NUMBER SYSTEMS THROUGHOUT HISTORY

Proto-numbers	One-two-many		Prehistory
No arithmetic			
Natural numbers	1, 2, 3, ...	Infinitely many!	Prehistory
Counting, same as ordering Addition, multiplication Limited subtraction, division			
Negative numbers, zero	0, -1, -2, ...	As many again	Ancient China, India: Europe 17th century
Subtraction			
Rational numbers	m/n	Doubly infinite array, "squashed" for ordering	Early history?
Division Lacking π, $\sqrt{2}$, etc			Pythagoras (<500BC)
Real numbers	"Usual numbers"	Innumerably more	Cantor, Meray (sequences)
Root extraction Limits, least upper bound		formal by 1880	Dedekind (cuts) Weierstrass (dec)
Complex (imaginary) numbers		Double array	R Bombelli (1572) [Distrust till geom interpretn ca 1800]
Extensive equation solvability Ordering lost!			
Quaternions		Double again	W R Hamilton (1843) [alone]
Commutative multiplication lost			
Cardinal numbers Ordinal numbers	0, 1, 2, ..., $\aleph_0$, $\aleph_1$, ... 0, 1, 2, ..., ω, $\omega+1$, $\omega+2$, ...		G Cantor (1883) [alone]
Transfinite (actually infinite!) Cardinals: Counting how many Ordinals: (Well-)Ordering Arithmetically primitive (operations lost)			
Ordinal numbers with "natural" arithmetic			Hessenberg [alone ?]
Same objects [!] Commutative but primitive arithmetic (cf nat nos)			
p-adic numbers			K Hensel (1899) [alone]
Hyperreal numbers (nonstandard analysis)			A Robinson (1966) [alone]
Full arithmetic Axiom of Choice (indefinite objects) Model theory (can have nonstandard anything)			
Surreal numbers			J H Conway (1976) [alone]
Full arithmetic Definite objects Earliness [!]			

A

CONSTRUCTION OF REAL NUMBER SYSTEM

(definitively by Edmund Landau)

NATURAL NUMBERS

Natural numbers: 1, 2, 3, ...

Peano Axioms:

$x \longrightarrow Sx$	(Successor function)
$Sx = Sy \implies x = y$	(Unique predecessor)
$Sx \neq 1$	(Nonsuccessor number)
$\{P(1) \ \& \ [P(x) \implies P(Sx)]\} \implies P(x)$	(Mathematical induction)

Addition:

$x + 1 = Sx$	(Recursive definition)
$x + Sy = S(x+y)$	
$x + y = y + x$	(Commutativity)
$(x + y) + z = x + (y + z)$	(Associativity)

Ordering:

$x < y$ if $x + z = y$ for some z	(Definition)
$x < y \ \& \ y < z \implies x < z$	(Transitivity)
$x = y$ or $x < y$ or $y < x$	(Trichotomy)

Multiplication:

$1y = y$	(Recursive definition)
$(Sx)y = xy + y$	
$xy = yx$	(Commutativity)
$(xy)z = x(yz)$	(Associativity)
$(x + y)z = xz + yz$	(Distributivity)

POSITIVE RATIONAL NUMBERS

Ratios: x/y Special value: $1/1 = 1$

Equivalence: $(xz)/(yz) = x/y$

Addition: $x/y + z/w = (xw + yz)/(yw)$ (Laws as above)

Multiplication: $(x/y)(z/w) = (xz)/(yw)$ (Laws as above)

Division (reciprocal): $1/(x/y) = y/x$ (Involutional)

$x < y \implies x < z < y$ for some z (Denseness)

POSITIVE REAL NUMBERS

Dedekind cuts (or other methods) (Laws as above)

Completeness: Every nonempty bounded set has least upper bound

FULL REAL NUMBER SYSTEM

Duplicate for negatives, adjoin zero (Laws as above)

