1 + u^4 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_3 + v_2 + 4\alpha uu_2 + 3\alpha u_1^2 + u_1 v + 2uv_1 \\ -v^2 + 6\alpha u^2 u_1 + u^2 v + \alpha u^4 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 + 2uv_1 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 - u_1^2 + 2u_1 v - v^2 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 - 2uu_2 - 2u_1^2 + 2u_1 v + 6uv_1 - 12u^2 u_1 \end{cases}$$

Case $n = 3$.

It is easy to show that if system (5) is homogeneous and has non-zero quadratic terms, then $W(u) = w = 1, 2, 3, 4, 5$. There are no homogeneous integrable systems (5) with $n = 3$, $w = 3, 4, 5$ and non-zero quadratic terms.

The most general homogeneous system (5) corresponding to $w = 2$ can be written as

$$\begin{aligned} u_t &= v_1, \\ v_t &= \alpha_1 u_5 + \alpha_2 v_3 + D_x[c_1 u u_2 + c_2 u_1^2 + c_5 u^3] + \\ &\quad + c_3 u v_1 + c_4 v u_1. \end{aligned} \tag{8}$$

Proposition 5. *If system (8) possesses an infinite hierarchy of higher symmetries then, up to re-scalings* $u \to \alpha u$, $x \to \beta x$, $t \to \gamma t$, *it is one of the list*

$$\begin{cases} u_t = v_1, \\ v_t = 2u_5 + v_3 + D_x[2uu_2 + u_1^2 + \frac{4}{27}u^3] \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = \frac{1}{5}u_5 + v_3 + D_x[uu_2 + uv + \frac{1}{3}u^3] \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = \frac{1}{5}u_5 + v_3 + D_x[2uu_2 + \frac{3}{2}u_1^2 + 2uv + \frac{4}{3}u^3] \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = v_3 + uv_1 + u_1 v \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = v_3 + 2uu_3 + 4u_1u_2 - 4u_1v - 8uv_1 - 24u^2u_1 \end{cases}$$

Homogeneous systems of equations (5) with $w = 1$ can be written in the form:

$$\begin{aligned} u_t &= v_1, \\ v_t &= \alpha_1 u_5 + \alpha_2 v_3 + c_1 u_2^2 + c_2 u_1 u_3 + c_3 u u_4 + c_4 u_2 v + c_5 u_1 v_1 + \\ &\quad + c_6 u v_2 + c_7 u_1^3 + c_8 u u_1 u_2 + c_9 u^2 u_3 + c_{10} u^2 v_1 + \\ &\quad + c_{11} u u_1 v + + c_{12} u^2 u_1^2 + c_{13} u^3 u_2 + c_{14} u^3 v + \\ &\quad + c_{15} u^4 u_1 + c_{16} u^6 \end{aligned} \tag{9}$$

Proposition 6. *If system (9) possesses infinitely many 4rd degree approximate symmetries then, up to re-scalings $u \to \alpha u$, $x \to \beta x$, $t \to \gamma t$,* *it is one of the equations in the following list*

$$\begin{cases} u_t = v_1 \\ v_t = 2u_5 + v_3 + u_2^2 + 2u_1u_3 + \frac{4}{27}u_1^3 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5}u_5 + v_3 + u_1u_3 + u_1v_1 + \frac{1}{3}u_1^3 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5}u_5 + v_3 + 2u_1u_3 + \frac{3}{2}u_2^2 + 2u_1v_1 + \frac{4}{3}u_1^3 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + u_1v_1 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_5 + v_3 + 10\alpha u_2^2 + 15\alpha u_1u_3 + 6\alpha uu_4 + vu_2 + \\ \quad +3u_1v_1 + 3uv_2 - v^2 + 15\alpha u_1^3 + 15\alpha u^2u_3 + \\ \quad +60\alpha uu_1u_2 + 3uu_1v + 3u^2v_1 + 45\alpha u^2u_1^2 + \\ \quad +20\alpha u^3u_2 + u^3v + 15\alpha u^4u_1 + \alpha u^6 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_5 + 6uu_4 + 15u_1u_3 + 10u_1^2 - v^2 + \\ +15u^2u_3 + 15u_1^3 + 60uu_1u_2 + 45u^2u_1^2 + \\ +20u^3u_2 + 15u^4u_1 + u^6 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + 3u_1v_1 + 3uv_2 + 3u^2v_1 \end{cases}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 - u_2^2 + 2u_2v - v^2 \end{cases}$$

Case $n = 5$.

