(I): Factorization of Linear Ordinary Differential Equations

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Outline

A teaser: Landau theorem, 1902

Topics to be discussed today

Loewy-Ore theory (and other flavours of factorization)

Classical algorithm of factorization (Beke)

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Linear ordinary differential (difference) operators:

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Beke E. (1894) gave an algorithm for factorization for the case $a_i \in Q(x)$.

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1990s: M. Bronstein, M. van Hoeij, M. Petkovšek (difference case), F. Schwarz, S. Ts. . . .

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Algorithms for decomposition:
M.Sosnin (2001), Gao et. al. (2003).

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$rGCD(L, M) = G \Longleftrightarrow L = L_1 \cdot G,\ M = M_1 \cdot G$
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Exercise: describe algorithms for GCDs and LCMs.

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$rLCM(L, M) = K$
$\iff \langle Sol(L), Sol(M) \rangle = \{u + v | Lu = 0, Mv = 0\} = Sol(K)$

Operator equations:

$$X \cdot L + Y \cdot M = B, \qquad L \cdot Z + M \cdot T = C$$

with unknown operators X, Y, Z, T are solvable iff respectively $rGCD(L, M)$ divides B on the right and $lGCD(L, M)$ divides C on the left.

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Q.: How one can find out if two given LODOs are similar?

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Such lattices are called *modular lattices* or *Dedekind structures*.

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There are notions of similarity, direct sums,
Kurosh & Ore theorems on direct sums ...

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Two operators P_1, P_2 generate the same morphism $\iff P_1 = P_2 (mod\ L)$.

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Because for morphisms:

1) P and L may have common solutions, i.e. nontrivial $rGCD(P, L)$. This means that the mapping of the solution space $Sol(L)$ by P may have a kernel $Sol(rGCD(P, L))$.
The morphism is not injective in this case.

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1) P and L may have common solutions, i.e. nontrivial $rGCD(P, L)$. This means that the mapping of the solution space $Sol(L)$ by P may have a kernel $Sol(rGCD(P, L))$.
The morphism is not injective in this case.

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Q.: Why do we need this “abstract nonsense”??

A.: this is *really needed* for the case of LPDEs!

Outline

A teaser: Landau theorem, 1902

Topics to be discussed today

Loewy-Ore theory (and other flavours of factorization)

Classical algorithm of factorization (Beke)

Classical algorithm of factorization (Beke) and its modern rivals

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(I) $L = L_1 \cdot (D - u(x)) \iff u = \frac{y'}{y}$, $y(x)$ is a solution of $Ly = 0$.
For $u \in k = Q(x)$ this means $y = \exp(\int u\,dx)$
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(II) $L = L_1 \cdot L_2$, $ord(L_2) = m$
$\Longrightarrow$ for some *associated* LODO $L_{(m)}$,
$L_{(m)} = \overline{L}_1 \cdot (D - \overline{u}(x))$