Subtraction

B

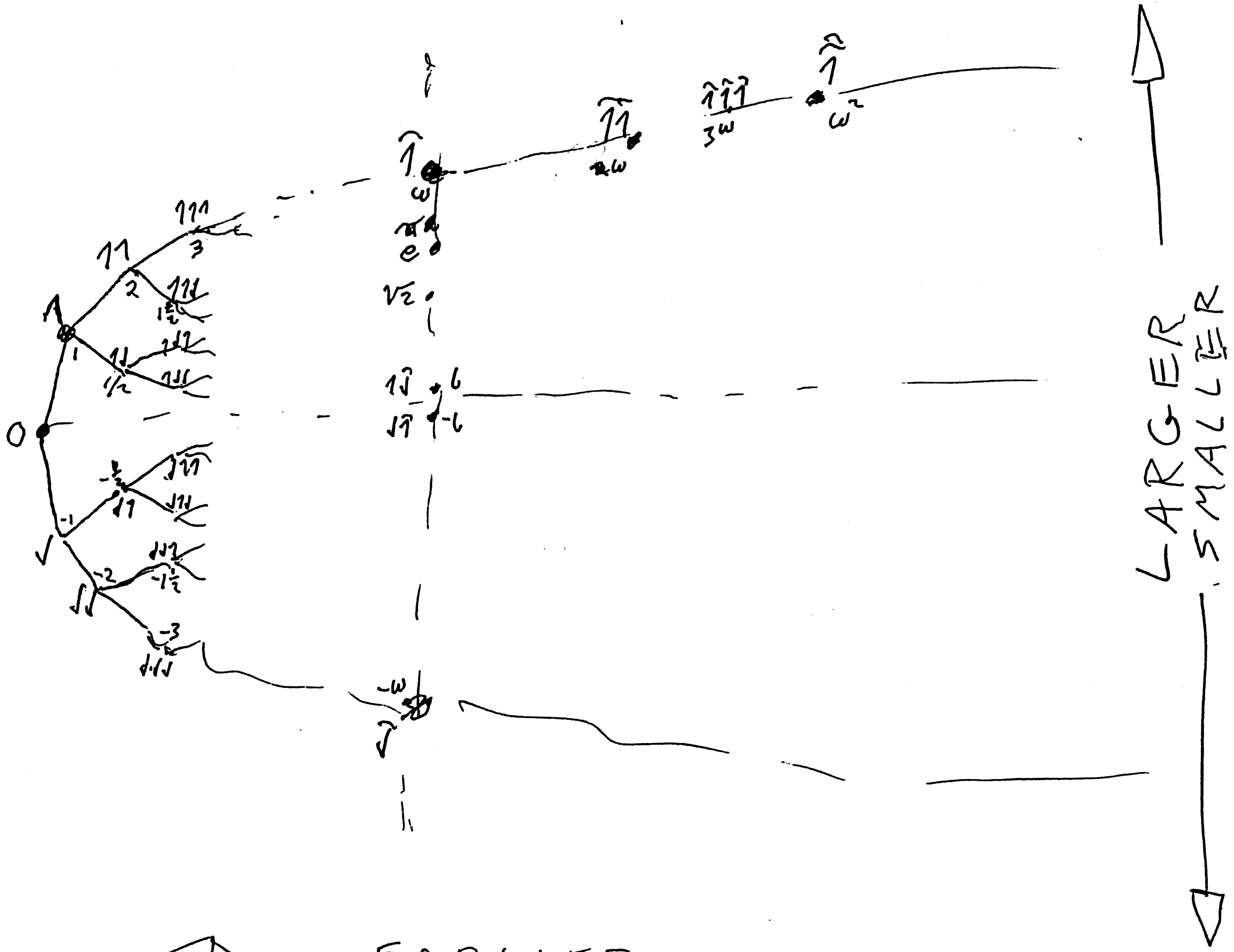
Brief definition

A surreal number
is a well ordered sequence
of binary choices.