Proposition 7. *The following systems posses infinite hierarchies of higher symmetries:*

$$\begin{cases} u_t = v_2 \\ v_t = \frac{9}{64}u_8 + v_5 + 3uu_6 + 9u_1u_5 + \frac{65}{4}u_2u_4 + \frac{35}{4}u_3^2 + \\ \quad +2u_1v_2 + 4uv_3 + 20u^2u_4 + 80uu_1u_3 + 60uu_2^2 + \\ \quad +88u_1^2u_2 + \frac{256}{5}u^3u_2 + \frac{384}{5}u^2u_1^2 + \frac{1024}{125}u^5 \end{cases} \tag{10}$$

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{54}u_9 + v_5 + \frac{5}{6}u_7u_1 + \frac{5}{3}u_6u_2 + \frac{5}{2}u_5u_3 + \frac{25}{12}u_4^2 - \\ \quad -5u_3v_1 - \frac{15}{2}u_2v_2 - 10u_1v_3 - \frac{45}{4}u_5u_1^2 - \frac{75}{2}u_1u_2u_4 - \\ \quad -\frac{75}{4}u_3^2u_1 - \frac{75}{4}u_2^2u_3 + \frac{45}{2}u_1^2v_1 + \frac{225}{4}u_3u_1^3 + \\ \quad +\frac{675}{8}u_2^2u_1^2 - \frac{405}{16}u_1^5 \end{cases} \tag{11}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_5 + 2u_2u_5 + 6u_3u_4 - 6u_3v - 22u_2v_1 - 30u_1v_2 - \\ \quad -20uv_3 + 96uu_1v + 96u^2v_1 - \\ \quad -2D_x[8u^2u_4 + 32uu_1u_3 + 13u_1^2u_2 + 24uu_2^2] + \\ \quad +120D_x[4u^3u_2 + 6u^2u_1^2] - 3840u^4u_1 \end{cases} \tag{12}$$

Camassa-Holm type equations equation

$$m = u - u_2, \quad m_t = cmu_1 + um_1$$

Camassa-Holm equation can be rewritten as

$$u_t = \Delta(-uu_3 + (c+1)uu_1 - cu_1u_2)\,, \quad c \neq 0\,, \tag{13}$$

where operator $\Delta = (1 - D^2)^{-1}$.

We extend the differential ring $\mathcal{R}$

$$\mathcal{R}^0_\Delta = \mathcal{R}\,, \quad \mathcal{R}^1_\Delta = \overline{\mathcal{R}^0_\Delta \bigcup \Delta(\mathcal{R}^0_\Delta)}\,, \quad \mathcal{R}^{n+1}_\Delta = \overline{\mathcal{R}^n_\Delta \bigcup \Delta(\mathcal{R}^n_\Delta)}\,,$$

Symbolic representation of operator Δ is $\Delta \to \frac{1}{1-\eta^2}$. The symbolic representation of elements of differential rings $\mathcal{R}^n_\Delta$ is obvious. For example if $A \in \mathcal{R}^0_\Delta$ and

$$A \to u^n a(\xi_1, ..., \xi_n) \Longrightarrow \Delta(A) \to u^n \frac{a(\xi_1, \ldots, \xi_n)}{1 - (\xi_1 + \cdots + \xi_n)^2}$$

.

Theorem 7. *(Mikhailov-VN) Equation*

$$u_t = \Delta(-uu_3 + (c+1)uu_1 - cu_1u_2)\,, \quad c \neq 0$$

is integrable if and only if $c = 2$ *or* $c = 3$