Even pithier

A number
is an ordinal sequence
of bits.

Number tree



EARLIER

LATER

LONGER

SHORTER

D

TABLE OF SURREAL NUMBERS — ARITHMETIC VS ARROW-SEQUENCE REPRESENTATIONS

Arithmetic	Arrow
0	0 [blank]
1	$\uparrow$
-1	$\downarrow$
2	$\uparrow\uparrow$
-2	$\downarrow\downarrow$
1/2	$\uparrow\downarrow$
-1/2	$\downarrow\uparrow$
3	$\uparrow\uparrow\uparrow$
1 1/2	$\uparrow\uparrow\downarrow$
2 1/2	$\uparrow\uparrow\uparrow\downarrow$
1/4	$\uparrow\downarrow\downarrow$
1/8	$\uparrow\downarrow\downarrow\downarrow$
2/3	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\ldots = \widehat{\uparrow\downarrow}$
1/3	$\uparrow\downarrow\widehat{\downarrow\uparrow}$
w	$\hat{\uparrow}$
ι	$\uparrow\hat{\downarrow}$
$w+1$	$\hat{\uparrow}\uparrow$
$w+2$	$\hat{\uparrow}\uparrow\uparrow$
$w+1/2$	$\hat{\uparrow}\uparrow\downarrow$
$w+\iota$	$\hat{\uparrow}\uparrow\hat{\downarrow}$
$2\,w$	$\hat{\uparrow}\hat{\uparrow}$
$3\,w$	$\hat{\uparrow}\hat{\uparrow}\hat{\uparrow}$
w^2	$\hat{\uparrow}\hat{\uparrow}\hat{\uparrow}\ldots = \hat{\hat{\uparrow}}$
w^3	$\hat{\hat{\hat{\uparrow}}}$
e^w	$\overset{\vdots}{\hat{\hat{\hat{\uparrow}}}} = \overset{\wedge}{\uparrow}$ [ordinal w^w]

Arithmetic	Arrow
$w-1$	$\hat{\uparrow}\downarrow$
$w-2$	$\hat{\uparrow}\downarrow\downarrow$
$w/2$	$\hat{\uparrow}\hat{\downarrow}$
$w/4$	$\hat{\uparrow}\hat{\downarrow}\hat{\downarrow}$
$w/3$	$\hat{\uparrow}\hat{\downarrow}\widehat{\hat{\downarrow}\hat{\uparrow}}$
$w^{1/2}$	$\hat{\uparrow}\hat{\hat{\downarrow}}$
$w^{1/2}-1$	$\hat{\uparrow}\hat{\hat{\downarrow}}\downarrow$
$2\,w^{1/2}$	$\hat{\uparrow}\hat{\hat{\downarrow}}\hat{\uparrow}$
$w^{1/2}/2$	$\hat{\uparrow}\hat{\hat{\downarrow}}\hat{\downarrow}$
$w^{1/2}/4$	$\hat{\uparrow}\hat{\hat{\downarrow}}\hat{\downarrow}\hat{\downarrow}$
$w^{1/4}$	$\hat{\uparrow}\hat{\hat{\downarrow}}\hat{\hat{\downarrow}}$
$w^{3/8}$	$\hat{\uparrow}\hat{\hat{\downarrow}}\hat{\hat{\downarrow}}\hat{\hat{\uparrow}}$
$w^{1/3}$	$\hat{\uparrow}\hat{\hat{\downarrow}}\widehat{\hat{\hat{\downarrow}}\hat{\hat{\uparrow}}}$
$\log w$	$\hat{\uparrow}\hat{\hat{\hat{\downarrow}}}$
$2\,\iota$	$\uparrow\hat{\downarrow}\uparrow$
$\iota/2$	$\uparrow\hat{\downarrow}\downarrow$
$\iota^{1/2}$	$\uparrow\hat{\downarrow}\hat{\uparrow}$
$\iota^{1/4}$	$\uparrow\widehat{\hat{\downarrow}\uparrow}\hat{\uparrow}$
$\iota/4$	$\uparrow\hat{\downarrow}\uparrow\uparrow$
ι^2	$\uparrow\widehat{\hat{\downarrow}}\hat{\downarrow}$
e^{-w}	$\uparrow\hat{\hat{\downarrow}}$
$1/\log w$	$\uparrow\hat{\downarrow}\hat{\hat{\uparrow}}$

SURREAL ARITHMETIC LAWS

Mostly like real number arithmetic

Commutative real closed field, of characteristic zero, with no divisors of zero

But not complete

Set of (usual) integers is bounded (e g by ω), but has no least upper bound (since any upper bound is a positive huge number, but can be diminished by 1 and remain positive huge)

Archimedean with respect to the whole numbers, but not with respect to the (usual) integers

Exponential and logarithm are defined and obey usual laws

Power x^y defined for $x>0$ as $\exp(y \log x)$

Usual laws of exponents apply

Every number has naturally defined unique "normal expansion"

$$\sum_{\alpha<\beta} r_\alpha \mu^{(\alpha)},$$

where α, β are ordinals,

the r_α are nonzero real numbers, and

the $\mu^{(\alpha)}$ are magnitudes decreasing with increasing α,

and every such form sums in a natural way to a definite number (having that form as its normal expansion).

These expansions can be combined and manipulated in all natural formal ways and the results agree with the same operations on the corresponding numbers.

I e, the correspondence is arithmetically perfect (J H Conway, and a deep theorem on well-quasi-ordered trees by J B Kruskal)

Other new arithmetic laws (analogous to usual analytic laws)

$\exp x > x^n$ for positive huge x and natural number n

$$\exp x = \sum_{n \in Nat} x^n/n! \quad \text{for tiny } x$$

BASIC REFERENCES FOR SURREAL NUMBERS

Surreal numbers: how two ex-students turned on
to pure mathematics and found total happiness:
a mathematical novelette
by Donald E Knuth
Addison-Wesley Publishing Co (1974)

On numbers and games
(London Mathematical Society monographs: no.6)
by John H Conway
Academic Press (1976)

An introduction to the theory of surreal numbers
(London Mathematical Society Lecture Note Series: 110)
by Harry Gonshor
Cambridge University Press (1986)

Foundations of analysis over surreal number fields
by Norman L Alling
North-Holland Publishing Co &
Elsevier Science Publishing Co (1987)

G

A SHORT COURSE IN THE ELEMENTARY THEORY OF SURREAL NUMBERS

A (surreal) number is a well ordered sequence of binary choices, up or down.

The numbers are size-ordered lexicographically, "up" greater than "down".

The numbers have a partial ordering of "earliness": a proper initial segment of a sequence of choices is earlier than the sequence.

Any consistent set of inequalities is satisfied by a unique earliest number.

The sum (addition) of two numbers is uniquely defined, recursively, as the earliest number consistent with (increasing) monotonicity in each argument.

The null sequence is the identity of addition, namely zero.

The sequence of opposite choices of a number (ups and downs interchanged) is its additive inverse, namely its negative.

A nonzero number is positive (negative) as its first choice is up (down).

All familiar elementary properties of addition apply (except archimedean).

The product (multiplication) of two numbers is uniquely defined, recursively, as the the earliest number consistent with distributivity over addition and positivity for positive factors (arguments).

A single up is the identity of multiplication (unity, the number one).

All familiar elementary properties of multiplication, in itself and with respect to addition, apply (except the archimedean property: there are infinite numbers, but any integer multiple of a finite number is finite).

An ordinal number is a sequence of only ups. The ordinal numbers obey the Hessenberg or so-called natural (commutative) arithmetic (rather than Cantor's noncommutative arithmetic).

A natural number is an ordinal number of finitely many choices.

The absolute value of a number is itself if positive, otherwise its negative.

A number is finite if its absolute value is less than some natural number, otherwise huge (infinite).

Any ordinal nonnatural number is huge. The earliest huge ordinal is omega, a simplest infinite sequence of ups.

A number is tiny (infinitesimal) if the absolute value of its product with every natural number is less than unity.

A usual ("real") number is one with either finitely many choices, or else fewest possible infinitely many choices with infinitely many ups and infinitely many downs.

A whole number (generalized integer) has no up immediately adjacent to a down.

"New" numbers, neither real nor ordinal, include: Omega minus one, omega followed by a down. Iota, the earliest positive tiny number, up followed by a simplest infinite sequence of downs.

A magnitude is a positive number not in finite ratio to any earlier number.

Every number has a unique normal expansion as a sum of terms, each a product of a nonzero usual number with a magnitude, the magnitudes well ordered with respect to "(much) greater than".

COMPARISON OF HYPERREAL NUMBERS WITH SURREAL NUMBERS

	Hyperreal numbers (A Robinson)	Surreal numbers (J H Conway)
Includes real numbers?	Yes	Yes
Includes usual arithmetic?	Yes	Yes
Includes infinite numbers?	Yes	Yes
Includes infinitesimal numbers?	Yes	Yes
Allows differential calculus?	Yes	Yes
Allows integral calculus?	Yes	Yes (?)
Constructive (without axiom of choice)?	No	Yes
Definite mathematical objects?	No	Yes
Helps prove standard analytic results?	Yes	Hopefully
$2^{1/2}$, etc	Irrational	"Rational"
Mathematical level (concepts, proofs, properties, operations, etc)	Sophisticated	Elementary
Delay in discovery (once accessible)	Short	Long

MEASURING GROWTH RATES OF FUNCTIONS

f(x) (as x --> oo)	Growth rate (number)	Growth rate (surreal form)
x	1	↑
x^2	2	↑↑
x^3	3	↑↑↑
constant	0	0 [= null sequence]
$x^{-1} = 1/x$	-1	↓
$x^{-2} = 1/x^2$	-2	↓↓
$x^{1/2} = \sqrt{x}$	1/2	↑↓
x^n	n	↑↑...↑ [n up-arrows]
$\exp x = e^x$	?	
log x [ln x]	?	
exp x	ω	↑↑↑... = $\widehat{\uparrow}$
log x	$\iota = 1/\omega$	↑↓↓↓... = ↑$\widehat{\downarrow}$
xe^x	$\omega+1$	↑↑↑...↑ = $\widehat{\uparrow}$↑

J

SOME SPECIAL CLASSES OF SURREAL NUMBERS

Counting numbers: 1, 2, 3, 4, ...

Natural numbers: 0, 1, 2, 3, ...

Integers: ... , -2, -1, 0, 1, 2, ...

Rationals: ratio of integers

Real numbers: finite arrow sequence, or simply infinite arrow sequence without pure tail of up's or pure tail of down's

Ordinals (with Hessenberg "natural" operations")

Whole numbers (generalized integers): earlier than any number < unit distance away

Huge (infinite) numbers: greater than every integer

Tiny (infinitesimal) numbers: absolute value < every positive real

Magnitudes: positive, and no earlier comparable number

Extravagances: positive huge, and > iterated exponential or < iterated logarithm of any earlier number

Simple: one-term normal form, product of magnitude by nonzero real number

Quanta: positive, and < any earlier positive number

SOME TYPES OF SORT-OF-NUMBER

Gaps: Dedekind cut of surreal number line not "at" a number

(e g $\infty = \widetilde{\uparrow\downarrow}^{\ell}$)

Virtual numbers: gap of width zero not "at" a number (e g $\widehat{\uparrow\downarrow}^{\ell}$)

Ghost infinitesimals: indefinite number, changeable with context, vastly smaller (in absolute value) than any other number appearing in context

Ghost infinite number: indefinite number, changeable with context, vastly larger (in absolute value) than any other number appearing in context

K

ARROW AND BINARY REPRESENTATIONS OF "REAL" SURREAL NUMBERS

A "real" or usual number is a surreal number whose arrow sequence is either:

1) of finite length, in which case the number is a dyadic rational, namely the ratio of an integer to a power of 2, or

2) of simple infinite length (ω length) with infinitely many up arrows and infinitely many down arrows, i e with no pure tail of ups nor pure tail of downs, in which case it is:

   a) a nondyadic rational (if it has a repeating finite sequence from some point on), or

   b) an irrational (if it has no repeating finite sequence of arrows from any point on).

Examples of real numbers:

$\downarrow\downarrow\uparrow\uparrow\downarrow = -11/8$ (case 1)

$\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\ldots = \uparrow\overset{\frown}{\downarrow\uparrow\uparrow\downarrow} = 4/5$ (case 2a)

$\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\ldots = 2^{1/2}$ (case 2b)

Examples of ω-length surreals that are not real numbers:

$\downarrow\downarrow\downarrow\ldots = \overset{\frown}{\downarrow} = -\omega$ (pure tail of downs)

$\uparrow\downarrow\uparrow\uparrow\uparrow\ldots = \uparrow\downarrow\overset{\frown}{\uparrow} = 1-\iota$ (pure tail of ups)

To convert an arrow sequence representing a real number to the usual binary expansion of that number:

A) If the number is negative (first arrow is a down), replace it by its absolute value (reverse all the arrows) and insert a minus sign in front.

B) If the number is an integer n , i e consists of n ups (perhaps preceded by a minus sign), write n as a binary integer in the usual way as a sequence of digits 0 and 1, and stop.

C) Otherwise, replace the first up-down arrow pair by the binary point (like the decimal point in decimal representation).

D) Write the number of up arrows before the point as a binary integer (as in step B).

E) Replace each down arrow after the point by 0 and each up arrow after the point by 1 .

F) If the sequence of digits is finite, append a final digit 1 , and stop.

Example: Start with $\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow$ (= -21/8): A converts it to $-\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow$;
B doesn't apply; C converts it to $-\uparrow\uparrow.\uparrow\downarrow$;
D converts it to $-10.\uparrow\downarrow$; E converts it to -10.10 ;
F converts it to final form -10.101 .

To convert from binary expansion to arrow representation, reverse the whole process.

L

QUOTES ABOUT NUMBERS

Isaac Newton (in Arithmeticae Universalis, 1707):

> By a number we understand not so much a multiple of a unit as an abstract quantity associated in a systematic way to some other quantity of the same kind that is taken as a unit. Numbers arise in three forms: integer, fraction and irrational. An integer is that which can be measured by unity; a fraction is a multiple of a portion of unity; an irrational number is incommensurable with unity.

Leopold Kronecker:

> God gave us the integers, all else is the work of man.

Leopold Kronecker (to Lindemann, quoted by H W Turnbull):

> Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are non-existent?
>
> [The reference is undoubtedly to Lindemann's 1882 proof that π is a transcendental number, i e cannot be described in purely algebraic terms.]

Bertrand Russell (at the start of an article titled "Definition of Number"):

> The question "What is a number?" is one which has been often asked, but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his _Grundlagen_der_Arithmetik_. Although this book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author in 1901.
>
> [He is referring to the cardinal numbers.]

Brooke's Law: Whenever a system becomes completely defined, some damn fool discovers something which either abolishes the system or expands it beyond recognition.

> [Found on Usenet.]

M

QUOTES ABOUT ASYMPTOTICS

G H Hardy, in Orders of Infinity (1910):

> No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.

van der Corput (1954):

> ... there reigns in pure asymptotics a — partially still unrevealed — harmony which enables us under general conditions to write down almost immediately the required asymptotic expansion.

N